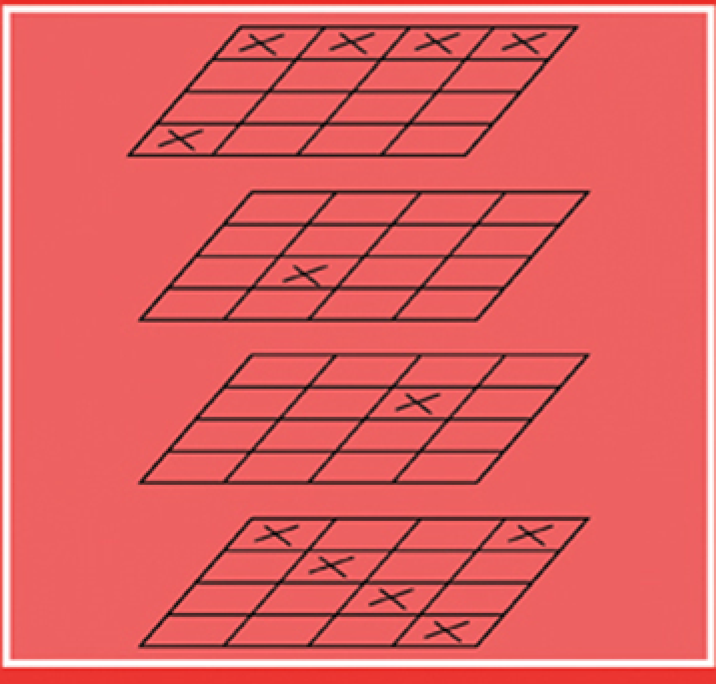
Encyclopedia of Mathematics and Its Applications 114

COMBINATORIAL GAMES

Tic-Tac-Toe Theory

József Beck



CAMBRIDGE

This page intentionally left blank

Combinatorial Games

Traditional game theory has been successful at developing strategy in games of incomplete information: when one player knows something that the other does not. But it has little to say about games of complete information, for example Tic-Tac-Toe, solitaire, and hex. This is the subject of Combinatorial Game Theory. Most board games are a challenge for mathematics: to analyze a position one has to examine the available options, and then the further options available after selecting any option, and so on. This leads to combinatorial chaos, where brute force study is impractical.

In this comprehensive volume, József Beck shows readers how to escape from the combinatorial chaos via the fake probabilistic method, a game-theoretic adaptation of the probabilistic method in combinatorics. Using this, the author is able to determine the exact results about infinite classes of many games, leading to the discovery of some striking new duality principles.

József Beck is a Professor in the Mathematics Department of Rutgers University. He has received the Fulkerson Prize for research in Discrete Mathematics and has written around 100 research publications. He is the co-author, with W. L. Chen, of the pioneering monograph *Irregularities of Distribution.*

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

50 F. Borceux *Handbook of Categorical Algebra I*
51 F. Borceux *Handbook of Categorical Algebra II*
52 F. Borceux *Handbook of Categorical Algebra III*
53 V. F. Kolchin *Random Graphs*
54 A. Katok and B. Hasselblatt *Introduction to the Modern Theory of Dynamical Systems*
55 V. N. Sachkov *Combinatorial Methods in Discrete Mathematics*
56 V. N. Sachkov *Probabilistic Methods in Discrete Mathematics*
57 P. M. Cohn *Skew Fields*
58 R. Gardner *Geometric Tomography*
59 G. A. Baker, Jr., and P. Graves-Morris *Padé Approximants, 2nd edn*
60 J. Krajicek *Bounded Arithmetic, Propositional Logic, and Complexity Theory*
61 H. Groemer *Geometric Applications of Fourier Series and Spherical Harmonics*
62 H. O. Fattorini *Infinite Dimensional Optimization and Control Theory*
63 A. C. Thompson *Minkowski Geometry*
64 R. B. Bapat and T. E. S. Raghavan *Nonnegative Matrices with Applications*
65 K. Engel *Sperner Theory*
66 D. Cvetkovic, P. Rowlinson and S. Simic *Eigenspaces of Graphs*
67 F. Bergeron, G. Labelle and P. Leroux *Combinatorial Species and Tree-Like Structures*
68 R. Goodman and N. Wallach *Representations and Invariants of the Classical Groups*
69 T. Beth, D. Jungnickel, and H. Lenz *Design Theory, I, 2nd edn*
70 A. Pietsch and J. Wenzel *Orthonormal Systems for Banach Space Geometry*
71 G. E. Andrews, R. Askey and R. Roy *Special Functions*
72 R. Ticciati *Quantum Field Theory for Mathematicians*
73 M. Stern *Semimodular Lattices*
74 I. Lasiecka and R. Triggiani *Control Theory for Partial Differential Equations I*
75 I. Lasiecka and R. Triggiani *Control Theory for Partial Differential Equations II*
76 A. A. Ivanov *Geometry of Sporadic Groups I*
77 A. Schinzel *Polymomials with Special Regard to Reducibility*
78 H. Lenz, T. Beth, and D. Jungnickel *Design Theory II, 2nd edn*
79 T. Palmer *Banach Algebras and the General Theory of ∗-Algebras II*
80 O. Stormark *Lie's Structural Approach to PDE Systems*
81 C. F. Dunkl and Y. Xu *Orthogonal Polynomials of Several Variables*
82 J. P. Mayberry *The Foundations of Mathematics in the Theory of Sets*
83 C. Foias *et al. Navier–Stokes Equations and Turbulence*
84 B. Polster and G. Steinke *Geometries on Surfaces*
85 R. B. Paris and D. Kaminski *Asymptotics and Mellin–Barnes Integrals*
86 R. McEliece *The Theory of Information and Coding, 2nd edn*
87 B. Magurn *Algebraic Introduction to K-Theory*
88 T. Mora *Solving Polynomial Equation Systems I*
89 K. Bichteler *Stochastic Integration with Jumps*
90 M. Lothaire *Algebraic Combinatorics on Words*
91 A. A. Ivanov and S. V. Shpectorov *Geometry of Sporadic Groups II*
92 P. McMullen and E. Schulte *Abstract Regular Polytopes*
93 G. Gierz *et al. Continuous Lattices and Domains*
94 S. Finch *Mathematical Constants*
95 Y. Jabri *The Mountain Pass Theorem*
96 G. Gasper and M. Rahman *Basic Hypergeometric Series, 2nd edn*
97 M. C. Pedicchio and W. Tholen (eds.) *Categorical Foundations*
98 M. E. H. Ismail *Classical and Quantum Orthogonal Polynomials in One Variable*
99 T. Mora *Solving Polynomial Equation Systems II*
100 E. Olivieri and M. Eulália Vares *Large Deviations and Metastability*
102 L. W. Beineke *et al.* (eds.) *Topics in Algebraic Graph Theory*
103 O. Staffans *Well-Posed Linear Systems*
105 M. Lothaire *Applied Combinatorics on Words*
106 A. Markoe *Analytic Tomography*
107 P. A. Martin *Multiple Scattering*
108 R. A. Brualdi *Combinatorial Matrix Classes*
110 M.-J. Lai and L. L. Schumaker *Spline Functions on Triangulations*
111 R. T. Curtis *Symmetric Generation of Groups*
112 H. Salzmann, T. Grundhöfer, H. Hähl and R. Löwen *The Classical Fields*
113 S. Peszat and J. Zabczyk *Stochastic Partial Differential Equations with Lévy Noise*

Combinatorial Games

Tic-Tac-Toe Theory

JÓZSEF BECK
Rutgers University

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521461009

First published 2008

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-46100-9 hardback

Transferred to digital printing 2009

Dedicated to
my mother who taught me how to play Nine Men's Morris ("Mill")

Contents

Preface

There is an old story about the inventor of Chess, which goes something like this. When the King learned the new game, he quickly fell in love with it, and invited the inventor to his palace. “I love your game,” said the King, “and to express my appreciation, I decided to grant your wish.” “Oh, thank you, Your Majesty,” began the inventor, “I am a humble man with a modest wish: just put one piece of rice on the first little square of the chess board, 2 pieces of rice on the second square, 4 pieces on the third square, 8 pieces on the fourth square, and so on; you double in each step.” “Oh, sure,” said the King, and immediately called for his servants, who started to bring in rice from the huge storage room of the palace. It didn’t take too long, however, to realize that the rice in the palace was not enough; in fact, as the court mathematician pointed out, even the rice produced by the whole world in the last thousand years wouldn’t be enough to fulfill the inventor’s wish ($2^{64} - 1$ pieces of rice). Then the King became so angry that he gave the order to execute the inventor. This is how the King discovered Combinatorial Chaos.

Of course, there is a less violent way to discover Combinatorial Chaos. Any attempt to analyze unsolved games like Chess, Go, Checkers, grown-up versions of Tic-Tac-Toe, Hex, etc., lead to the same conclusion: we get quickly lost in millions and millions of cases, and feel shipwrecked in the middle of the ocean.

To be fair, the hopelessness of Combinatorial Chaos has a positive side: it keeps the games alive for competition.

Is it really hopeless to escape from Combinatorial Chaos? The reader is surely wondering: “How about Game Theory?” “Can Game Theory help here?” Traditional Game Theory focuses on games of *incomplete information* (like Poker where neither player can see the opponent’s cards) and says very little about Combinatorial Games such as Chess, Go, etc. Here the term Combinatorial Game means a 2-player zero-sum game of skill (no chance moves) with *complete information*, and the payoff function has 3 values only: win, draw, and loss.

The “very little” that Traditional Game Theory can say is the following piece of advice: try a backtracking algorithm on the game-tree. Unfortunately, backtracking leads to mindless exponential-time computations and doesn't give any insight; this is better than nothing, but not much. Consequently, computers provide remarkably little help here; for example, we can easily simulate a *random play* on a computer, but it is impossible to simulate an *optimal play* (due to the enormous complexity of the computations). We simply have no data available for these games; no data to extrapolate, no data to search for patterns.

The 3-dimensional $5 \times 5 \times 5$ version of Tic-Tac-Toe, for instance, has about 3^{125} positions (each one of the 5^3 cells has 3 options: either marked by the first player, or marked by the second player, or unmarked), and backtracking on a graph of 3^{125} vertices ("position graph") takes at least 3^{125} steps, which is roughly the third power of the "chaos" the chess-loving King was facing above. No wonder the $5 \times 5 \times 5 = 5^3$ Tic-Tac-Toe is unsolved!

It is even more shocking that we know only two(!) explicit winning strategies in the whole class of $n \times n \times \cdots \times n = n^d$ Tic-Tac-Toe games: the 3^3 version (which has an easy winning strategy) and the 4^3 version (which has an extremely complicated winning strategy).

If traditional Game Theory doesn't help, and the computer doesn't really help either, then what can we do? The objective of this book is exactly to show an escape from Combinatorial Chaos, to win a battle in a hopeless war. This "victory" on the class of Tic-Tac-Toe-like games is demonstrated. Tic-Tac-Toe itself is for children (a very simple game really), but there are many grown-up versions, such as the $4 \times 4 \times 4 = 4^3$ game, and, in general, the $n \times n \times \cdots \times n = n^d$ hypercube versions, which are anything but simple. Besides hypercube Tic-Tac-Toe, we study Clique Games, Arithmetic Progression Games, and many more games motivated by Ramsey Theory. These "Tic-Tac-Toe-like games" form a very interesting sub-class of Combinatorial Games: these are games for which the standard algebraic methods fail to work. The main result of the book is that for some infinite families of natural "Tic-Tac-Toe-like games with (at least) 2-dimensional goals" we know the exact value of the phase transition between "Weak Win" and "Strong Draw." We call these thresholds Clique Achievement Numbers, Lattice Achievement Numbers, and in the Reverse Games, Clique Avoidance Numbers and Lattice Avoidance Numbers. These are game-theoretic analogues of the Ramsey Numbers and Van der Waerden Numbers. Unlike the Ramsey Theory thresholds, which are hopeless in the sense that the best-known upper and lower bounds are very far from each other, here we can find the exact values of the game numbers. For precise statements see Sections 6, 8, 9, and 12.

To prove these exact results we develop a "fake probabilistic method" (we don't do case studies!); the name *Tic-Tac-Toe theory* in the title of the book actually refers to this "fake probabilistic method." The "fake probabilistic method" has two steps: (1) randomization and (2) derandomization. *Randomization* is a game-theoretic adaptation of the so-called Probabilistic Method ("Erdős Theory"); *derandomization* means to apply potential functions ("resource count"). The Probabilistic Method (usually) gives existence only; the potential technique, on the other hand, supplies explicit strategies. What is more, many of our explicit winning and drawing strategies are very efficient combinatorial algorithms (in fact, the most efficient ones that we know).

The "fake probabilistic method" is not the first theory of Combinatorial Games. There is already a well-known and successful theory: the *addition theory* of "Nim-like compound games." It is an algebraic theory designed to handle complicated games which are, or eventually turn out to be, compounds of several very simple games. "Nim-like compound games" is the subject of the first volume of the remarkable *Winning Ways for your Mathematical Plays* written by *Berlekamp, Conway*, and *Guy* (published in 1982). Volume 1 was called *Theory*, and volume 2 had the more prosaic name of *Case Studies*. As stated by the authors: "there are lots of games for which the theories we have now developed are useful, and even more for which they are not." The family of Tic-Tac-Toe-like games – briefly discussed in Chapter 22 of the *Winning Ways* (vol. 2) – definitely belongs to this latter class. By largely extending Chapter 22, and systematically using the "fake probabilistic method" – which is completely missing(!) from the *Winning Ways* – in this book an attempt is made to upgrade the Case Studies to a Quantitative Theory.

The algebraic and probabilistic approaches represent two entirely different viewpoints, which apparently complement each other. In contrast to the *local* viewpoint of the addition theory, the "fake probabilistic method" is a *global* theory for games which do *not* decompose into simple sub-games, and remain as single coherent entities throughout play. A given position P is evaluated by a score-system which has some natural probabilistic interpretation such as the "loss probability in the randomized game starting from position P." Optimizing the score-system is how we cut short the exhaustive search, and construct efficient ("polynomial time") strategies.

The "fake probabilistic method" works best for large values of the parameters – a consequence of the underlying "laws of large numbers." The "addition theory," on the other hand, works best for little games.

The pioneering papers of the subject are:

1. *Regularity and Positional Games*, by A. W. Hales and R. I. Jewett from 1963;
2. *On a Combinatorial Game*, by P. Erdős and J. Selfridge from 1973;
3. *Biased Positional Games* by V. Chvátal and P. Erdős from 1978; and, as a guiding motivation,
4. the Erdős–Lovász 2-Coloring Theorem from 1975.

The first discovered fundamental connections such as "strategy stealing and Ramsey Theory" and "pairing strategy and Matching Theory", and introduced our basic game class ("positional games"). The last three papers (Erdős with different co-authors) initiated and motivated the "games, randomization, derandomization" viewpoint, the core idea of the book. What is developed here is a far-reaching extension of these ideas – it took 25 years hard labor to work out the details. The majority of the results are published here for the first time.

Being an enthusiastic teacher myself, I tried to write the book in a lecture series format that I would like to use myself in the classroom. Each section is basically an independent lecture; most of them can be covered in the usual 80-minute time frame.

Beside the Theory the book contains dozens of challenging Exercises. The reader is advised to find the solutions to the exercises all by him/herself.

The notation is standard. For example, $c, c_0, c_1, c_2, \ldots$ denote, as usual, positive absolute constants (that I could but do not care to determine); "$a_n = o(1)$" and "$a_n = O(1)$" mean that $a_n \to 0$ and $|a_n| < c$ as $n \to \infty$; and, similarly, "$f(n) = o(g(n))$" and "$f(n) = O(g(n))$" mean that $f(n)/g(n) \to 0$ and $|f(n)/g(n)| < c$ as $n \to \infty$. Also $\log x$, $\log_2 x$, and $\log_3 x$ stand for, respectively, the natural logarithm, the base 2 logarithm, and the base 3 logarithm of x.

There are two informal sections: *A summary of the book in a nutshell* at the beginning, and *An informal introduction to Game Theory* at the end of the book in Appendix D. Both are easy reading; we highly recommend the reader to start the book with these two sections.

Last but not least, I would like to thank the Harold H. Martin Chair at Rutgers University and the National Science Foundation for the research grants supporting my work.

A summary of the book in a nutshell

Mathematics is spectacularly successful at making generalizations: the more than 2000-year old arithmetic and geometry were developed into the monumental fields of calculus, modern algebra, topology, algebraic geometry, and so on. On the other hand, mathematics could say remarkably little about nontraditional complex systems. A good example is the notorious "$3n+1$ problem." If n is even, take $n/2$, if n is odd, take $(3n+1)/2$; show that, starting from an arbitrary positive integer n and applying the two rules repeatedly, eventually we end up with the periodic sequence 1,2,1,2,1,2,.... The problem was raised in the 1930s, and after 70 years of diligent research it is still completely hopeless!

Next consider some games. Tic-Tac-Toe is an easy game, so let's switch to the 3-space. The $3 \times 3 \times 3$ Tic-Tac-Toe is a trivial first player win, the $4 \times 4 \times 4$ Tic-Tac-Toe is a very difficult first player win (computer-assisted proof by O. Patashnik in the late 1970s), and the $5 \times 5 \times 5$ Tic-Tac-Toe is a hopeless open problem (it is conjectured to be a draw game). Note that there is a general recipe to analyze games: perform backtracking on the game-tree (or position graph). For the $5 \times 5 \times 5$ Tic-Tac-Toe this requires about 3^{125} steps, which is totally intractable.

We face the same "combinatorial chaos" with the game of Hex. Hex was invented in the early 1940s by Piet Hein (Denmark), since when it has become very popular, especially among mathematicians. The board is a rhombus of hexagons of size $n \times n$; the two players, White (who starts) and Black, take two pairs of opposite sides of the board. The two players alternately put their pieces on unoccupied hexagons (White has white pieces and Black has black pieces). White (Black) wins if his pieces connect his opposite sides of the board.

In the late 1940s John Nash (*A Beautiful Mind*) proved, by a pioneering application of the Strategy Stealing Argument, that Hex is a first player win. The notorious open problem is to find an *explicit* winning strategy. It remains open for every $n \geq 8$. Note that the standard size of Hex is $n = 11$, which has about 3^{121} different positions.

What is common in the $3n+1$ problem, the $5 \times 5 \times 5$ Tic-Tac-Toe, and Hex? They all have extremely simple rules, which unexpectedly lead to chaos: exhibiting unpredictable behavior, without any clear order, without any pattern. These three problems form a good sample, representing a large part (perhaps even the majority) of the applied world problems. Mathematics gave up on these kinds of problems, sending them to the dump called "combinatorial chaos." Is there an escape from the combinatorial chaos?

It is safe to say that understanding/handling combinatorial chaos is one of the main problems of modern mathematics. However, the two game classes (n^d Tic-Tac-Toe and $n \times n$ Hex) represent a bigger challenge, they are even more hopeless, than the $3n+1$ problem. For the $3n+1$ problem we can at least carry out computer experimentation; for example, it is known that the conjecture is true for every $n \leq 10^{16}$ (a huge data bank is available): we can search the millions of solved cases for hidden patterns; we can try to extrapolate (which, unfortunately, has not led us anywhere yet).

For the game classes, on the other hand, only a half-dozen cases are solved. Computers do not help: it is easy to simulate a *random play*, but it is impossible to simulate an *optimal play* – this hopelessness leaves the games alive for competition. We simply have no data available; it is impossible to search for patterns if there are no data. (For example, we know only two(!) explicit winning strategies in the whole class of $n \times n \times \cdots \times n = n^d$ Tic-Tac-Toe games: the 3^3 version, which has an easy winning strategy, and the 4^3 version, which has an extremely complicated winning strategy.) These Combinatorial Games represent a humiliating challenge for mathematics!

Note that the subject of Game Theory was created by the Hungarian–American mathematician John von Neumann in a pioneering paper from 1928 and in the well-known book *Theory of Games and Economic Behavior* jointly written with the economist Oscar Morgenstern in 1944. By the way, the main motivation of von Neumann was to understand the role of bluffing in Poker. (von Neumann didn't care, or at least had nothing to say, about combinatorial chaos; the von Neumann–Morgenstern book completely avoids the subject!) Poker is a card game of incomplete information: the game is interesting because neither player knows the opponent's cards. In 1928 von Neumann proved his famous minimax theorem, stating that in games of incomplete information either player has an optimal strategy. This optimal strategy is typically a randomized ("mixed") strategy (to make up for the lack of information).

Traditional Game Theory doesn't say much about games of complete information like Chess, Go, Checkers, and grown-up versions of Tic-Tac-Toe; this is the subject of Combinatorial Game Theory. So far Combinatorial Game Theory has developed in two directions:

(I) the theory of "Nim-like games," which means games that fall apart into simple subgames in the course of a play, and
(II) the theory of "Tic-Tac-Toe-like games," which is about games that do not fall apart, but remain a coherent entity during the course of a play.

Direction (I) is discussed in the first volume of the well-known book *Winning Ways* by Berlekamp, Conway, and Guy from 1982. Direction (II) is discussed in this book.

As I said before, the main challenge of Combinatorial Game Theory is to handle combinatorial chaos. To analyze a position in a game (say, in Chess), it is important to examine the options, and all the options of the options, and all the options of the options of the options, and so on. This explains the exponential nature of the game tree, and any intensive case study is clearly impractical even for very simple games, like the $5 \times 5 \times 5$ Tic-Tac-Toe. There are dozens of similar games, where there is a clearcut natural conjecture about which player has a winning strategy, but the proof is hopelessly out of reach (for example, 5-in-a-row in the plane, the status of "Snaky" in Animal Tic-Tac-Toe, Kaplansky's 4-in-a-line game, Hex in a board of size at least 8×8, and so on, see Section 4).

Direction (I), "Nim-like games," basically avoids the challenge of chaos by restricting itself to games with simple components, where an "addition theory" can work. Direction (II) is a desperate attempt to handle combinatorial chaos.

The first challenge of direction (II) is to pinpoint the reasons why these games are hopeless. Chess, Tic-Tac-Toe and its variants, Hex, and the rest are all "Who-does-it-first?" games (which player gives the first checkmate, who gets the first 3-in-a-row, etc.). "Who-does-it-first?" reflects competition, a key ingredient of game playing, but it is not the most fundamental question. The most fundamental question is "What are the achievable configurations, achievable, but not necessarily first?" and the complementary question "What are the impossible configurations?" Drawing the line between "doable" and "impossible" (doable, but not necessarily first!) is the primary task of direction (II). First we have to clearly understand "what is doable"; "what is doable first" is a secondary question. "Doing-it-first" is the ordinary win concept; it is reasonable, therefore, to call "doing it, but not necessarily first" a Weak Win. If a player fails to achieve a Weak Win, we say the opponent forced (at least) a Strong Draw.

The first idea is to switch from ordinary win to Weak Win; the second idea of direction (II) is to carefully define its subject: "generalized Tic-Tac-Toe." Why "generalized Tic-Tac-Toe"? "Tic-Tac-Toe-like games" are the simplest case in the sense that they are static games. Unlike Chess, Go, and Checkers, where the players repeatedly relocate or even remove pieces from the board ("dynamic games"), in Tic-Tac-Toe and Hex the players make permanent marks on the board, and relocating or removing a mark is illegal. (Chess is particularly complicated. There are 6 types of pieces: King, Queen, Bishop, Knight, Rook, Pawn, and each one has its own set of rules of "how to move the piece." The instructions of playing Tic-Tac-Toe is just a couple of lines, but the "instructions of playing Chess" is several pages long.) The "relative" simplicity of games such as "Tic-Tac-Toe" makes them ideal candidates for a mathematical theory.

What does "generalized Tic-Tac-Toe" mean? Nobody knows what "generalized Chess" or "generalized Go" are supposed to mean, but (almost) everybody would agree on what "generalized Tic-Tac-Toe" should mean. In Tic-Tac-Toe the "board" is a $3 \times 3 = 9$ element set, and there are 8 "winning triplets." Similarly, "generalized Tic-Tac-Toe" can be played on an arbitrary finite hypergraph, where the hyperedges are called "winning sets," the union set is the "board," the players alternately occupy elements of the "board." Ordinary win means that a player can occupy a whole "winning set" first; Weak Win simply means to occupy a whole winning set, but not necessarily first.

How can direction (II) deal with combinatorial chaos? The exhaustive search through the exponentially large game-tree takes an *enormous* amount of time (usually more than the age of the universe). A desperate(!) attempt to make up for the lack of time is to study the *random walk* on the game-tree; that is, to study the *randomized game* where both players play randomly.

The extremely surprising message of direction (II) is that the probabilistic analysis of the randomized game can often be *converted* into optimal Weak Win and Strong Draw strategies via potential arguments. It is basically a game-theoretic adaptation of the so-called Probabilistic Method in Combinatorics ("Erdős Theory"); this is why we refer to it as a "fake probabilistic method."

The fake probabilistic method is considered a mathematical paradox. It is a "paradox" because Game Theory is about *perfect* players, and it is shocking that a play between *random generators* ("dumb players") has anything to do with a play between perfect players! "Poker and randomness" is a natural combination: mixed strategy (i.e. random choice among deterministic strategies) is necessary to make up for the lack of complete information. On the other hand, "Tic-Tac-Toe and randomness" sounds like a mismatch. To explain the connection between "Tic-Tac-Toe" and "randomness" requires a longer analysis.

First note that the connection is not trivial in the sense that an optimal strategy is never a "random play." In fact, a "random play" usually leads to a quick, catastrophic defeat. It is a simple general fact that for games of "complete information" the optimal strategies are always deterministic ("pure"). The fake probabilistic method is employed to *find* an explicit deterministic optimal strategy. This is where the connection is: the fake probabilistic method is *motivated* by traditional Probability Theory, but eventually it is *derandomized* by *potential arguments*. In other words, we eventually get rid of Probability Theory completely, but the intermediate "probabilistic step" is an absolutely crucial, inevitable part of the understanding process.

The fake probabilistic method consists of the following main chapters:

(i) game-theoretic first moment,
(ii) game-theoretic second and higher moments,
(iii) game-theoretic independence.

By using the fake probabilistic method, we can find the *exact* solution of infinitely many natural "Ramseyish" games, thought to be completely hopeless before, like some Clique Games, 2-dimensional van der Waerden games, and some "sub-space" versions of multi-dimensional Tic-Tac-Toe (the goal sets are at least "2-dimensional").

As said before, nobody knows how to win a "who-does-it-first game." We have much more luck with Weak Win where "doing it first" is ignored. A Weak Win Game, or simply a Weak Game, is played on an arbitrary finite hypergraph, the two players are called Maker and Breaker (alternative names are Builder and Blocker). To achieve an ordinary win a player has to "build and block" at the same time. In a Weak Game these two jobs are separated, which makes the analysis somewhat *easier*, but not *easy*. For example, the notoriously difficult Hex is clearly equivalent to a Weak Game, but it doesn't help to find an explicit first player's winning strategy.

What we have been discussing so far was the achievement version. The Reverse Game (meaning the avoidance version) is equally interesting, or perhaps even more interesting.

The general definition of the *Reverse Weak Game* goes as follows. As usual, it is played on an arbitrary finite hypergraph. One player is a kind of "anti-builder": he wants to avoid occupying a whole winning set – we call him Avoider. The other player is a kind of "anti-blocker": he wants to force the reluctant Avoider to build a winning set – "anti-blocker" is officially called Forcer.

Why "Ramseyish" games? Well, Ramsey Theory gives some partial information about ordinary win. We have a chance, therefore, to compare what we know about ordinary win with that of Weak Win.

The first step in the fake probabilistic method is to describe the majority play, and then, in the second step, try to find a connection between the majority play and the optimal play (the surprising part is that it works!).

The best way to illustrate this is to study the Weak and Reverse Weak versions of the (K_n, K_q) Clique Game: the players alternately take new edges of the complete graph K_n; Maker's goal is to occupy a large clique K_q; Breaker wants to stop Maker. In the Reverse Game, Forcer wants to force the reluctant Avoider to occupy a K_q.

If $q = q(n)$ is "very small" in terms of n, then Maker (or Forcer) can easily win. On the other hand, if $q = q(n)$ is "not so small" in terms of n, then Breaker (or Avoider) can easily win. Where is the game-theoretic breaking point? We call the breaking point the Clique Achievement (Avoidance) Number.

For "small" ns no one knows the answer, but for "large" ns we know the exact value of the breaking point! Indeed, assume that n is sufficiently large like $n \geq 2^{10^{10}}$. If we take the lower integral part

$$q = \lfloor 2\log_2 n - 2\log_2 \log_2 n + 2\log_2 e - 3 \rfloor$$

(base 2 logarithm), then Maker (or Forcer) wins. On the other hand, if we take the upper integral part

$$q = \lceil 2\log_2 n - 2\log_2 \log_2 n + 2\log_2 e - 3 \rceil,$$

then Breaker (or Avoider) wins.

For example, if $n = 2^{10^{10}}$, then

$$2\log_2 n - 2\log_2 \log_2 n + 2\log_2 e - 3 =$$

$$= 2 \cdot 10^{10} - 66.4385 + 2.8854 - 3 = 19,999,999,933.446,$$

and so the largest clique size that Maker can build (Forcer can force Avoider to build) is $19,999,999,933$.

This level of accuracy is even more striking because for smaller values of n we do not know the Clique Achievement Number. For example, if $n = 20$, then it can be either 4 or 5 or 6 (which one?); if $n = 100$, then it can be either 5 or 6 or 7 or 8 or 9 (which one?); if $n = 2^{100}$, then it can be either 99 or 100 or 101 or … or 188 (which one?), that is there are 90 possible candidates. (Even less is known about the small Avoidance Numbers.) We will (probably!) never know the exact values of these game numbers for $n = 20$, or for $n = 100$, or for $n = 2^{100}$, but we know the exact value for a monster number such as $n = 2^{10^{10}}$. This is truly surprising! This is the complete opposite of the usual induction way of discovering patterns from the small cases (the method of direction (I)).

The explanation for this unusual phenomenon comes from our technique: the fake probabilistic method. Probability Theory is a collections of Laws of Large Numbers. Converting the probabilistic arguments into a potential strategy leads to certain "error terms"; these "error terms" become negligible compared to the "main term" if the board is large.

It is also very surprising that the Weak Clique Game and the *Reverse* Weak Clique Game have *exactly* the same breaking point: Clique Achievement Number = Clique Avoidance Number. This contradicts common sense. We would expect that an eager Maker in the "straight" game has a good chance to build a larger clique than a reluctant Avoider in the Reverse version, but this "natural" expectation turns out to be wrong. We cannot give any *a priori* reason why the two breaking points coincide. All that can be said is that the highly technical proof of the "straight" case (around 30 pages) can be easily adapted (like *maximum* is replaced by *minimum*) to yield the same breaking point for the Reverse Game, but this is hardly the answer that we are looking for.

What is the mysterious expression $2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 3$? An expert of the theory of Random Graphs immediately recognizes that $2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 3$ is exactly 2 less than the Clique Number of the symmetric Random Graph $\mathbf{R}(K_n, 1/2)$ (1/2 is the edge probability).

A combination of the first and second moment methods (standard Probability Theory) shows that the Clique Number $\omega(\mathbf{R}(K_n, 1/2))$ of the Random Graph has a very strong concentration. Typically it is concentrated on a *single* integer with probability $\to 1$ as $n \to \infty$ (and even in the worst case there are at most two values). Indeed, the expected number of q-cliques in $\mathbf{R}(K_n, 1/2)$ equals

$$f(q) = f_n(q) = \binom{n}{q} 2^{-\binom{q}{2}}.$$

The function $f(q)$ drops under 1 around $q \approx 2\log_2 n$. The real solution of the equation $f(q) = 1$ is

$$q = 2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 1 + o(1), \tag{1}$$

which is exactly 2 more than the game-theoretic breaking point

$$q = 2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 3 + o(1) \tag{2}$$

mentioned above.

To build a clique K_q of size (1) by Maker (or Avoider in the Reverse Game) on the board K_n is the majority outcome. The majority play outcome differs from the optimal play outcome by a mere additive constant 2.

The strong concentration of the Clique Number of the Random Graph is not that terribly surprising as it seems at first sight. Indeed, $f(q)$ is a very rapidly changing function

$$\frac{f(q)}{f(q+1)} = \frac{q+1}{n-q} 2^q = n^{1+o(1)}$$

if $q \approx 2\log_2 n$. On an intuitive level, it is explained by the obvious fact that if q switches to $q+1$, then $\binom{q}{2}$ switches to $\binom{q+1}{2} = \binom{q}{2} + q$, which is a large "square-root size" increase.

Is there a "reasonable" variant of the Clique Game for which the breaking point is exactly (1), i.e. the Clique Number of the Random Graph? The answer is "yes," and the game is a "Picker–Chooser game." To motivate the "Picker–Chooser game," note that the alternating Tic-Tac-Toe-like play splits the board into two equal (or almost equal) parts. But there are many other ways to divide the board into two equal parts. The "I-cut-you'll-choose way" (motivated by how a couple shares a single piece of cake after dinner) goes as follows: in each move, Picker picks two previously unselected points of the board, Chooser chooses one of them, and the other one goes back to Picker. In the *Picker–Chooser* game Picker is the builder (i.e. he wants to occupy a whole winning set) and Chooser is the blocker (i.e. his goal is to mark every winning set).

When Chooser is the builder and Picker is the blocker, we call it the *Chooser–Picker* game.

The proof of the theorem that the "majority clique number" (1) is the exact value of the breaking point for the (K_n, K_q) Picker–Chooser Clique Game (where of course the "points" are the edges of K_n) is based on the concepts of:

(a) game-theoretic first moment; and
(b) game-theoretic second moment.

The proof is far from trivial, but not so terribly difficult either (because Picker has so much control of the game). It is a perfect stepping stone before conquering the much more challenging Weak and Reverse Weak, and also the Chooser–Picker versions. The last three Clique Games all have the *same* breaking point, namely (2). What is (2)?

Well, (2) is the real solution of the equation

$$\binom{n}{q} 2^{-\binom{q}{2}} = f(q) = \frac{\binom{n}{2}}{2\binom{q}{2}}. \tag{3}$$

The intuitive meaning of (3) is that the overwhelming majority of the edges of the random graph are covered by exactly one copy of K_q. In other words, the Random Graph may have a large number of copies of K_q, but they are well-spread (uncrowded); in fact, there is room enough to be typically pairwise edge-disjoint. This suggests the following *intuition*. Assume that we are at a "last stage" of playing a Clique Game where Maker (playing the Weak Game) has a large number of "almost complete" K_qs: "almost complete" in the sense that, (a) in each "almost complete" K_q all but *two edges* are occupied by Maker, (b) all of these edge-pairs are unoccupied yet, and (c) these extremely dangerous K_qs are pairwise edge-disjoint. If (a)–(b)–(c) hold, then Breaker can still escape from losing: he can block these disjoint unoccupied edge-pairs by a simple Pairing Strategy! It is exactly the Pairing Strategy that distinguishes the Picker–Chooser game from the rest of the bunch. Indeed, in each of the Weak, Reverse Weak, and Chooser–Picker games, "blocker" can easily *win* the Disjoint Game (meaning the trivial game where the winning sets are disjoint and contain at least two elements each) by employing a Pairing Strategy. In sharp contrast, in the Picker–Chooser version Chooser always loses a "sufficiently large" Disjoint Game (more precisely, if there are at least 2^n disjoint n-element winning sets, then Picker wins the Picker–Chooser game).

This is the best intuitive explanation that we know to understand breaking point (2). This intuition requests the "Random Graph heuristic," i.e., to (artificially!) introduce a random structure in order to understand a deterministic game of complete information.

But the connection is much deeper than that. To *prove* that (2) is the exact value of the game-theoretic breaking point, one requires a fake probabilistic method. The main steps of the proof are:

(i) game-theoretic first moment,
(ii) game-theoretic higher moments (involving "self-improving potentials"), and
(iii) game-theoretic independence.

Developing (i)–(iii) is a long and difficult task. The word "fake" in the fake probabilistic method refers to the fact that, when an optimal strategy is actually defined, the "probabilistic part" completely disappears. It is a metamorphosis: as a caterpillar turns into a butterfly, the probabilistic arguments are similarly converted into (deterministic) potential arguments.

Note that potential arguments are widely used in puzzles ("one-player games"). A well-known example is Conway's *Solitaire Army* puzzle: arrange men behind a line and then by playing "jump and remove", horizontally or vertically, move a man as far across the line as possible. Conway's beautiful "golden ratio" proof, a striking potential argument, shows that it is *impossible* to send a man forward 5 (4 is possible). Conway's result is from the early 1960s. (It is worthwhile to mention the new result that if "to jump a man diagonally" is permitted, then 5 is replaced by 9; in other words, it is impossible to send a man forward 9, but 8 is possible. The proof is similar, but the details are substantially more complicated.)

It is quite natural to use potential arguments to describe *impossible configurations* (as Conway did). It is more surprising that potential arguments are equally useful to describe *achievable configurations* (i.e. Maker's Weak Win) as well. But the biggest surprise of all is that the Maker's Building Criterions and the Breaker's Blocking Criterions often coincide, yielding *exact* solutions of several seemingly hopeless Ramseyish games. There is, however, a fundamental difference: Conway's argument works for small values such as 5, but the fake probabilistic method gives sharp results only for "large values" of the parameters (we refer to this mysterious phenomenon as a "game-theoretic law of large numbers").

These exact solutions all depend on the concept of "game-theoretic independence" – another striking connection with Probability Theory. What is game-theoretic independence? There is a trivial and a non-trivial interpretation of game-theoretic independence.

The "trivial" (but still very useful) interpretation is about *disjoint* games. Consider a set of hypergraphs with the property that, in each one, Breaker (as the second player) has a strategy to block (mark) every winning set. If the hypergraphs are pairwise disjoint (in the strong sense that the "boards" are disjoint), then, of course, Breaker can block the union hypergraph as well. Disjointness guarantees that in any component either player can play independently from the rest of the components. For example, the concept of the pairing strategy is based on this simple observation.

In the "non-trivial" interpretation, the initial game does *not* fall apart into disjoint components. Instead Breaker can *force* that eventually, in a much later stage of the play, the family of unblocked (yet) hyperedges does fall apart into much smaller (disjoint) components. This is how Breaker can eventually finish the job of blocking the whole initial hypergraph, namely "blocking componentwise" in the "small" components.

A convincing probabilistic intuition behind the non-trivial version is the well-known Local Lemma (or Lovász Local Lemma). The Local Lemma is a remarkable probabilistic sieve argument to prove the *existence* of certain very complicated structures that we are unable to construct directly.

A typical application of the Local Lemma goes as follows:

Erdős–Lovász 2-Coloring Theorem (1975). *Let* $\mathcal{F} = \{A_1, A_2, A_3, \ldots\}$ *be an n-uniform hypergraph. Suppose that each* A_i *intersects at most* 2^{n-3} *other* $A_j \in \mathcal{F}$ *("local size"). Then there is a 2-coloring of the "board"* $V = \bigcup_i A_i$ *such that no* $A_i \in \mathcal{F}$ *is monochromatic.*

The conclusion (almost!) means that there exists a *drawing terminal position* (we have cheated a little bit: in a drawing terminal position, the two color classes have equal size). The very surprising message of the Erdős–Lovász 2-Coloring Theorem is that the "global size" of hypergraph $\mathcal{F}$ is irrelevant (it can even be infinite!), only the "local size" matters.

Of course, the existence of a single (or even several) drawing terminal position does *not* guarantee the existence of a *drawing strategy*. But perhaps it is still true that under the Erdős–Lovász condition (or under some similar but slightly weaker local condition), Breaker (or Avoider, or Picker) has a blocking strategy, i.e. he can block every winning set in the Weak (or Reverse Weak, or Chooser–Picker) game on $\mathcal{F}$. We refer to this "blocking draw" as a Strong Draw.

This is a wonderful problem; we call it the Neighborhood Conjecture. Unfortunately, the conjecture is still open in general, in spite of all efforts trying to prove it during the last 25 years.

We know, however, several partial results, which lead to interesting applications. A very important special case, when the conjecture is "nearly proved," is the class of Almost Disjoint hypergraphs: where any two hyperedges have at most one common point. This is certainly the case for "lines," the winning sets of the n^d Tic-Tac-Toe.

What do we know about the multidimensional n^d Tic-Tac-Toe? We know that it is a draw game even if the dimension d is as large as $d = c_1 n^2 / \log n$, i.e. nearly quadratic in terms of (the winning size) n. What is more, the draw is a Strong Draw: the second player can mark every winning line (if they play till the whole board is occupied). Note that this bound is nearly best possible: if $d > c_2 n^2$, then the second player *cannot* force a Strong Draw.

How is it that for the Clique Game we know the *exact* value of the breaking point, but for the multidimensional Tic-Tac-Toe we could not even find the asymptotic truth (due to the extra factor of $\log n$ in the denominator)? The answer is somewhat technical. The winning lines in the multidimensional n^d Tic-Tac-Toe form an extremely *irregular* hypergraph: the maximum degree is *much* larger than the average degree. This is why one cannot apply the Blocking Criterions directly to the "n^d hypergraph." First we have to employ a Truncation Procedure to bring the maximum degree close to the average degree, and the price that we pay for this degree reduction is the loss of a factor of $\log n$.

However, if we consider the n^d *Torus* Tic-Tac-Toe, then the corresponding hypergraph becomes perfectly uniform (the torus is a *group*). For example, every point of the n^d *Torus* Tic-Tac-Toe has $(3^d - 1)/2$ winning lines passing through it. This uniformity explains why for the n^d *Torus* Tic-Tac-Toe we can prove asymptotically sharp thresholds.

A "winning line" in the n^d Tic-Tac-Toe is a set of n points on a straight line forming an n-term Arithmetic Progression. This motivates the "Arithmetic Progression Game": the board is the interval $1, 2, \ldots, N$, and the goal is to build an n-term Arithmetic Progression. The corresponding hypergraph is "nearly regular"; this is why we can prove asymptotically sharp results.

Let us return to the n^d Torus Tic-Tac-Toe. If the "winning line" is replaced by "winning plane" (or "winning subspace of dimension ≥ 2" in general), then we can go far beyond "asymptotically sharp": we can even determine the *exact* value of the game-theoretic threshold, as in the Clique Game. For example, a "winning plane" is an $n \times n$ lattice in the n^d Torus. This is another rapidly changing 2-dimensional configuration: if n switches to $n+1$, then $n \times n$ switches to $(n+1) \times (n+1)$, which is again a "square-root size" increase just as in the case of the cliques. This formal similarity to the Clique Game (both have "2-dimensional goals") explains why there is a chance to find the *exact* value of the game-theoretic breaking point (the actual proofs are rather different).

It is very difficult to visualize the d-dimensional torus if d is large; here is an easier version: a game with 2-dimensional goal sets played on the plane.

Two-dimensional Arithmetic Progression Game. A natural way to obtain a 2-dimensional arithmetic progression (AP) is to take the Cartesian product. The Cartesian product of two q-term APs with the same gap is a $q \times q$ Aligned Square Lattice.

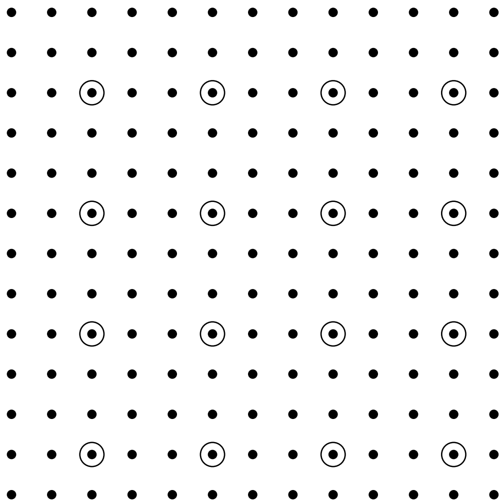


Figure 1 4×4 Aligned Square Lattice on a 13×13 board

Let ($N \times N$, $q \times q$ Square Lattice) denote the game where the board is the $N \times N$ chessboard, and the winning sets are the $q \times q$ Aligned Square Lattices (see Figure 1 for $N = 13$, $q = 4$, and for a particular 4×4 winning set). Again we know the exact value of the game-theoretic breaking point: if

$$q = \left\lfloor \sqrt{\log_2 N} + o(1) \right\rfloor,$$

then Maker can always build a $q \times q$ Aligned Square Lattice, and this is the best that Maker can achieve. Breaker can always prevent Maker from building a $(q+1) \times (q+1)$ Aligned Square Lattice. Again the error term $o(1)$ becomes negligible if N is large. For example, $N = 2^{10^{40}+10^{20}}$ is large enough, and then

$$\sqrt{\log_2 N} = \sqrt{10^{40} + 10^{20}} = 10^{20} + \frac{1}{2} + O(10^{-20}),$$

so $\sqrt{\log_2 N}$ is not too close to an integer (in fact, it is almost exactly in the middle), which guarantees that $q = 10^{20}$ is the largest Aligned Square Lattice size that Maker can build.

Similarly, $q = 10^{20}$ is the largest Aligned Square Lattice size that Forcer can force Avoider to build.

Here is an interesting detour: consider (say) the biased (2:1) avoidance version where Avoider takes 2 points and Forcer takes 1 point of the $N \times N$ board per move. Then again we know the exact value of the game-theoretic breaking point: if

$$q = \left\lfloor \sqrt{\log_{\frac{3}{2}} N} + o(1) \right\rfloor,$$

then Forcer can always force Avoider to build a $q \times q$ Aligned Square Lattice, and this is the best that Forcer can achieve. Avoider can always avoid building a $(q+1) \times (q+1)$ Aligned Square Lattice. Notice that the base of the logarithm changed from 2 to 3/2.

How about the biased (2:1) achievement version where Maker takes 2 points and Breaker takes 1 point of the $N \times N$ board per move? Then we know the following lower bound: if

$$q = \left\lfloor \sqrt{\log_{\frac{3}{2}} N + 2\log_2 N} + o(1) \right\rfloor,$$

then Maker can always build a $q \times q$ Aligned Square Lattice. We conjecture (but cannot prove) that this is the best that topdog Maker can achieve (i.e. Breaker can always prevent Maker from building a $(q+1) \times (q+1)$ Aligned Square Lattice). Notice that in the biased (2:1) game (eager) Maker can build a substantially larger Aligned Square Lattice than (reluctant) Avoider; the ratio of the corresponding qs is at least as large as

$$\frac{\sqrt{\frac{1}{\log(3/2)} + \frac{2}{\log 2}}}{\sqrt{\frac{1}{\log(3/2)}}} = 1.473.$$

This makes the equality Achievement Number = Avoidance Number in the fair (1:1) games even more surprising.

We can prove the exact formulas only for large board size, such as K_N with $N \geq 2^{10^{10}}$ (Clique Game) and the $N \times N$ grid with $N \geq 2^{10^{40}}$ (Square Lattice Game), but we are convinced that the exact formulas give the truth even for small board sizes like 100 and 1000.

We summarize the meaning of "game-theoretic independence" in the (1:1) game as follows. It is about games such as Tic-Tac-Toe for which the local size is much smaller than the global size. Even if the game starts out as a coherent entity, either player can force it to develop into smaller, local size composites. A sort of intuitive explanation behind it is the Erdős–Lovász 2-Coloring Theorem, which itself is a sophisticated application of statistical independence. Game-theoretic independence is about how to sequentialize statistical independence.

Here we stop the informal discussion, and begin the formal treatment. It is going to be a long journey.

Part A

Weak Win and Strong Draw

Games belong to the oldest experiences of mankind, well before the appearance of any kind of serious mathematics. ("Serious mathematics" is in fact very young: Euclid's *Elements* is less than three-thousand years old.) The playing of games has long been a natural instinct of all humans, and is why the *solving* of games is a natural instinct of mathematicians. Recreational mathematics is a vast collection of all kinds of clever observations ("pre-theorems") about games and puzzles, the perfect empirical background for a mathematical theory. It is well-known that games of chance played an absolutely crucial role in the early development of Probability Theory. Similarly, Graph Theory grew out of puzzles (i.e. 1-player games) such as the famous Königsberg bridge problem, solved by Euler ("Euler trail"), or Hamilton's roundtrip puzzle on the graph of the dodecahedron ("Hamilton cycle problem"). Unlike these two very successful theories, we still do not have a really satisfying quantitative theory of games of pure skill with complete information, or as they are usually called nowadays: Combinatorial Games. Using technical terms, Combinatorial Games are 2-player zero-sum games, mostly finite, with complete information and no chance moves, and the payoff function has three values $\pm 1, 0$ as the first player wins or loses the play, or it ends in a draw.

Combinatorial Game Theory attempts to answer the questions of "who wins," "how to win," and "how long does it take to win." Naturally "win" means "forced win," i.e. a "winning strategy."

Note that Graph Theory and Combinatorial Game Theory face the very same challenge: combinatorial chaos. Given a general graph G, the most natural questions are: what is the chromatic number of G? What is the length of the longest path in G? In particular, does G contain a Hamiltonian path, or a Hamiltonian cycle? What is the size of the largest complete subgraph of G? All that Graph Theory can say is "try out everything," i.e. the brute force approach, which leads to combinatorial chaos.

Similarly, to find a winning strategy in a general game (of complete information) all we can do is backtracking of the enormous game-tree, or position-graph, which also leads to combinatorial chaos.

How do we escape from the combinatorial chaos? In particular, when and how can a player win in a game such as Tic-Tac-Toe? And, of course, what are the "Tic-Tac-Toe like games"? This is the subject of Part A. We start slowly: in Chapter I we discuss many concrete examples and prove a few simple (but important!) theorems. In Chapter II we formulate the main results, and prove a few more simple theorems. The hard proofs come later in Parts C and D.

Chapter I
Win vs. Weak Win

Chess, Tic-Tac-Toe, and Hex are among the most well-known games of complete information with no chance move. What is common in these apparently very different games? In either game the player that wins is the one who achieves a "winning configuration" first. A "winning configuration" in Tic-Tac-Toe is a "3-in-a-row," in Hex it is a "connecting chain of hexagons," and in Chess it is a "capture of the opponent's King" (called a checkmate).

The objective of other well-known games of complete information like Checkers and Go is more complicated. In Checkers the goal is to be the first player either to capture all of the opponent's pieces (checkers) or to build a position where the opponent cannot make a move. The capture of a single piece (jumping over) is a "mini-win configuration," and, similarly, an arrangement where the opponent cannot make a move is a "winning configuration."

In Go the goal is to capture as many stones of the opponent as possible ("capturing" means to "surround a set of opponent's stones by a connected set").

These games are clearly very different, but the basic question is always the same: "Which player can achieve a winning configuration **first**?".

The bad news is that no one knows *how* to achieve a winning configuration first, except by exhaustive case study. There is no general theorem whatsoever answering the question of *how*. The well-known strategy stealing argument gives a partial answer to *when*, but doesn't say a word about *how*. (Note that "doing it first" means competition, a key characteristic of game playing.)

For example, the $4 \times 4 \times 4 = 4^3$ Tic-Tac-Toe is a first player's win, but the winning strategy is extremely complicated: it is the size of a phone-book (computer-assisted task due to O. Patashnik). The $5 \times 5 \times 5 = 5^3$ version is expected to be a draw, but no one can prove it.

In principle, we could search *all* strategies, but it is absurdly long: the total number of strategies is a double exponential function of the board-size. The exhaustive search through all positions (backtracking the "game tree," or the "position graph"), which is officially called Backward Labeling, is more efficient, but still requires exponential time ("hard").

Doing it first is hopeless, but if we ignore "first," then an even more fundamental question arises: "What can a player achieve by his own moves against an adversary?" "Which configurations are achievable (but not necessarily first)?" Or the equivalent complementary question: "What are the impossible configurations?"

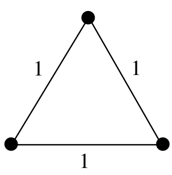
To see where our general concepts (to be defined in Section 5) come from, in Sections 1–4 we first inspect some particular games.

1
Illustration: every finite point set in the plane is a Weak Winner

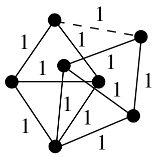
1. Building a congruent copy of a given point set. The first two sections of Chapter I discuss an amusing game. The objective is to demonstrate the power of the potential technique – the basic method of the book – with a simple example. Also it gives us the opportunity for an early introduction to some useful Potential Criterions (see Theorems 1.2–1.4).

To motivate our concrete game, we start with a trivial observation: every 2-coloring of the vertices of an equilateral triangle of side length 1 yields a side where both endpoints have the same color (and have distance 1).

This was trivial, but how about 3 colors instead of 2? The triangle doesn't work, we need a more sophisticated geometric graph: the so-called "7-point Moser-graph" in the plane – which has 11 edges, each of length 1 – will do the job.



equilateral
triangle



7-point
Moser-graph

The Moser-graph has the combinatorial property that every 3-coloring of the vertices yields an edge where both endpoints have the same color (and have distance 1).

How about 4 colors instead of 3? Does there exist a geometric graph in the plane such that every edge has length 1, and every 4-coloring of the vertices of this graph yields an edge where both endpoints have the same color? This innocent-looking question was unsolved for more than 50 years, and became a famous problem under the name of *the chromatic number of the plane*. Note that the answer to the question is negative for 7 colors; nothing is known about 4, 5, and 6 colors.

An interesting branch of Ramsey Theory, called Euclidean Ramsey Theory, studies the following more general problem: consider a finite set of points X in some Euclidean space $\mathbf{R}^d$. We would like to decide whether or not for every partition of $X = A_1 \cup A_2 \cup \cdots \cup A_r$ into r subsets, it is always true that some A_i contains a congruent copy of some given point set S (the "goal set"). The partition $X = A_1 \cup A_2 \cup \cdots \cup A_r$ is often called an r-coloring of X, where $A_1, \ldots, A_r$ are the color classes. For example, if X is the 7-point Moser-graph in the plane, the "goal set" S consists of two points a unit distance apart, and $r = 3$. Then the answer to the question above is "yes."

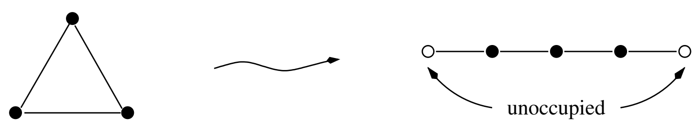
Unfortunately Euclidean Ramsey Theory is very under-developed: it has many interesting conjectures, but hardly any general result (see e.g. Chapter 11 in the *Handbook of Combinatorics*). Here we study a game-theoretic version, and prove a very general result in a surprisingly elementary way.

The game-theoretic version goes as follows: there are two players, called Maker and Breaker, who alternately select new points from some Euclidean space $\mathbf{R}^d$. Maker marks his points red and Breaker marks his points blue. Maker's goal is to build a congruent copy of some given point set S, Breaker's goal is simply to stop Maker (Breaker doesn't want to build).

The board of the game is infinite, in fact uncountable, so how to define the *length* of the game? It is reasonable to assume that the length of the game is $\leq \omega$, where ω denotes, as usual, the first infinite ordinal number. In other words, if Maker cannot win in a finite number of moves, then the players take turns for every natural number, and the play declares that Breaker wins (a draw is impossible). We call this game the "S-building game in $\mathbf{R}^d$."

Example 1: Let the goal set $S = S_3$ be a 3-term arithmetic progression (A.P.) where the gap is 1, and let $d = 1$ (we play on the line). Can Maker win? The answer is an easy "no." Indeed, divide the infinite line into disjoint pairs of points at distance 1 apart – by using this pairing strategy (if a player takes one member of the pair, the opponent takes the other one) Breaker can prevent Maker from building a congruent copy of S_3. This example already convinces us that the 1-dimensional case is not very interesting: Maker can build hardly anything. In sharp contrast, the 2-dimensional case will turn out to be very "rich": Maker can build a congruent copy of *any* given finite plane set, even if he is the underdog!

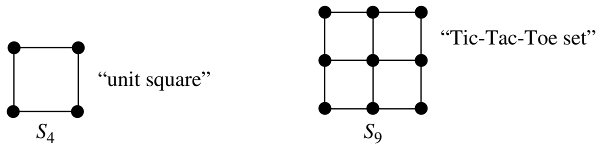
But before proving this surprisingly general theorem, let us see more concrete examples. Of course, in the plane Maker can easily build a congruent copy of the 3-term A.P. S_3 in 3 moves, and also the 4-term A.P. S_4 (the gap is 1) in 5 moves. Indeed, the trick is to start with an equilateral triangle, which has 3 ways to be extended to a "virgin configuration."



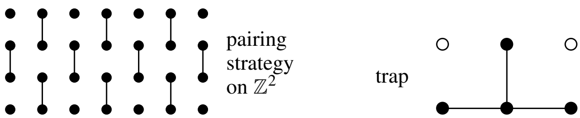
How about the 5-term and 6-term A.P.s S_5 and S_6 (the gap is always 1)?

We challenge the reader to spend some time with these concrete goal sets S_5 and S_6 before reading the proof of the general theorem below.

Example 2: Let the goal set be the 4 vertices of the "unit square" $S = S_4$



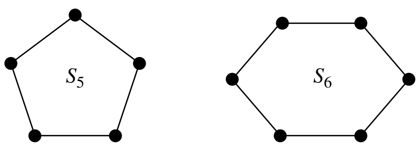
A simple pairing strategy shows that Maker cannot build a "unit square" S_4 on the infinite grid $\mathbb{Z}^2$, but he can easily do it on the whole plain. The trick is to get a trap



We challenge the reader to show that Maker can always build a congruent copy of the "unit square" S_4 in the plane in his 6th move (or before).

Example 3: Let us switch from the 2×2 S_4 to the 3×3 goal set $S = S_9$. We call S_9 the "Tic-Tac-Toe set." Let $d = 2$; can Maker win? If "yes," how long does it take to win?

Example 4: Example 2 was about the "unit square"; how about the regular pentagon $S = S_5$ or the regular hexagon $S = S_6$?



Let $d = 2$; can Maker win? If "yes," how long does it take to win? We challenge the reader to answer these questions before reading the rest of the section.

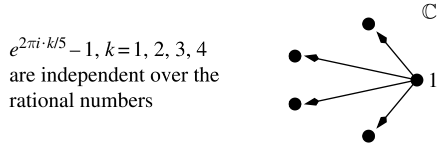
2. A positive result. The objective is to prove:

Theorem 1.1 *Let S be an arbitrary finite point set in the plane, and consider the following S-building game: 2 players, called Maker and Breaker, alternately pick new points from the plane, each picks 1 point per move; Maker's goal is to build a congruent copy of S in a finite number of moves, and Breaker's goal is to stop Maker.*

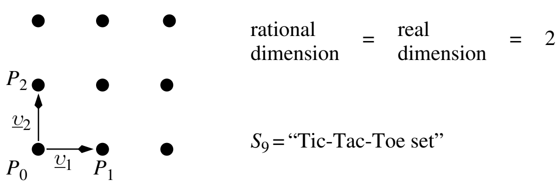
Given an arbitrary finite point set S in the plane, Maker always has a winning strategy in the S-building game in the plane.

Proof. We label the points of S as follows: $S = \{P_0, P_1, \ldots, P_k\}$, i.e. S is a $(k+1)$-element set. We pick P_0 (an arbitrary element of S) and consider the k vectors $v_j = \vec{P_0 P_j}$, $j = 1, 2, \ldots, k$ starting from P_0 and ending at P_j. The k planar vectors $v_1, v_2, \ldots, v_k$ may or may not be linearly independent over the *rationals*. □

Remark. It is important to emphasize that the field of *rational numbers* is being discussed and not *real numbers*. For example, if S is the regular pentagon in the unit circle with $P_0 = 1$ ("complex plane"), then the 4 vectors v_1, v_2, v_3, v_4 are linearly independent over the rational numbers (because the cyclotomic field $\mathbf{Q}(e^{2\pi i/5})$ with $i = \sqrt{-1}$ a 4-dimensional vector space over $\mathbf{Q}$), but, of course, the same set of 4 vectors are *not* linearly independent over the real numbers (since the dimension of the plane is 2). In other words, the "rational dimension" is 4, but the "real dimension" is 2.



On the other hand, for the 9-element "Tic-Tac-Toe set" $S = S_9$ (see Example 3), the 8 vectors $v_1, v_2, \ldots, v_8$ have the form $kv_1 + lv_2$, $k \in \{0, 1, 2\}$, $l \in \{0, 1, 2\}$, $k + l \geq 1$, implying that the rational and the real dimensions coincide: either one is 2.



For an arbitrary point set S with $|S| = k+1$, let $m = m(S)$ denote the maximum number of vectors among $v_j = \vec{P_0 P_j}$, $j = 1, 2, \ldots, k$, which are linearly independent over the *rational* numbers. Note that m may have any value in the interval $1 \leq m \leq k$.

For notational convenience assume that v_j, $j = 1, 2, \ldots, m$ are linearly independent over the *rational numbers*; then, of course, the rest can be expressed as

$$v_{m+i} = \sum_{j=1}^{m} \alpha_{m+i}^{(j)} v_j, \quad i = 1, 2, \ldots, k-m, \tag{1.1}$$

where the coefficients $\alpha_{m+i}^{(j)}$ are all rational numbers.

The basic idea of the proof is to involve a very large number of *rotated* copies of the given goal set S. The lack of rotation makes the 1-dimensional game disappointingly restrictive, and the possibility of using rotation makes the 2-dimensional game rich.

We are going to define a large number of angles

$$0 < \theta_1 < \theta_2 < \theta_3 < \cdots < \theta_i < \cdots < \theta_r < 2\pi, \tag{1.2}$$

where $r = r(S)$ is an integral parameter to be specified later (r will depend on goal set S only).

Let Rot_θ denote the operation "rotated by angle θ"; for example, $Rot_{\theta_i} v_j$ denotes the rotated copy of vector v_j, rotated by angle θ_i. For notational convenience write $v_{j,i} = Rot_{\theta_i} v_j$, including the case $\theta_0 = 0$, that is $v_{j,0} = v_j$.

The existence of the desired angles θ_i in (1.2) is guaranteed by:

Lemma 1: *For every integer $r \geq 1$, we can find r real-valued angles $0 = \theta_0 < \theta_1 < \cdots < \theta_i < \cdots < \theta_r < 2\pi$ such that the $m(r+1)$ vectors $v_{j,i}$ ($1 \leq j \leq m$, $0 \leq i \leq r$) are linearly independent over the rational numbers.*

Proof. We use the well-known fact that the set of rational numbers is countable, but the set of real numbers is uncountable. We proceed by induction on r; we start with $r = 1$. Assume that there exist rational numbers $a_{j,i}$ such that

$$\sum_{i=0}^{1} \sum_{j=1}^{m} a_{j,i} v_{j,i} = 0, \tag{1.3}$$

where in the right-hand side of (1.3) 0 stands for the "zero vector"; then $\sum_{j=1}^{m} a_{j,1} v_{j,1} = -\sum_{j=1}^{m} a_{j,0} v_{j,0}$, or, equivalently, $Rot_{\theta_1} u = w$, where both vectors u and w belong to the set

$$Y = \left\{ \sum_{j=1}^{m} a_j v_j : \text{ every } a_j \text{ is rational and } \sum_{j=1}^{m} a_j^2 \neq 0 \right\}. \tag{1.4}$$

Since set Y defined in (1.4) is countable, there is only a countable set of solutions for the equation $Rot_\theta u = w$, $u \in Y$, $w \in Y$ in variable θ.

Choosing a real number $\theta = \theta_1$ in $0 < \theta_1 < 2\pi$, which is *not* a solution, we conclude that

$$\sum_{i=0}^{1} \sum_{j=1}^{m} a_{j,i} v_{j,i} = 0, \quad \text{where } \sum_{i=0}^{1} \sum_{j=1}^{m} a_{j,i}^2 \neq 0,$$

can never happen, proving Lemma 1 for $r = 1$.

The general case goes very similarly. Assume that we already constructed $r-1$ (≥ 1) angles $0 = \theta_0 < \theta_1 < \cdots < \theta_i < \cdots < \theta(r-1) < 2\pi$ such that the mr vectors

$v_{j,i}$ $(1 \le j \le m,\ 0 \le i \le r-1)$ are linearly independent over the rationals. Assume that there exist rational coefficients $a_{j,i}$ such that

$$\sum_{i=0}^{r}\sum_{j=1}^{m} a_{j,i} v_{j,i} = 0, \tag{1.5}$$

where in the right-hand side of (1.5) 0 stands for the "zero vector"; then $\sum_{j=1}^{m} a_{j,r} v_{j,r} = -\sum_{i=0}^{r-1}\sum_{j=1}^{m} a_{j,i} v_{j,i}$, or, equivalently, $Rot_{\theta_1} u = w$, where $u \in Y$ (see (1.4)) and w belongs to the set

$$Z = \left\{ \sum_{i=0}^{r-1}\sum_{j=1}^{m} b_{j,i} v_{j,i} : \text{ every } b_{j,i} \text{ is rational and } \sum_{i=0}^{r-1}\sum_{j=1}^{m} b_{j,i}^2 \neq 0 \right\}.$$

Since both Y and Z are countable, there is only a countable set of solutions for the equation $Rot_\theta u = w$, $u \in Y$, $w \in Z$ in variable θ.

By choosing a real number $\theta = \theta_r$ in $\theta_{r-1} < \theta_r < 2\pi$, which is *not* a solution, Lemma 1 follows. □

The heart of the proof is the following "very regular, lattice-like construction" (a finite point set in the plane)

$$X = X(r; D; N) = \left\{ \sum_{i=0}^{r}\sum_{j=1}^{m} \frac{d_{j,i}}{D} v_{j,i} : \text{ every } d_{j,i} \text{ is an integer with } |d_{j,i}| \le N \right\}, \tag{1.6}$$

where both new integral parameters D and N will be specified later (together with $r = r(S)$).

Notice that the "very regular, lattice-like plane set" (1.6) is the projection of an $m(r+1)$-dimensional $(2N+1) \times \cdots \times (2N+1) = (2N+1)^{m(r+1)}$ hypercube to the plane.

The key property of point set $X = X(r; D; N)$ is that it is very "rich" in congruent copies of goal set S. □

Lemma 2: *Point set X – defined in (1.6) – has the following two properties:*

(a) the cardinality $|X|$ of set X is exactly $(2N+1)^{m(r+1)}$;
(b) set X contains at least $(2N+1-C_0)^{m(r+1)} \cdot (r+1)$ distinct congruent copies of goal set S, where $C_0 = C_0(S)$ is an absolute constant depending only on goal set S, but entirely independent of the parameters (r, D, and N) of the proof.

Proof. By Lemma 1 the $m(r+1)$ vectors $v_{j,i}$ $(1 \le j \le m,\ 0 \le i \le r)$ are linearly independent over the rationals. So different vector sums in (1.6) determine different points in the plane, which immediately proves part (a).

To prove part (b), fix an arbitrary integer i_0 in $0 \le i_0 \le r$, and estimate from below the number of translated copies of $Rot_{\theta_{i_0}} S$ in set X (i.e. angle θ_{i_0} is fixed). Select an arbitrary point

$$Q_0 = \sum_{i=0}^{r} \sum_{j=1}^{m} \frac{d_{j,i}}{D} v_{j,i} \in X, \tag{1.7}$$

and consider the k points $Q_l = Q_0 + v_{l,i_0}$, where $l = 1, 2, \ldots, k$. The $(k+1)$-element set $\{Q_0, Q_1, Q_2, \ldots, Q_k\}$ is certainly a translated copy of $Rot_{\theta_{i_0}} S$, but when can we guarantee that set $\{Q_0, Q_1, Q_2, \ldots, Q_k\}$ is *inside* X? Visualizing plane set X as a "hypercube," the answer becomes very simple: the set $\{Q_0, Q_1, Q_2, \ldots, Q_k\}$ is inside X if point Q_0 is "far from the border of the hypercube." The following elementary calculations make this vague statement more precise.

We divide the k points $Q_1, Q_2, \ldots, Q_k$ into two parts. First, consider an arbitrary Q_l with $1 \le l \le m$: by definition

$$Q_l = \sum_{i=0}^{r} \sum_{j=1}^{m} \frac{d_{j,i} + \delta(l, i_0; j, i) D}{D} v_{j,i}, \tag{1.8}$$

where $\delta(l, i_0; j, i) = 1$ if $(l, i_0) = (j, i)$ and $\delta(l, i_0; j, i) = 0$ if $(l, i_0) \ne (j, i)$.

If $m + 1 \le l \le k$, then by (1.1)

$$v_l = \sum_{j=1}^{m} \alpha_l^{(j)} v_j = \frac{1}{D} \sum_{j=1}^{m} C_l^{(j)} v_j, \tag{1.9}$$

where D is the least common denominator of all rational coefficients $\alpha_l^{(j)}$, and, of course, all $C_l^{(j)}$ are integers. So if $m + 1 \le l \le k$, then by (1.9)

$$Q_l = \sum_{i=0}^{r} \sum_{j=1}^{m} \frac{d_{j,i} + \delta(i_0; i) C_l^{(j)}}{D} v_{j,i}, \tag{1.10}$$

where $\delta(i_0; i) = 1$ if $i_0 = i$ and $\delta(i_0; i) = 0$ if $i_0) \ne i$. Let

$$C^* = \max_{1 \le j \le m} \max_{m+1 \le l \le k} |C_l^{(j)}|, \quad \text{and } C^{**} = \max\{C^*, D\}. \tag{1.11}$$

Now if $|d_{j,i_0}| \le N - C^{**}$ holds for every $j = 1, 2, \ldots, m$ (meaning "the point Q_0 is far from the border of hypercube X"), then by (1.6)–(1.11) the set $\{Q_0, Q_1, Q_2, \ldots, Q_k\}$ is *inside* X. We recall that $\{Q_0, Q_1, Q_2, \ldots, Q_k\}$ is a translated copy of $Rot_{\theta_{i_0}} S$; therefore, if the inequality $|d_{j,i}| \le N - C^{**}$ holds for every $j = 1, 2, \ldots, m$ and $i = 0, 1, \ldots, r$, then by (1.6) the point $Q_0 \in X$ (defined by (1.7)) is contained in at least $(r+1)$ distinct copies of goal set S (namely, in a translated copy of $Rot_{\theta_i} S$ with $i = 0, 1, \ldots, r$).

Let $\#[S \subset X]$ denote the total number of congruent copies of goal set S; the previous argument gives the lower bound

$$\#[S \subset X] \ge (2(N - C^{**}) + 1)^{m(r+1)} \cdot (r+1) = (2N + 1 - C_0)^{m(r+1)} \cdot (r+1), \tag{1.12}$$

where $C_0 = 2C^{**}$, completing the proof of Lemma 2. □

The fact that "set X is rich in congruent copies of goal set S" is expressed in quantitative terms as follows (see Lemma 2 and (1.12))

$$\frac{\#[S \subset X]}{|X|} \geq \left(1 - \frac{C_0}{2N+1}\right)^{m(r+1)} \cdot (r+1). \tag{1.13}$$

In (1.13) $C_0 = 2C^{**}$ and m ($\leq k = |S| - 1$) are "fixed" (i.e. they depend on goal set S only), but at this stage parameters r and N are completely arbitrary. It is crucial to see that parameters r and N can be specified in such a way that

$$\frac{\#[S \subset X]}{|X|} \geq \frac{r+1}{2} \geq \text{"arbitrarily large."} \tag{1.14}$$

Indeed, for every "arbitrarily large" value of r there is a sufficiently large value of N such that

$$\left(1 - \frac{C_0}{2N+1}\right)^{m(r+1)} \geq \frac{1}{2},$$

and then (1.13) implies (1.14).

We emphasize that inequality (1.14) is the key quantitative property of our point set X (see (1.6)).

After these preparations we are now ready to explain how Maker is able to build a congruent copy of the given goal set S. The reader is probably expecting a quick greedy algorithm, but what we are going to do here is in fact a slow indirect procedure:

(i) Maker will stay strictly inside (the huge!) set X;
(ii) Maker will always choose his next point by optimizing an appropriate *potential function* (we define it below);
(iii) whenever set X is completely exhausted, Maker will end up with a congruent copy of goal set S.

Steps (i)–(iii) describe Maker's indirect building strategy. Of course, Breaker doesn't know about Maker's plan to stay inside the set X (Breaker doesn't know about the set X at all!), and Maker doesn't know in advance whether Breaker's next move will be inside or outside of X – but these are all irrelevant, Maker will own a congruent copy of S anyway.

The main question remains: "What kind of potential function does Maker use?" Maker will use a natural Power-of-Two Scoring System. As far as we know, the first appearance of the Power-of-Two Scoring System is in a short but important paper of Erdős and Selfridge [1973], see Theorem 1.4 below. We will return to the "potential technique" and the Erdős–Selfridge Theorem in Section 10.

3. Potential criterions. It is convenient to introduce the hypergraph $\mathcal{F}$ of "winning sets": a $(k+1)$-element subset $A \subset X$ of set X (defined in (1.6)) is a hyperedge

$A \in \mathcal{F}$ if and only if A is a congruent copy of goal set S. Hypergraph $\mathcal{F}$ is $(k+1)$-uniform with size $|\mathcal{F}| = \#[S \subset X]$; we refer to X as the "board," and call $\mathcal{F}$ the "family of winning sets." We will apply the following very general hypergraph result; it plays the role of our "Lemma 3," but for later applications we prefer to call it a "theorem."

Theorem 1.2 *Let $(V, \mathcal{F})$ be a finite hypergraph: V is an arbitrary finite set, and $\mathcal{F}$ is an arbitrary family of subsets of V. The Maker–Breaker Game on $(V, \mathcal{F})$ is defined as follows: the two players, called Maker and Breaker, alternately occupy previously unoccupied elements of "board" V (the elements are called "points"); Maker's goal is to occupy a whole "winning set" $A \in \mathcal{F}$, Breaker's goal is to stop Maker. If $\mathcal{F}$ is n-uniform and*

$$\frac{|\mathcal{F}|}{|V|} > 2^{n-3} \cdot \Delta_2(\mathcal{F}),$$

where $\Delta_2(\mathcal{F})$ is the Max Pair-Degree, then Maker, as the first player, has a winning strategy in the Maker–Breaker Game on $(V, \mathcal{F})$.

The Max Pair-Degree is defined as follows: assume that, fixing any 2 distinct points of board V, there are $\leq \Delta_2(\mathcal{F})$ winning sets $A \in \mathcal{F}$ containing both points, and equality occurs for some point pair. Then we call $\Delta_2(\mathcal{F})$ the Max Pair-Degree of $\mathcal{F}$.

In particular, for **Almost Disjoint** *hypergraphs, where any two hyperedges have at most one common point (like a family of "lines"), the condition simplifies to $|\mathcal{F}| > 2^{n-3}|V|$.*

Remark. If $\mathcal{F}$ is n-uniform, then $\frac{|\mathcal{F}|}{|V|}$ is $\frac{1}{n}$ times the *Average Degree*. Indeed, this equality follows from the easy identity $n|\mathcal{F}| = \text{AverDeg}(\mathcal{F})|V|$.

The hypothesis of Theorem 1.2 is a simple Density Condition: in a "dense" hypergraph Maker can always occupy a whole winning set.

First we explain how Theorem 1.2 completes the proof of Theorem 1.1, and discuss the proof of Theorem 1.2 later. Of course, we apply Theorem 1.2 with $V = X$, where X is defined in (1.6), and $\mathcal{F}$ is the $(k+1)$-uniform hypergraph such that a $(k+1)$-element subset $A \subset X$ of set X (see (1.6)) is a hyperedge $A \in \mathcal{F}$ if and only if A is a congruent copy of goal set S. There is, however, an almost trivial formal difficulty in the application of Theorem 1.2 that we have to point out: in Theorem 1.2 Breaker always replies in set X, but in the "S-building game" Breaker may or may not reply in set X (Breaker has the whole plane to choose from). We can easily overcome this formal difficulty by introducing "fake moves": whenever Breaker's move is outside of set X, Maker chooses an arbitrary unoccupied point in X and declares this fake move to be "Breaker's move." If later Breaker actually occupies this fake point (i.e. the "fake move" becomes a "real move"), then Maker chooses another unoccupied point and declares this fake move

"Breaker's move," and so on. By using the simple trick of "fake moves," there is no difficulty whatsoever in applying Theorem 1.2 to the "S-building game in the plane." As we said before, we choose $V = X$ (see (1.6)) and

$$\mathcal{F} = \{A \subset X: \; A \text{ is a congruent copy of } S\}.$$

Clearly $n = |S| = k+1$, but what is the Max Pair-Degree $\Delta_2(\mathcal{F})$? The exact value is a difficult question, but we don't need that – the following trivial upper bound suffices

$$\Delta_2(\mathcal{F}) \le \binom{|S|}{2} = \binom{k+1}{2}. \tag{1.15}$$

Theorem 1.2 applies if (see Lemma 2)

$$\frac{|\mathcal{F}|}{|V|} = \frac{\#[S \subset X]}{|X|} \ge \left(1 - \frac{C_0}{2N+1}\right)^{m(r+1)} \cdot (r+1) > 2^{n-3} \cdot \Delta_2(\mathcal{F}). \tag{1.16}$$

(1.16) is satisfied if (see (1.15))

$$\left(1 - \frac{C_0}{2N+1}\right)^{m(r+1)} \cdot (r+1) > 2^{n-3} \cdot \Delta_2(\mathcal{F}) \ge \binom{k+1}{2} 2^{k-2}. \tag{1.17}$$

Here C_0 and $m(\le k = |S| - 1)$ are fixed (in the sense that they depend only on set S), but parameters r and N are completely free. Now let $r = (k+1)^2 2^{k-2}$, then inequality (1.17) holds if N is sufficiently large. Applying Theorem 1.2 (see (1.16)–(1.17)) the proof of Theorem 1.1 is complete. □

It remains, of course, to prove Theorem 1.2.

Proof of Theorem 1.2. Assume we are in the middle of a play where Maker (the first player) already occupies $x_1, x_2, \ldots, x_i$, and Breaker (the second player) occupies $y_1, y_2, \ldots, y_i$. The question is how to choose Maker's next point x_{i+1}. Those winning sets that contain at least one $y_j (j \le i)$ are "useless" for Maker; we call them "dead sets." The winning sets which are not "dead" (yet) are called "survivors." The "survivors" have a chance to be completely occupied by Maker. What is the total "winning chance" of the position? We evaluate the given position by the following "opportunity function" (measuring the *opportunity* of winning): $T_i = \sum_{s \in S_i} 2^{n-u_s}$, where u_s is the number of unoccupied points of the "survivor" A_s ($s \in S_i =$"index-set of the survivors," and index i indicates that we are at the stage of choosing the $(i+1)$st point x_{i+1} of Maker. Note that the "opportunity" can be much greater than 1 (i.e. it is *not* a probability), but it is always non-negative.

A natural choice for x_{i+1} is to maximize the "winning chance" T_{i+1} at the next turn. Let x_{i+1} and y_{i+1} denote the next moves of the 2 players. What is their effect on T_{i+1}? Well, first x_{i+1} doubles the "chances" for each "survivor" $A_s \ni x_{i+1}$, i.e. we have to add the sum $\sum_{s \in S_i:\; x_{i+1} \in A_s} 2^{n-u_s}$ to $\ni T_i$.

On the other hand, y_{i+1} "kills" all the "survivors" $A_s \ni y_{i+1}$, which means we have to subtract the sum

$$\sum_{s\in S_i:\ y_{i+1}\in A_s} 2^{n-u_s}$$

from T_i.

Warning: we have to make a correction to those "survivors" A_s that contain both x_{i+1} and y_{i+1}. These "survivors" A_s were "doubled" first and "killed" second. So what we have to subtract from T_i is not

$$\sum_{s\in S_i:\ \{x_{i+1},y_{i+1}\}\subset A_s} 2^{n-u_s}$$

but the twice as large

$$\sum_{s\in S_i:\ \{x_{i+1},y_{i+1}\}\subset A_s} 2^{n-u_s+1}.$$

It follows that

$$T_{i+1} = T_i + \sum_{s\in S_i:\ x_{i+1}\in A_s} 2^{n-u_s} - \sum_{s\in S_i:\ y_{i+1}\in A_s} 2^{n-u_s} - \sum_{s\in S_i:\ \{x_{i+1},y_{i+1}\}\subset A_s} 2^{n-u_s}.$$

Now the natural choice for x_{i+1} is the unoccupied z for which $\sum_{s\in S_i:\ z\in A_s} 2^{n-u_s}$ attains its maximum. Then clearly

$$T_{i+1} \geq T_i - \sum_{s\in S_i:\ \{x_{i+1},y_{i+1}\}\subset A_s} 2^{n-u_s}.$$

We trivally have

$$\sum_{s\in S_i:\ \{x_{i+1},y_{i+1}\}\subset A_s} 2^{n-u_s} \leq \Delta_2 \cdot 2^{n-2}.$$

Indeed, there are at most Δ_2 winning sets A_s containing the given 2 points $\{x_{i+1}, y_{i+1}\}$, and $2^{n-u_s} \leq 2^{n-2}$, since x_{i+1} and y_{i+1} were definitely unoccupied points at the previous turn.

Therefore

$$T_{i+1} \geq T_i - \Delta_2 2^{n-2}. \tag{1.18}$$

What happens at the end? Let ℓ denote the number of turns, i.e. the ℓth turn is the last one. Clearly $\ell = |V|/2$. Inequality $T_\ell = T_{last} > 0$ means that at the end Breaker could not "kill" (block) all the winning sets. In other words, $T_\ell = T_{last} > 0$ means that Maker was indeed able to occupy a whole winning set.

So all what we have to check is that $T_\ell = T_{last} > 0$. But this is trivial; indeed, $T_{start} = T_0 = |\mathcal{F}|$, so we have

$$T_{last} \geq |\mathcal{F}| - \frac{|V|}{2}\Delta_2 2^{n-2}. \tag{1.19}$$

It follows that, if $|\mathcal{F}| > 2^{n-3}|V|\Delta_2$, then $T_{last} > 0$, implying that at the end of the play Maker was able to completely occupy a winning set. □

Under the condition of Theorem 1.2, Maker can occupy a whole winning set $A \in \mathcal{F}$, but how long does it take for Maker to do this? The minimum number of moves needed against a perfect opponent is called the Move Number.

A straightforward adaptation of the proof technique of Theorem 1.2 gives a simple but very interesting lower bound on the Move Number.

Move Number. Assume that $\mathcal{F}$ is an n-uniform hypergraph with Max Pair-Degree $\Delta_2(\mathcal{F})$; how long does it take for Maker to occupy a whole $A \in \mathcal{F}$? The following definition is crucial: a hyperedge $A \in \mathcal{F}$ becomes *visible* if Maker has at least 2 marks in it. Note that "2" is critical in the sense that 2 points determine at most $\Delta_2(\mathcal{F})$ hyperedges. We follow the previous proof applied for the *visible sets*. Assume that we are in the middle of a play: $x_1, x_2, \ldots, x_i$ denote the points of Maker and $y_1, y_2, \ldots, y_{i-1}$ denote the points of Breaker up to this stage (we consider the "worse case" where Breaker is the second player). The "danger function" is defined as

$$D_i = \sum_{A\in\mathcal{F}:\ **} 2^{|A\cap\{x_1,x_2,\ldots,x_i\}|}$$

where $**$ means the double requirement $A \cap \{y_1, y_2, \ldots, y_{i-1}\} = \emptyset$ and $|A \cap \{x_1, x_2, \ldots, x_i\}| \geq 2$; Breaker chooses that new point $y = y_i$ for which the sum

$$D_i(y) = \sum_{y\in A\in\mathcal{F}:\ **} 2^{|A\cap\{x_1,x_2,\ldots,x_i\}|}$$

attains its maximum. What is the effect of the consecutive moves y_i and x_{i+1}? We clearly have

$$D_{i+1} = D_i - D_i(y_i) + D_i(x_{i+1}) - \sum_{\{y_i,x_{i+1}\}\subset A\in\mathcal{F}:\ **} 2^{|A\cap\{x_1,x_2,\ldots,x_i\}|} + \sum_{A\in\mathcal{F}:\ ***} 2^2,$$

where $***$ means the triple requirement $x_{i+1} \in A$, $|A \cap \{x_1, x_2, \ldots, x_i\}| = 1$, and $A \cap \{y_1, y_2, \ldots, y_i\} = \emptyset$. Since 2 points determine at most $\Delta_2(\mathcal{F})$ hyperedges, we obtain the simple inequality

$$D_{i+1} \leq D_i - D_i(y_i) + D_i(x_{i+1}) + \sum_{A\in\mathcal{F}:\ ***} 2^2 \leq D_i + 4i \cdot \Delta_2(\mathcal{F}).$$

Now assume that Maker can occupy a whole winning set for the first time at his Mth move (M is the Move Number); then

$$2^n \leq D_M \leq D_{M-1} + 4\Delta_2(\mathcal{F})(M-1) \leq D_{M-2} + 4\Delta_2(\mathcal{F})(M-1+M-2) \leq \cdots$$
$$\leq D_1 + 4\Delta_2(\mathcal{F})(1+2+\ldots+(M-1)) = D_1 + 2\Delta_2(\mathcal{F})M(M-1).$$

Notice that $D_1 = 0$ (for at the beginning Maker does not have 2 points yet), so $2^n \leq D_M \leq D_1 + 2\Delta_2(\mathcal{F})M(M-1) = 2\Delta_2(\mathcal{F})M(M-1)$, implying the following: *exponential* lower bound.

Theorem 1.3 *If the underlying hypergraph is n-uniform and the Max Pair-Degree is $\Delta_2(\mathcal{F})$, then playing the Maker–Breaker game on hypergraph $\mathcal{F}$, it takes at least*

$$\frac{2^{(n-1)/2}}{\sqrt{\Delta_2(\mathcal{F})}}$$

moves for Maker (the first player) to occupy a whole winning set.

What it means is that there is no quick win in an Almost Disjoint, or nearly Almost Disjoint, hypergraph. Even if the first player has a winning strategy, the second player can postpone it for an exponentially long time! Exponential time practically means "it takes forever."

The potential proof technique of Theorems 1.2–1.3 shows a striking similarity with the proof of the well-known Erdős–Selfridge Theorem. This is not an accident: the Erdős–Selfridge Theorem was the pioneering result, and Theorems 1.2–1.3 were discovered after reading the Erdős–Selfridge paper [1973], see Beck [1981a] and [1981b].

The Erdős–Selfridge Theorem is a draw criterion, and proceeds as follows.

Theorem 1.4 *("Erdős–Selfridge Theorem") Let $\mathcal{F}$ be an n-uniform hypergraph, and assume that $|\mathcal{F}| + MaxDeg(\mathcal{F}) < 2^n$, where $MaxDeg(\mathcal{F})$ denotes the maximum degree of hypergraph $\mathcal{F}$. Then playing the Maker–Breaker game on $\mathcal{F}$, Breaker (the second player) can put his mark in every $A \in \mathcal{F}$.*

Remark. If the second player can block every winning set, then the first player can also block every winning set (hint: either use the general "strategy stealing" argument, or simply repeat the whole proof); consequently, the "generalized Tic-Tac-Toe game" on $\mathcal{F}$ is a draw.

Exercise 1.1 *Prove the Erdős–Selfridge Theorem.*

We will return to Theorem 1.4 in Section 10, where we include a proof, and show several generalizations and applications.

For many more quick applications of Theorems 1.2–1.4, see Chapter III (Sections 13–15). The reader can jump ahead and read these self-contained applications right now.

What we do in this book is a "hypergraph theory with a game-theoretic motivation." Almost Disjoint hypergraphs play a central role in the theory. The main task is to develop sophisticated "reinforcements" of the simple hypergraph criterions Theorems 1.2–1.4. For example, see Theorem 24.2 (which is an "advanced" building criterion) and Theorems 34.1, 37.5, and 40.1 (the three "ugly" blocking criterions).

2
Analyzing the proof of Theorem 1.1

1. How long does it take to build? Let us return to Example 1: in view of Theorem 1.1, Maker can always build a congruent copy of the 5-term A.P. S_5 in the plane

but how long does it take to build S_5? Analyzing the proof of Theorem 1.1 in this very simple special case, we have

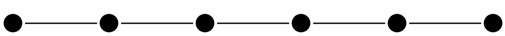
$$k = 4,\ m = 1,\ D = 1,\ C^* = C^{**} = 4,\ C_0 = 2C^{**} = 8,\ \Delta_2 = 4,$$

and the key inequality (1.17) is

$$\left(1 - \frac{8}{2N+1}\right)^{r+1} \cdot (r+1) > 2^{4-2} \cdot \Delta_2 = 2^2 \cdot 4. \tag{2.1}$$

By choosing (say) $r = 43$ and $N = 154$, inequality (2.1) holds. It follows that, if Maker restricts himself to set X (see (1.6)), then, when X is exhausted, Maker will certainly own a congruent copy of the 5-term A.P. S_5. Since $|X| = (2N+1)^{m(r+1)}$ (see Lemma 2 (a)), this gives about $309^{44} \approx 10^{110}$ moves even for a very simple goal set like S_5. This bound is rather disappointing!

How about the 6-term A.P. S_6?



The proof of Theorem 1.1 gives

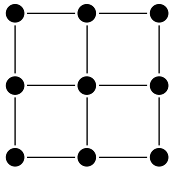
$$k = 5,\ m = 1,\ D = 1,\ C^* = C^{**} = 5,\ C_0 = 2C^{**} = 10,\ \Delta_2 = 5,$$

and the key inequality (1.17) is

$$\left(1 - \frac{10}{2N+1}\right)^{r+1} \cdot (r+1) > 2^{5-2} \cdot \Delta_2 = 2^3 \cdot 5. \tag{2.2}$$

By choosing (say) $r = 108$ and $N = 491$, inequality (2.2) holds. Since $|X| = (2N+1)^{m(r+1)}$ (see Lemma 2 (a)), the argument gives about $983^{109} \approx 10^{327}$ moves for goal set S_6.

Next we switch to Example 3, and consider the 3×3 "Tic-Tac-Toe set" S_9; how long does it take to build a congruent copy of S_9?



The proof of Theorem 1.1 gives

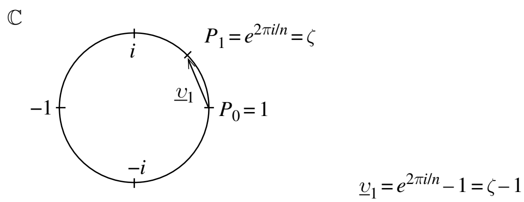
$$k = 8,\ m = 2,\ D = 1,\ C^* = C^{**} = 2,\ C_0 = 2C^{**} = 4,\ \Delta_2 = 12,$$

and the key inequality (1.17) is

$$\left(1 - \frac{4}{2N+1}\right)^{2(r+1)} \cdot (r+1) > 2^{8-2} \cdot \Delta_2 = 2^6 \cdot 12,$$

which is satisfied with $r = 2087$ and $N = 10\,450$, so $|X| = (2N+1)^{m(r+1)} = 20\,901^{4176} \approx 10^{18,041}$ moves suffice to build a congruent copy of goal set S_9.

Example 4 is about the regular pentagon and the regular hexagon; the corresponding "rational dimensions" turn out to be $m = 4$ and $m = 2$, respectively. Why 4 and 2? The best is to understand the general case: the regular n-gon for arbitrary $n \geq 3$. Consider the regular n-gon in the unit circle of the complex plane where $P_0 = 1$



By using the notation of the proof of Theorem 1.1, we have $v_j = \zeta^j - 1$, $1 \leq j \leq n-1$, where $\zeta = e^{2\pi i/n}$ (and, of course, $i = \sqrt{-1}$). For later application note that

$$\sum_{j=1}^{n-1} v_j = \sum_{j=1}^{n-1} (\zeta^j - 1) = -1 - (n-1) = -n. \tag{2.3}$$

Note that the algebraic number field $\mathbf{Q}(\zeta) = \mathbf{Q}(e^{2\pi i/n})$ is called the cyclotomic field of the nth roots of unity. It is known from algebraic number theory that $\mathbf{Q}(\zeta)$ is a $\phi(n)$-dimensional vector space over the rationals $\mathbf{Q}$, where $\phi(n)$ is Euler's function: $\phi(n)$ is the number of integers t in $1 \le t \le n$ which are relatively prime to n. We have the elementary product formula

$$\phi(n) = n \prod_{p|n:\ p=prime} \left(1 - \frac{1}{p}\right);$$

for example, $\phi(5) = 4$ and $\phi(6) = 2$.

Since $0 = \zeta^n - 1 = (\zeta - 1)(1 + \zeta + \zeta^2 + \ldots + \zeta^{n-1})$, we have $\zeta^{n-1} = -(1 + \zeta + \zeta^2 + \ldots + \zeta^{n-2})$, so the cyclotomic field $\mathbf{Q}(\zeta)$ is generated by $1, \zeta, \zeta^2, \ldots, \zeta^{n-2}$. The $n-1$ generators $1, \zeta, \zeta^2, \ldots, \zeta^{n-2}$ can be expressed in terms of the $n-1$ vectors $v_1, v_2, \ldots, v_{n-1}$; indeed, by (2.3) we have

$$1 = -\frac{v_1 + v_2 + \cdots + v_{n-1}}{n}, \quad \zeta^j = v_j - \frac{v_1 + v_2 + \cdots + v_{n-1}}{n}, \ 1 \le j \le n-2.$$

Therefore, in the special case of the regular n-gon, the "rational dimension" of vector set $\{v_1, \ldots, v_{n-1}\}$, denoted by m in the proof of Theorem 1.1, is exactly the Euler's function $m = \phi(n)$.

For the regular pentagon (see Example 4) the proof of Theorem 1.1 gives

$$k = 4,\ m = \phi(5) = 4,\ D = C^* = C^{**} = 1,\ C_0 = 2C^{**} = 2,\ \Delta_2 = 5,$$

and the key inequality (1.17) is

$$\left(1 - \frac{2}{2N+1}\right)^{4(r+1)} \cdot (r+1) > 2^{4-2} \cdot \Delta_2 = 2^2 \cdot 5,$$

which is satisfied with $r = 54$ and $N = 110$, so $|X| = (2N+1)^{m(r+1)} = 221^{220} \approx 10^{516}$ moves suffice to build a congruent copy of a regular pentagon.

Finally consider the regular hexagon: the proof of Theorem 1.1 gives

$$k = 5,\ m = \phi(6) = 2,\ D = 1,\ C^* = C^{**} = 2,\ C_0 = 2C^{**} = 4,\ \Delta_2 = 6,$$

and the key inequality (1.17) is

$$\left(1 - \frac{4}{2N+1}\right)^{2(r+1)} \cdot (r+1) > 2^{5-2} \cdot \Delta_2 = 2^3 \cdot 6,$$

which is satisfied with $r = 130$ and $N = 393$, so $|X| = (2N+1)^{m(r+1)} = 787^{262} \approx 10^{759}$ moves suffice to build a congruent copy of a regular hexagon.

The wonderful thing about Theorem 1.1 is that it is strikingly general. Yet there is an obvious handicap: these upper bounds to the Move Number are all ridiculously large. We are convinced that Maker can build each one of the concrete goal sets listed in Examples 1–4 in (say) less than 1000 moves, but do not have the slightest idea how to prove it. The problem is that any kind of brute force case study becomes hopelessly complicated.

By playing on the plane, Maker can build a congruent copy of a given 3-term A.P. in 3 moves, a 4-term A.P. in 5 moves, an equilateral triangle in 3 moves, a square in 6 moves. These are economical, or nearly economical, strategies. How about some more complicated goal sets such as a regular 30-gon or a regular 40-gon ("polygons")? Let $\mathcal{F}$ denote the family of all congruent copies of a given regular 30-gon in the plane. The Max Pair-Degree of hypergraph $\mathcal{F}$ is 2 (a regular polygon uniquely defines a circle, and for the family of all circles of fixed radius the Max Pair-Degree is obviously 2). Applying Theorem 1.3 we obtain that Maker needs at least $2^{15-1} = 2^{14} > 16\,000$ moves to build a given regular 30-gon. For the regular 40-gon the same argument gives at least $2^{20-1} = 2^{19} > 524\,000$ moves. Pretty big numbers!

Note that for an arbitrary n-element goal set S in the plane the corresponding Max Pair-Degree is trivially less than $\binom{n}{2} < n^2/2$ (a non-trivial bound comes from estimating the maximum repetition of the same distance; this is Erdős's famous Unit Distance Problem). Thus Theorem 1.3 gives the general lower bound $\geq \frac{1}{n}2^{n/2}$ for the Move Number. This is exponentially large, meaning that for large goal sets the "building process" is extremely slow; anything but economical!

The trivial upper bound $\leq \binom{n}{2}$ for the Max Pair-Degree (see above) can be improved to $\leq 4n^{4/3}$; this is the current record in the Unit Distance Problem (in Combinatorial Geometry), see Székely's elegant paper [1997]. Replacing the trivial bound $\binom{n}{2}$ with the hard bound $4n^{4/3}$ makes only a slight ("logarithmic") improvement in the given exponential lower bound for the Move Number.

2. Effective vs. ineffective. Let us return to Theorem 1.1. One weakness is the very poor upper bound for the Move Number (such as $\leq 10^{18\,041}$ moves for the 9-element "Tic-Tac-Toe set" S_9 in Example 3), but an even more fundamental weakness is the lack of an upper bound depending only on $|S|$ (the size of the given goal set S). The appearance of constant C_0 in key inequality (1.17) makes the upper bound "ineffective"!

What does "ineffective" mean here? What is "wrong" with constant C_0? The obvious problem is that a rational number $\alpha = C/D$ is "finite but not bounded", i.e. the numerator C and the denominator D can be arbitrarily large. Indeed, in view of (1.1), the rational coefficients $\alpha_l^{(j)}$ in $v_l = \sum_{j=1}^{m} \alpha_l^{(j)} v_j$, $\quad l = m+1, m+2, \ldots, k$, may have arbitrarily large common denominator D and arbitrarily large numerators: $\alpha_l^{(j)} = C_l^{(j)}/D$, which implies that

$$C^* = \max_{1\leq j\leq m} \max_{m+1\leq l\leq k} |C_l^{(j)}| \quad \text{and } C^{**} = \max\{C^*, D\},$$

are both "finite but not bounded in terms of S." Since $C_0 = 2C^{**}$ and C_0 show up in the key inequality (1.17), the original proof of Theorem 1.1 does not give any hint of how to bound the Move Number in terms of the single parameter $|S|$.

We have to *modify* the proof of Theorem 1.1 to obtain the following *effective* version.

Theorem 2.1 *Consider the "S-building game in the plane" introduced in Theorem 1.1: Maker can always build a congruent copy of any given finite point set S in at most*

$$2^{2^{|S|^2}} \text{ moves if } |S| \geq 10.$$

Question: Can we replace the doubly exponential upper bound in Theorem 2.1 by a plain exponential bound? Notice that "plain exponential" is necessary.

For the sake of completeness we include a **proof of Theorem 2.1.** A reader in a rush is advised to skip the technical proof below, and to jump ahead to Theorem 2.2.

As said before, we are going to modify the original proof of Theorem 1.1, but the beginning of the proof remains the same. Consider the $k = |S| - 1$ vectors $v_1, \ldots, v_k$, and again assume that among these k vectors exactly the first m (with some $1 \leq m \leq k$) are linearly independent over the rationals; so $v_1, \ldots, v_m$ are linearly independent over the rationals, and the rest can be written in the form

$$v_l = \sum_{j=1}^{m} \alpha_l^{(j)} v_j, \quad l = m+1, m+2, \ldots, k$$

with rational coefficients

$$\alpha_l^{(j)} = \frac{A_1(j,l)}{B_1(j,l)};$$

here $A_1(j,l)$ and $B_1(j,l)$ are relatively prime integers. Write

$$C_l = \max_{1 \leq j \leq m} \{|A_1(j,l)|, |B_1(j,l)|\}, \quad l = m+1, m+2, \ldots, k. \tag{2.4}$$

For notational convenience we can assume

$$C_{m+1} \geq C_{m+2} \geq \cdots \geq C_k \tag{2.5}$$

(if (2.5) is violated, then simply rearrange the lower indexes!), and take the largest one $M_1 = C_{m+1}$.

Next we basically repeat the previous step with the one larger set $\{v_1, v_2, \ldots, v_m, v_{m+1}\}$ instead of $\{v_1, v_2, \ldots, v_m\}$: for every $l = m+2, m+3, \ldots, k$ consider the solutions of the linear equation

$$v_l = \sum_{j=1}^{m+1} \beta_l^{(j)} v_j, \tag{2.6}$$

with *rational* coefficients

$$\beta_l^{(j)} = \frac{A_2(j,l)}{B_2(j,l)};$$

here $A_2(j,l)$ and $B_2(j,l)$ are relatively prime integers. Since the vector set $\{v_1, v_2, \ldots, v_{m+1}\}$ is not linearly independent any more, system (2.6) may have several solutions; one solution comes from the previous step $\beta_l^{(j)} = \alpha_l^{(j)}$ $(j = 1, \ldots, m)$, $\beta_l^{(m+1)} = 0$ for every $l = m+2, m+3, \ldots, k$. Since (2.6) may have many solutions, we optimize: for every $l = m+2, m+3, \ldots, k$ we choose that particular solution of (2.6) for which

$$C'_l = \max_{1 \le j \le m+1} \{|A_2(j,l)|, |B_2(j,l)|\}$$

attains its *minimum*. For notational convenience we can assume

$$C'_{m+2} \ge C'_{m+3} \ge \cdots \ge C'_k \tag{2.7}$$

(if (2.7) is violated, then simply rearrange the lower indexes!), and take the largest one $M_2 = C'_{m+2}$. Then, of course, $M_1 \ge M_2$.

Again we repeat the previous step with the one longer set $\{v_1, v_2, \ldots, v_{m+1}, v_{m+2}\}$ instead of $\{v_1, v_2, \ldots, v_{m+1}\}$: for every $l = m+3, m+4, \ldots, k$ consider the solutions of the linear equation

$$v_l = \sum_{j=1}^{m+1} \gamma_l^{(j)} v_j \tag{2.8}$$

with *rational* coefficients

$$\gamma_l^{(j)} = \frac{A_3(j,l)}{B_3(j,l)};$$

here $A_3(j,l)$ and $B_3(j,l)$ are relatively prime integers. Since the vector set $\{v_1, \ldots, v_{m+2}\}$ is not linearly independent, system (2.8) may have several solutions; one solution comes from the previous step $\gamma_l^{(j)} = \beta_l^{(j)}$ $(j = 1, \ldots, m+1)$, $\gamma_l^{(m+2)} = 0$ for every $l = m+3, m+4, \ldots, k$. Since (2.8) may have many solutions, we optimize: for every $l = m+3, m+4, \ldots, k$ we choose that particular solution for which

$$C''_l = \max_{1 \le j \le m+2} \{|A_3(j,l)|, |B_3(j,l)|\}$$

attains its *minimum*. For notational convenience, we can assume

$$C''_{m+3} \ge C''_{m+3} \ge \cdots \ge C''_k \tag{2.9}$$

(if (2.9) is violated, then simply rearrange the lower indexes!), and take the largest one $M_3 = C''_{m+3}$. Then, of course, $M_1 \ge M_2 \ge M_3$.

By repeating this argument, we obtain a decreasing sequence

$$M_1 \ge M_2 \ge M_3 \ge \ldots \ge M_{k-m}. \tag{2.10}$$

Notice that sequence (2.10) depends only on the given point set S; the elements of the sequence can be *arbitrarily large* (since a rational number is "finite but unbounded").

We are going to define a decreasing sequence of *constants*

$$\Omega_1 > \Omega_2 > \Omega_3 > \ldots > \Omega_{k-m} \tag{2.11}$$

depending only on the *size* $|S| = k+1$ of S (the explicit form of (2.11) is "ugly," see (2.19) and (2.20) later), and compare the "arbitrary" sequence (2.10) with the "constant" sequence (2.11).

Assume that there is an index $\nu \geq 1$ such that

$$M_\nu > \Omega_\nu \quad \text{but} \quad M_{\nu+1} \leq \Omega_{\nu+1}. \tag{2.12}$$

We modify our "board set" X (see (1.6)): let

$$\widetilde{X} = \widetilde{X}(r; D; N) = \left\{ \sum_{i=0}^{r} \sum_{j=1}^{m+\nu} \frac{d_{j,i}}{D} v_{j,i} : \text{ every } d_{j,i} \text{ is an integer with } |d_{j,i}| \leq N \right\}; \tag{2.13}$$

here again r, D, and N are unspecified integral parameters.

Notice that (2.13) is the projection of an $(m+\nu)(r+1)$-dimensional $(2N+1) \times \cdots \times (2N+1) = (2N+1)^{(m+\nu)(r+1)}$ hypercube; the value of $\nu(\geq 1)$ is defined by (2.12). The meaning of (2.13) is that, although the "extra" vectors $v_{m+1}, \ldots, v_{m+\nu}$ are not rationally independent of $v_1, \ldots, v_m$, we still handle the $m+\nu$ vectors $v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+\nu}$ like an independent vector set (because any dependence among them requires rationals with too large numerator/denominator).

We define parameters N and D in (2.13) as follows

$$N = \Omega_\nu \quad \text{and} \quad D = \prod_{j=1}^{m+\nu} \prod_{l=m+\nu+1}^{k} B_{\nu+1}(j, l), \tag{2.14}$$

i.e. D is the "product of the denominators" showing up in the $(\nu+1)$st step of the iterated procedure above (see (2.5)–(2.10)). By (2.12) and (2.14)

$$D \leq (\Omega_{\nu+1})^{k^2}. \tag{2.15}$$

By repeating the proof of Lemma 2 in Section 1, we get the following "effective" analogue.

Lemma 1: *Point set $\widetilde{X}$ – defined in (2.13) – has the following 2 properties:*

(a) the cardinality $|\widetilde{X}|$ of set $\widetilde{X}$ is exactly $(2N+1)^{(m+\nu)(r+1)}$;
(b) set $\widetilde{X}$ contains at least $(2N+1-\widetilde{C})^{(m+\nu)(r+1)} \cdot (r+1)$ distinct congruent copies of goal set S, where

$$\widetilde{C} = 2\,(\Omega_{\nu+1})^{k^2}$$

is an "effective" constant.

We apply Theorem 1.2 to the new "board" $V = \widetilde{X}$ (see (2.13)) with the simple trick of "fake moves," and, of course

$$\mathcal{F} = \{A \subset \widetilde{X} : \ A \text{ is a congruent copy of } S\};$$

clearly $n = |S| = k+1$.

Theorem 1.2 applies if (see Lemma 1)

$$\frac{|\mathcal{F}|}{|V|} = \frac{\#[S \subset \widetilde{X}]}{|\widetilde{X}|} \geq \left(1 - \frac{\widetilde{C}}{2N+1}\right)^{(m+\nu)(r+1)} \cdot (r+1) > 2^{n-3} \cdot \Delta_2(\mathcal{F}). \tag{2.16}$$

(2.16) is satisfied if

$$\left(1-\frac{\widetilde{C}}{2N+1}\right)^{(m+\nu)(r+1)}\cdot(r+1)>2^{n-3}\cdot\Delta_2(\mathcal{F})\geq\binom{k+1}{2}2^{k-2}. \tag{2.17}$$

By choosing $r=(k+1)^2 2^k$ and

$$N\geq(k+1)^3 2^k\left(\Omega_{\nu+1}\right)^{k^2}, \tag{2.18}$$

inequality (2.17) holds.

Now it is clear how to define the constants in (2.12). We proceed backward; we start with the last one Ω_{k-m}: let

$$\Omega_{k-m}=(k+1)^3 2^k, \tag{2.19}$$

and define the backward recurrence relation

$$\Omega_\nu=(k+1)^3 2^k\left(\Omega_{\nu+1}\right)^{k^2}. \tag{2.20}$$

Formulas (2.19)–(2.20) are clearly motivated by (2.14) and (2.18).

Now we are ready to complete the proof of Theorem 2.1: Theorem 1.2 implies that, staying in $\widetilde{X}$ as long as possible, at the end Maker will own a congruent copy of goal set S; this gives the following upper bound for the Move Number (see (2.14))

$$\leq|\widetilde{X}|=(2N+1)^{(m+\nu)(r+1)}=(2\Omega_\nu+1)^{(m+\nu)(r+1)}\leq(2\Omega_\nu+1)^{(k+1)^3 2^k}. \tag{2.21}$$

By (2.20) the first (i.e. largest) member of the constant sequence in (2.12) is less than

$$\Omega_1\leq\left(\left(\left(2^{2k}\right)^{2^2}\right)^{2k^2}\cdots\right)^{2k^2}=2^{2k\cdot 2k^2\cdot 2k^2\cdots 2k^2}\leq 2^{(2k^2)^{k+1}},$$

so, by (2.20), the "Move Number" is less than

$$\left(2^{(2k^2)^{k+1}}\right)^{(k+1)^3 2^k}<2^{2^{(k+1)^2}}\quad\text{if}\quad|S|=k+1\geq 10. \tag{2.22}$$

Finally, consider the last case when there is *no* index $\nu\geq 1$ such that $M_\nu>\Omega_\nu$ but $M_{\nu+1}\leq\Omega_{\nu+1}$, i.e. when (2.2) fails. Then

$$\Omega_1\geq M_1,\Omega_2\geq M_2,\ldots,\Omega_{k-m}\geq M_{k-m},$$

so the original proof of Theorem 1.1 already gives the "effective" upper bound (2.22). This completes the proof of Theorem 2.1. □

3. The biased version. Last question: What happens if Maker is the "underdog"? More precisely, what happens in the "biased" case where Maker takes 1 point per move but Breaker takes several, say, b (≥ 2) points per move? For example, let (say) $b=100$. Can Maker still build a congruent copy of any given finite point set S in the plane? The answer is "yes"; all what we need to do is to replace the "fair" (1:1) type building criterion Theorem 1.2 with the following "biased" $(p:q)$ version (see Beck [1982]):

Theorem 2.2 ("biased building") *If*

$$\sum_{A\in\mathcal{F}}\left(\frac{p+q}{p}\right)^{-|A|} > p^2\cdot q^2\cdot(p+q)^{-3}\cdot\Delta_2(\mathcal{F})\cdot|V(\mathcal{F})|,$$

where $\Delta_2(\mathcal{F})$ is the Max Pair-Degree of hypergraph $\mathcal{F}$, and $V(\mathcal{F})$ is the board, then the first player can occupy a whole winning set $A\in\mathcal{F}$ in the biased $(p:q)$ play on $\mathcal{F}$ (the first player takes p new points and the second player takes q new points per move).

Applying Theorem 2.2 instead of Theorem 1.2 we immediately obtain the following "biased" version of Theorem 2.1.

Theorem 2.3 *Let S be an arbitrary finite set of points in the Euclidean plane, let $b\geq 1$ be an arbitrary integer, and consider the $(1:b)$ version of the S-building game where Maker is the underdog: Maker and Breaker alternately pick new points in the plane, Maker picks one point per move, Breaker picks $b\geq 1$ point(s) per move; Maker's goal is to build a congruent copy of S in a finite number of moves, and Breaker's goal is to stop Maker. For every finite S and $b\geq 1$, Maker can build a congruent copy of S in less than*

$$(b+1)^{(b+1)^{|S|^2}} \quad \text{moves if } |S|\geq 10.$$

Not surprisingly, the proof of the "biased criterion" Theorem 2.2 is very similar to that of the "fair" Theorem 1.2. Assume we are in the middle of a $(p:q)$ play, Maker (the first player) already occupies

$$X(i)=\{x_1^{(1)},\ldots,x_1^{(p)},x_2^{(1)},\ldots,x_2^{(p)},\ldots,x_i^{(1)},\ldots,x_i^{(p)}\}$$

and Breaker (the second player) occupies

$$Y(i)=\{y_1^{(1)},\ldots,y_1^{(q)},y_2^{(1)},\ldots,y_2^{(q)},\ldots,y_i^{(1)},\ldots,y_i^{(q)}\};$$

at this stage of the play the "weight" $w_i(A)$ of an $A\in\mathcal{F}$ is either 0 or an integral power of $\frac{p+q}{p}$. More precisely, either (1) $w_i(A)=0$ if $A\cap Y(i)\neq\emptyset$ (meaning that $A\in\mathcal{F}$ is a "dead set"), or (2)

$$w_i(A)=\left(\frac{p+q}{p}\right)^{|A\cap X(i)|} \quad \text{if} \quad A\cap Y(i)=\emptyset$$

(meaning that $A\in\mathcal{F}$ is a "survivor"). Maker evaluates his chances to win by using the following "opportunity function"

$$T_i=\sum_{A\in\mathcal{F}}w_i(A). \tag{2.23}$$

We stop here and challenge the reader to complete the proof of Theorem 2.2. Just in case, we include the whole proof at the end of Section 20.

It is very important to see what motivates "opportunity function" (2.23). The motivation is "probabilistic," and goes as follows. Assume we are right after the ith turn of the play with sets $X(i)$ and $Y(i)$ defined above, and consider a Random 2-Coloring of the unoccupied points of board V with odds $(p:q)$, meaning that, we color a point red (Maker's color) with probability $\frac{p}{p+q}$ and color a point blue (Breaker's color) with probability $\frac{q}{p+q}$, the points are colored independently of each other. Then the renormalized "opportunity function" (see (2.23))

$$\left(\frac{p}{p+q}\right)^n T_i = \left(\frac{p}{p+q}\right)^n \sum_{A\in\mathcal{F}} w_i(A)$$

is exactly the *expected number of completely red winning sets* (red means: Maker's own). In other words, a probabilistic argument motivates the choice of our potential function! We refer to this proof technique as a "fake probabilistic method." We will return to the probabilistic motivation later in great detail.

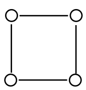
Theorem 2.2 is the biased version of Theorem 1.2; the biased version of Theorem 1.4 will be discussed, with several interesting applications, in Section 20 (see Theorem 20.1); a biased version of Theorem 1.3 will be applied in Section 15 (see Lemma 2 there).

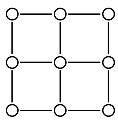
3

Examples: Tic-Tac-Toe games

1. Weak Winners and Winners. The game that we have been studying in Sections 1–2 (the "S-building game in the plane," where S is a given finite point set) was a Maker–Breaker game. One player – called Maker – wanted to build a goal set (namely, a congruent copy of S), and the other player – called Breaker – simply wanted to stop Maker. Tic-Tac-Toe and its variants are very different: they are *not* Maker–Breaker games, they are games where both players want to build, and the player declared the winner is the player who occupies a whole goal set *first*. The main question is "Who does it first?" instead of "Can Maker do it or not?".

The awful truth is that we know almost nothing about "Who does it first?" games. For example, consider the "Who does it first?" version of the S-building game in Section 1 – we restrict ourselves to the fair (1:1) version and assume that Maker is the first player. We know that Maker can always build a congruent copy of the 2×2 goal set S_4



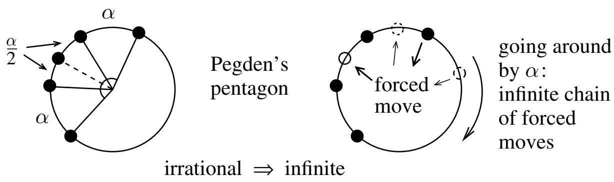


in ≤ 6 moves. What is more, the same case study shows that Maker can always build a congruent copy of S_4 *first* (again in ≤ 6 moves). But how about the 3×3 goal set S_9 in Example 3? Can Maker build S_9 *first*? If "yes," how long does it take to build S_9 *first*?

If Maker can do it, but not necessarily first, we call it a *Weak Winner;* if Maker can do it first, we call it a *Winner*.

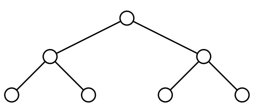
So the previous question goes as follows: "Is S_9 a Winner?" Is the regular pentagon a Winner? Is the regular hexagon a Winner? we don't know!

The message of Section 1 is that "every finite point set in the plane is a Weak Winner" – explaining the title of the section. Is it true that every finite point set in the plane is a Winner? The answer is "No." For example, the following irregular pentagon is *not* a Winner; the key property is that angle α is an irrational multiple of π

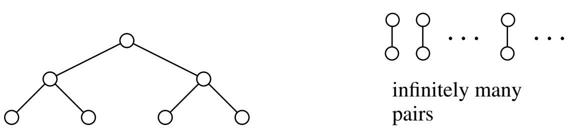


This example is due to Wesley Pegden (Ph.D. student at Rutgers). Pegden's pentagon consists of 4 consecutive points with common gap α plus the point in the middle. The idea of the proof is that the second player can always threaten the first player with an *infinite* sequence of forced moves, namely with irrational rotations of angle α along a fixed circle, before the first player can complete his own copy of the goal pentagon. More precisely, at an early stage of the play, the second player can achieve that on some circle he owns 3 points from some consecutive 4 with gap α, and he can take the 4th point in his next move. Then of course the first player is forced to take the middle point. After that the second player takes the 5th point with gap α, which again forces the first player to take the middle point of the last four; after that the second player takes the 6th point with gap α, which again forces the first player to take the middle point of the last four; and so on. We challenge the reader to clarify this intuition, and to give a precise proof that Pegden's irrational pentagon is not a Winner. The proof requires some case study(!) which we skip here, see Pegden [2005].

The underlying idea of Pegden's irrational pentagon construction is illustrated on the following oversimplified "abstract" hypergraph game. Consider a binary tree of 3 levels; the players take vertices



the "winning sets" are the 4 full-length branches (3-sets) of the binary tree. This is a simple first player win game; however, adding infinitely many disjoint 2-element "extra" winning sets to the hypergraph enables the second player to postpone his inevitable loss by infinitely many moves!



In other words, by adding the extra 2-sets, the first player cannot force a finite win anymore.

Pegden's clever construction is one example (a pentagon); are there infinitely many examples of "Weak Winner≠Winner" with arbitrarily large size (number of points)? In general, we can ask:

Open Problem 3.1 *Is there a finite procedure to decide whether or not a given finite point set S in the plane is a Winner? In other words, is there a way to characterize those finite point sets S in the plane for which Maker, as the first player, can always build a congruent copy of S in the plane* **first** *(i.e. before the opponent could complete his own copy of S)?*

We think Open Problem 3.1 is totally hopeless even for medium size point sets S, especially that "building is exponentially slow." What we are referring to here is Theorem 1.3, which gives the exponential lower bound $\geq \frac{1}{n}2^{n/2}$ for the Move Number of an arbitrary S with $|S| = n$.

2. Tic-Tac-Toe games on the plane. The S-building game was an artificial example, constructed mainly for illustration purposes. It is time now to talk about a "real" game: Tic-Tac-Toe and its closest variants. We begin with Tic-Tac-Toe itself, arguably the simplest, oldest, and most popular board game in the world.

Why is Tic-Tac-Toe so popular? Well, there are many reasons, but we think the best explanation is that it is the simplest example to demonstrate the difference between "Weak Win" and ordinary "win"! This will be explained below.

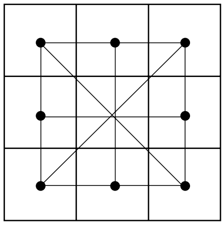
Let us emphasize again: we can say very little about ordinary win in general, nothing other than exhaustive search; the main subject of the book is "Weak Win."

Now let's return to Tic-Tac-Toe. The rules of Tic-Tac-Toe (called Noughts-and-Crosses in the UK) are well-known; we quote Dudeney: "Every child knows how to play this game; whichever player first gets three in a line, wins with the exulting cry:

Tit, tat, toe, My last go;
Three jolly butcher boys All in a row."

The **Tic-Tac-Toe** board is a big square which is partitioned into $3 \times 3 = 9$ congruent small squares. The first player starts by putting an X in one of the 9 small squares, the second player puts an O into any other small square, and then they alternate X and O in the remaining empty squares until one player wins by getting three of his own squares in a line (horizontally, vertically, or diagonally). If neither player gets three in a line, the play ends in a draw. Figure 3.1 below shows the board and the eight winning triplets of Tic-Tac-Toe.

Figure 3.2 below shows a "typical" play: $X_1, \ldots, X_5$ denote the moves of the first player, and $O_1, \ldots, O_4$ denote the moves of the second player in this order. This particular play ends in a draw.



3^2

Figure 3.1

O_3	X_2	O_1
X_4	X_1	O_4
X_5	O_2	X_3

Figure 3.2

X_2	O_1	
X_3	X_1	?
?		O_2

Figure 3.3

Figure 3.3 above shows another play in which the second player's opening move O_1 was a "mistake": the first player gets a "winning trap" and wins in his 4th move.

Every child "knows" that Tic-Tac-Toe is a *draw game*, i.e. either player can force a draw. We mathematicians have higher standards: we want a *proof*. Unfortunately, there is nothing to be proud of about the proof that we are going to give below. The proof is an ugly case study (anyone knows an elegant proof?).

The first half of the statement is easy though: the first player can always force a draw by an easy pairing strategy. Indeed, the first player opens with the center, which blocks 4 winning lines, and the remaining 4 lines are easily blocked by the following pairing

$$\begin{bmatrix} | & - & - \\ | & X & | \\ - & - & | \end{bmatrix}$$

This means the first player can always put his mark in every winning set, no matter what the opponent is doing. By contrast, the second player *cannot* put his mark in every winning set. In other words, the second player cannot prevent the first player from occupying a winning set. The reader is probably wondering, "Wait a minute, this seems to contradict the fact that Tic-Tac-Toe is a draw!" But, of course, there is no real contradiction here: the first player can occupy a whole winning set, but he cannot occupy it first if the opponent plays rationally. Indeed, let the first player's opening move be the center

Case 1

O_1		
	X_1	

Case 2

	O_1	
	X_1	

Then the second player has two options: either he takes a corner, or a side. In either case the first player occupies X_2, X_3, X_4, and completes a winning triplet. Of course, this way the opponent's winning triplet (O_1, O_2, O_3 if the second player plays rationally) comes first; notice that here we *changed the rule* and assumed that the players do *not* quit playing even after some winning set is completely occupied by either player, they just keep playing till the whole board is completed. We refer to this as the Full Play Convention.

Case 1

O_1		
	X_1	
X_2	X_3	X_4

Case 2

	O_1	
	X_1	
X_2	X_4	X_3

Occupying a whole winning set, but not necessarily first, is what we call a *Weak Win*. We have just learned that in Tic-Tac-Toe the first player can achieve a Weak Win (assuming the Full Play Convention!).

The complement of Weak Win is called a *Strong Draw*. Tic-Tac-Toe is a draw game (we prove this fact below) but not a Strong Draw. We sometimes refer to this property – draw but not a Strong Draw – as a "Delicate Draw."

Tic-Tac-Toe is a "3-in-a-row" game on a 3×3 board. A straightforward 2-dimensional generalization is the "n-in-a-row" game on an $n \times n$ board; we call it the $n \times n$ Tic-Tac-Toe, or simply the n^2 game. The n^2 game has $2n+2$ winning sets: n horizontals, n verticals, and 2 diagonals, each one of size n. The n^2 games are rather dull: the 2^2 game is a trivial first player win, and the rest of the n^2 games ($n \geq 3$) are all simple draw games – see Theorems 3.1–3.3 below. We begin with:

Theorem 3.1 *Ordinary* 3^2 *Tic-Tac-Toe is a draw but not a Strong Draw.*

The only **proof** that we know is a not too long but still unpleasant(!) case study. However, it seems ridiculous to write a whole book about games such as Tic-Tac-Toe, and not to solve Tic-Tac-Toe itself. To emphasize: by including this case study an exception was made; the book is *not* about case studies.

We have already proved that the first player can force a draw in the 3^2 game; in fact, by a pairing strategy. It remains to give the second player's drawing strategy. Of course, we are not interested in how poorly the second player can play; all that we care about is the second player's optimal play. Therefore, when we describe a second player's drawing strategy, a substantial reduction in the size of the case study comes from the following two assumptions:

(i) each player completes a winning triplet if he/she can;
(ii) each player prevents the opponent from doing so in his/her next move.

It is either player's best interest to follow rules (i) and (ii). A second source of reduction comes from using the *symmetries* of the board. We label the 9 little squares in the following natural way

1	2	3
4	5	6
7	8	9

The center ("5") is the "strongest point": it is the only cell contained by 4 winning lines. The second player's drawing strategy has 3 parts according, as the opponent's opening move is in the center (threatening four winning triplets), or in a corner (threatening 3 winning triplets), or on a side (threatening two winning triplets). If the first player starts in the center, then the second player's best response is in the corner, say in 1. The second player can force a draw in the $5 \to 1$ end game. This part of second player's drawing strategy is the following

$5 \to 1$
- $2 \to 8 \to 3 \to 7$ (winning trap: the second player wins)
  - $8 \to 4 \to 6$ (Draw: next the second player can block the 3, 5, 7-line)
  - $8 \to 6 \to 4 \to 7 \to 3 \to 9$ (Draw)
  - $8 \to 7 \to 3$ (Draw: next the second player can block the 4, 5, 6-line)
  - $8 \to 9 \to 4 \to 7 \to 3 \to 6$ (Draw)
- $3 \to 7 \to 4 \to 6$ (Draw: next the second player can block the 2, 5, 8-line)
- $6 \to 4 \to 7 \to 3 \to 2 \to 8 \to 9$ (Draw)
- $9 \to 3 \to 2 \to 8$ (Draw: next the second player can block the 4, 5, 6-line)
- 4: same as $5 \to 1 \to 2$
- 7: same as $5 \to 1 \to 3$
- 8: same as $5 \to 1 \to 6$

If the first player does not start in the center, then, not surprisingly, the second player's best response is in the center. If the first player starts in the corner, say in 1, then the second player's drawing strategy is the following

$1 \to 5$

- $2 \to 3 \to 7 \to 4 \to 6$ (Draw)
- $3 \to 2 \to 8 \to 4 \to 6 \to 9 \to 7$ (Draw)
- $6 \to 2 \to 8 \to 7 \to 3 \to 9 \to 4$ (Draw)
- $9 \to 2 \to 8 \to 7 \to 3 \to 6 \to 4$ (Draw)
- 4: $\to$ same as $1 \to 5 \to 2$
- 7: $\to$ same as $1 \to 5 \to 3$
- 8: $\to$ same as $1 \to 5 \to 6$

Finally, if the first player starts on the side, say in 2, then the second player's drawing strategy is the following

$2 \to 5$

- $4 \to 1 \to 9 \to 3 \to 7$ (Draw)
- $7 \to 4 \to 6 \to 9 \to 1 \to 3 \to 8$ (Draw)
- $8 \to 1 \to 9 \to 7 \to$ (winning trap: second player wins)
- 1: $\to$ same as $1 \to 5 \to 2$
- 3: $\to$ same as $1 \to 5 \to 2$
- 6: $\to$ same as $1 \to 5 \to 4$
- 9: $\to$ same as $1 \to 5 \to 7$

This completes the case study, and Theorem 3.1 follows. □

The 4^2 Tic-Tac-Toe is a "less interesting" draw: neither player can force a Weak Win (assuming the Full Play Convention). First we show that the first player can put his mark in every winning set (4-in-a-row) of the 4^2 board, i.e. he can force a Strong Draw. In fact, this Strong Draw is a Pairing Strategy Draw: first player's opening move is in the middle

$$\begin{bmatrix} * & - & | & - \\ | & X_1 & / & | \\ | & / & - & - \\ - & - & | & | \end{bmatrix}$$

X_1 in the middle blocks 3 winning lines, 7 winning lines remain unblocked, and we have $4^2 - 1 = 15$ cells to block them. An explicit pairing strategy is given on the picture above (the first player doesn't even need the asterisk-marked cell in the upper-left corner).

Next we show how the second player can put his mark in every winning set ("Strong Draw"). This case is more difficult since the second player *cannot* have a Pairing Strategy Draw. Indeed, the cell/line ratio is *less* than 2 (there are $4^2 = 16$

cells and $4+4+2=10$ winning lines), so it is impossible that every winning line owns a “private pair of cells.”

Nevertheless the second player can block by using a combination of 3 different pairing strategies! Indeed, apart from symmetry there are three possible opening moves of the first player, marked by X. The second player’s reply is 0, and for the rest of the play the second player employs the pairing strategy direction marked in the corresponding picture (for every move of the first player in a marked square the second player takes the similarly marked square in the direction indicated by the mark).

$$\begin{bmatrix} X & - & | & - \\ | & O & / & | \\ | & / & - & - \\ - & - & | & | \end{bmatrix} \qquad \begin{bmatrix} | & X & - & - \\ | & O & / & | \\ - & / & | & - \\ - & - & | & | \end{bmatrix} \qquad \begin{bmatrix} \backslash & | & - & - \\ | & X & O & | \\ - & - & \backslash & | \\ | & | & - & - \end{bmatrix}$$

This elegant argument is due to *David Galvin*.

This shows that the 4^2 game comes very close to having a Pairing Strategy Draw: *after* the first player’s opening move the second player can always employ a draw-forcing pairing strategy.

Theorem 3.2 *The* 4^2*-game is a Strong Draw, but not a Pairing Strategy Draw (because the second player cannot force a draw by a single pairing strategy).*

The n^2 Tic-Tac-Toe with $n \geq 5$ is particularly simple: either player can force a draw by a pairing strategy; in fact, both players may use the same pairing.

Case $n=5$

In the 5^2-game either player can force a draw by employing the following “pairing”

$$\begin{bmatrix} 11 & 1 & 8 & 1 & 12 \\ 6 & 2 & 2 & 9 & 10 \\ 3 & 7 & * & 9 & 3 \\ 6 & 7 & 4 & 4 & 10 \\ 12 & 5 & 8 & 5 & 11 \end{bmatrix}$$

Note that every winning line has its own *pair* (“private pair”): the first row has two 1s, the second row has two 2s, …, the fifth row has two 5s, the first column has two 6s, …, the fifth column has two 10s, the diagonal of slope -1 has two 11s, and finally, the other diagonal of slope 1 has two 12s. Either player can occupy at least 1 point from each winning line: indeed, whenever the first (second) player occupies a numbered cell, the opponent takes the other cell of the same number. (In the first player’s strategy, the opening move is the asterisk-marked center; in the second player’s strategy, if the first player takes the center, then the second player may take any other cell.)

By using this *pairing strategy* either player can block every winning set, implying that 5^2 is a draw game.

A more suggestive way to indicate the same pairing strategy for the 5^2-game is shown below: all we have to do is to make sure that for every move of the opponent in a marked square we take the similarly marked square in the direction indicated by the mark.

$$\begin{bmatrix} \backslash & - & | & - & / \\ | & - & - & | & | \\ - & | & * & | & - \\ | & | & - & - & | \\ / & - & | & - & \backslash \end{bmatrix}$$

Case $n = 6$

The 6^2-game is another Pairing Strategy Draw. An explicit pairing goes as follows

$$\begin{bmatrix} 13 & 1 & 9 & 10 & 1 & 14 \\ 7 & * & 2 & 2 & * & 12 \\ 3 & 8 & * & * & 11 & 3 \\ 4 & 8 & * & * & 11 & 4 \\ 7 & * & 5 & 5 & * & 12 \\ 14 & 6 & 9 & 10 & 6 & 13 \end{bmatrix}$$

Whenever the first (second) player occupies a numbered cell, the opponent takes the other cell of the same number. We do not even need the eight asterisk-marked cells in the 2 diagonals.

Case $n \geq 7$

By using the special cases $n = 5$ and 6, the reader can easily solve all n^2-games with $n \geq 7$.

Exercise 3.1 *Let $n \geq 5$ be an integer. Show that if the n^2-game has a Pairing Strategy Draw, then the $(n+2)^2$-game also has a Pairing Strategy Draw.*

Theorem 3.3 *The $n \times n$ Tic-Tac-Toe is a Pairing Strategy Draw for every $n \geq 5$.*

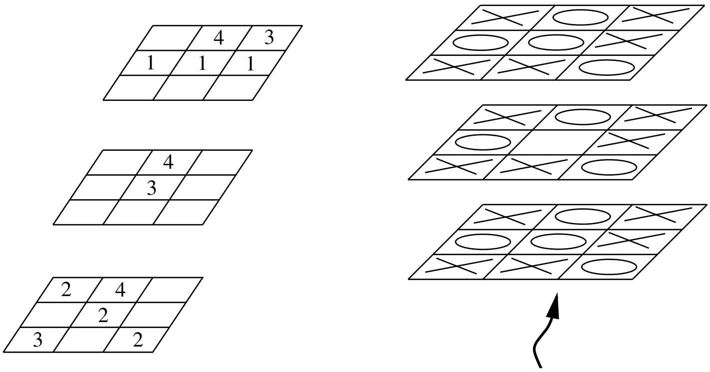
Ordinary 3^2 Tic-Tac-Toe turns out to be the most interesting member of the (dull) family of n^2-games. This is where the "phase transition" happens: 2^2 is a trivial first player win (the first player always wins in his second move), 3^2 is a draw, and the rest – the n^2-games with $n \geq 4$ – are all drawn, too.

This gives a complete understanding of the 2-dimensional $n \times n$ Tic-Tac-Toe.

3. Tic-Tac-Toe in higher dimensions. Unfortunately, we know much less about the 3-dimensional $n \times n \times n = n^3$ Tic-Tac-Toe. The $2 \times 2 \times 2 = 2^3$-game (just like the 2^2-game) is a trivial win. Every play has the same outcome: first player win.

The 3^3 Tic-Tac-Toe is a less trivial but still easy win. Every play has only two possible outcomes: either a (1) first player win, or a (2) second player win. That is, no draw play is possible. Thus the first player can force a win – this follows from a well-known general argument called "Strategy Stealing." We return to "Strategy Stealing" later in Section 5 (We recommend the reader to finish the proof him/herself).

In the 3^3-game there are 49 winning lines (3-in-a-row): 4 *space-diagonals* (joining opposite corners of the cube), 18 *plane-diagonals* (in fact 12 of them are on the 6 faces joining opposite corners of some face, and 6 more plane-diagonals inside the cube), and 27 axis-parallel lines (parallel with one of the 3 coordinate axes). We illustrate 4 particular winning lines in the left-hand side of the figure below.



Exercise 3.2 *Find an explicit first player winning strategy in the* 3^3*-game.*

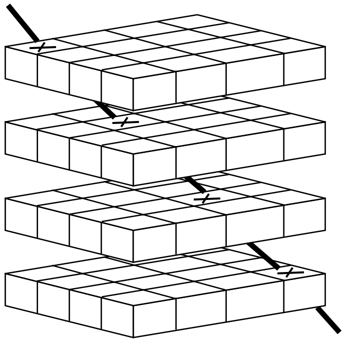
Exercise 3.3

(a) *Show that no draw is possible in the* 3^3*-game: every play must have a winner.*
(b) *Show that every 2-coloring of the cells in the* 3^3*-board yields a monochromatic winning line.*

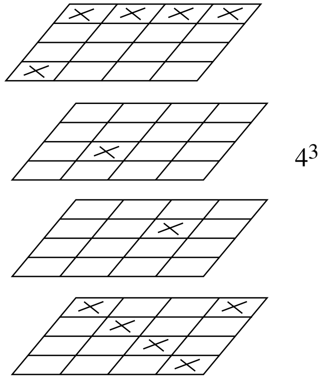
Exercise 3.4 *Consider the Reverse version of the* 3^3*-game: the player who gets 3-in-a-row first is the loser. Which player has a winning strategy?*

The right-hand side of the figure above demonstrates an interesting property of the 3^3-game. We already know that Drawing Position cannot exist (see Exercise 3.3 (b)), *but* if we remove the center and the 13 winning lines going through the center, then the "truncated" 3^3-hypergraph of $3^3 - 1 = 26$ points and $49 - 13 = 36$ winning lines has a *Proper 2-coloring*, i.e. there is no monochromatic winning line.

It is interesting to note that the sizes of the two color classes differ by 2, so this is *not* a Drawing Terminal Position. In fact, a Drawing Terminal Position (i.e. Proper *Halving* 2-Coloring) cannot even exist!



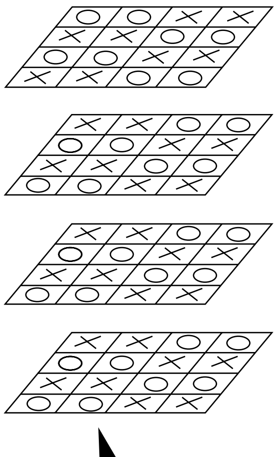
Qubic $= 4^3$ Tic-Tac-Toe



Three typical winning sets in $4 \times 4 \times 4$ game

The first variant of Tic-Tac-Toe, which is truly exciting to play, is the 3-dimensional $4 \times 4 \times 4 = 4^3$ game: it was marketed by Parker Brothers as *Qubic*; henceforth we often refer to it as such. A remarkable property of Qubic is that it has a Drawing Terminal Position, but the first player can nevertheless force a win.

In Qubic there are 4 *space-diagonals* (joining opposite corners of the cube), 24 *plane-diagonals* (in fact 12 of them are on the 6 faces joining opposite corners of some face, and 12 more plane-diagonals inside the cube), and 48 axis-parallel lines (parallel with one of the 3 coordinate axes); altogether 76 4-in-a-row. As said before, Qubic has a Drawing Terminal Position: we can put 32 Xs and 32 Os on the board such that every 4-in-a-row contains both marks – see the figure below.



Proper Halving 2-Coloring
of 4^3 Tic-Tac-Toe

Qubic is a first player win just like the 3^3 game, but there is a big difference: the winning strategy in Qubic is extremely complicated! The first explicit winning strategy was found by Oren Patashnik in 1977, and it was a celebrated victory for Computer Science (and Artificial Intelligence). The solution involved a most intricate human–computer interaction; for the details we refer the reader to Patashnik's fascinating survey paper [1980]. Patashnik's solution employes hundreds of long sequences of *forced moves*. A sequence of *forced moves* means that the second player must continually block first player's 3-in-a-line until at some move the first player has a *winning trap*: the second player must simultaneously block two such 3-in-a-line, this is clearly impossible, so the first player wins. Patashnik's solution contains a "dictionary" of 2 929 "strategic moves." The first player forces a win as follows:

(1) if he can make 4-in-a-row with this move, he does it;
(2) he blocks the opponent's 3-in-a-line if he must;
(3) he looks for a sequence of forced moves, and employs it if he finds one;
(4) otherwise he consults Patashnik's dictionary.

After this brief discussion of the 4^3 game ("Qubic"), we switch to the general case, called *hypercube Tic-Tac-Toe*, which is formally defined as follows.

n^d **hypercube Tic-Tac-Toe** or simply the n^d **game**. The board V of the n^d game is the d-dimensional hypercube of size $n \times \cdots \times n = n^d$, that is, the set of d-tuples

$$V = \left\{ \mathbf{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d : \ 1 \le a_j \le n \text{ for each } 1 \le j \le d \right\}.$$

The winning sets of the n^d-game are the n-in-a-line sets, i.e. the n-element sequences

$$\left(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)} \right)$$

of the board V such that, for each j, the sequence $a_j^{(1)}, a_j^{(2)}, \ldots, a_j^{(n)}$ composed of the jth coordinates is either $1, 2, 3, \ldots, n$ ("increasing"), or $n, n-1, n-2, \ldots, 1$ ("decreasing"), or a *constant*. The two players alternately put their marks (X and O) in the previously unmarked cells (i.e. unit cubes) of the d-dimensional solid hypercube n^d of side n. Each player marks one cell per move. The winner is the player to occupy a whole winning set *first*, i.e. to have n of his marks in an n-in-a-line *first*. In other words, the winning sets are exactly the n-in-a-line in the n^d hypercube; here, of course, each elementary "cell" is identified with its own center. If neither player gets n-in-a-line, the play is a draw. The special case $n = 3, d = 2$ gives ordinary Tic-Tac-Toe. Note that in higher dimensions most of the n-in-a-line are some kind of diagonal.

The winning sets in the n^d game are "lines," or "winning lines." The number of winning lines in the 3^2 and 4^3 games are 8 and 76. In the general case we have an elegant short formula for the number of "winning lines," see Theorem 3.4 (a) below. In the rest of the book we often call the cells "points" (identifying a cell with its own center).

Theorem 3.4

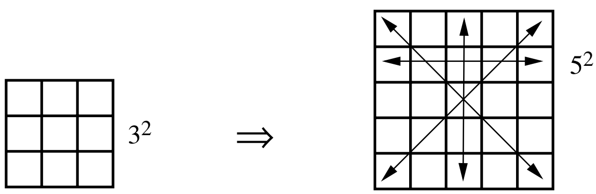
(a) The total number of winning lines in the n^d-game is $\big((n+2)^d - n^d\big)/2$.
(b) If n is odd, there are at most $(3^d - 1)/2$ winning lines through any point, and this is attained only at the center of the board. In other words, the maximum degree of the n^d-hypergraph is $(3^d - 1)/2$.
(c) If n is even ("when the board does not have a center"), the maximum degree drops to $2^d - 1$, and equality occurs if there is a common $c \in \{1, \ldots, n\}$ such that every coordinate c_j equals either c or $n+1-c$ ($j = 1, 2, \ldots, d$).

Proof. To prove (a) note that for each $j \in \{1, 2, \ldots, d\}$, the sequence $a_j^{(1)}, a_j^{(2)}, \ldots, a_j^{(n)}$ composed of the jth coordinates of the points on a winning line is either strictly *increasing* from 1 to n, or strictly *decreasing* from n to 1, or a *constant* $c = c_j \in \{1, 2, \ldots, n\}$. Since for each coordinate we have $(n+2)$ possibilities $\{1, 2, \ldots, n, \textit{increasing}, \textit{decreasing}\}$, this gives $(n+2)^d$, but we have to subtract n^d because at least one coordinate must change. Finally, we have to divide by 2, since every line has two orientations.

An alternative geometric/intuitive way of getting the formula $\big((n+2)^d - n^d\big)/2$ goes as follows. Imagine the board n^d is surrounded by an additional layer of cells, one cell thick. This new object is a cube

$$(n+2) \times (n+2) \times \cdots \times (n+2) = (n+2)^d.$$

It is easy to see that every winning line of the n^d-board extends to a uniquely determined *pair* of cells in the new surface layer. So the total number of lines is $((n+2)^d - n^d)/2$.



Why does ordinary Tic-Tac-Toe have 8 winning triplets?

$$8 = \frac{5^2 - 3^2}{2}$$

Next we prove (b): let n be odd. Given a point $\mathbf{c} = (c_1, c_2, \ldots, c_d) \in n^d$, for each $j \in \{1, 2, \ldots, d\}$ there are three options: the jth coordinates of the points on an *oriented* line containing $\mathbf{c}$:

(1) either increase from 1 to n,
(2) or decrease from n to 1,
(3) or remain constant c_j.

Since every line has two orientations, and it is impossible that all coordinates remain constant, the maximum degree is $\leq (3^d - 1)/2$, and we have equality for the center (only).

This suggests that the center of the board is probably the best opening move (n is odd).

Finally, assume that n is even. Let $\mathbf{c} = (c_1, c_2, \ldots, c_d) \in n^d$ be a point, and consider the family of those n-in-a-line which contain $\mathbf{c}$. Fixing a proper subset index-set $I \subset \{1, 2, \ldots, d\}$, there is at most *one* n-in-a-line in this family for which the jth coordinates of the points on the line remain constant c_j for each $j \in I$, and increase or decrease for each $j \notin I$. So the maximum degree is $\leq \sum_{i=0}^{d-1} \binom{d}{i} = 2^d - 1$, and equality occurs if for some fixed $c \in \{1, \ldots, n\}$ every coordinate c_j equals c or $n + 1 - c$ $(j = 1, 2, \ldots, d)$. □

4. Where is the phase transition? We know that the n^2 games are rather dull (with the possible exception of ordinary 3^2 Tic-Tac-Toe itself); the 3^3 game is too easy, the 4^3 game is very interesting and difficult, but it is completely solved; how about the next one, the 5^3 game? Is it true that 5^3 is a draw game? How about the 5^4 game? Is it true that 5^4 is a first player win? Unfortunately these are hopeless questions.

Open Problem 3.2 *Is it true that 5^3 Tic-Tac-Toe is a draw game? Is it true that 5^4 Tic-Tac-Toe is a first player win?*

Very little is known about the n^d games with $d \geq 3$, especially about winning. We know that the first player can achieve a 4-in-a-row first in the 3-space (4^3 Tic-Tac-Toe); how about achieving a 5-in-a-row? In other words, the first player wants a winning strategy in some 5^d Tic-Tac-Toe. Let d_0 denote the smallest dimension d when the first player has a forced win in the 5^d game; how small is d_0? (A famous result in Ramsey Theory, called the Hales–Jewett Theorem, see Section 7, guarantees that d_0 is finite.) The second question in Open Problem 3.2 suggests that $d_0 = 4$, but what can we actually prove? Can we prove that $d_0 \leq 1000$? No, we cannot. Can we prove that $d_0 \leq 1000^{1000}$? No, we cannot. Can we prove that $d_0 \leq 1000^{1000^{1000}}$? No, we cannot prove that either. Even if we iterate this 1000 times, we still cannot prove that this "1000-tower" is an upper bound on d_0. Unfortunately, the best-known upper bound on d_0 is embarrassingly poor. For more about d_0, see Section 7.

Another major problem is the following. We know an explicit dimension d_0 such that in the 5^{d_0} Tic-Tac-Toe the first player has a winning strategy: (1) it is bad enough that the smallest d_0 we know is enormous, but (2) it is even worse that the proof does not give the slightest hint how the winning strategy actually looks (!), see Theorems 5.1 and 6.1 later.

Next we mention two conjectures about hypercube Tic-Tac-Toe (published in Patashnik [1980]), which represent a very interesting but failed(!) attempt to describe the "phase transition" from draw to win in simple terms. The first one, called "modification of Gammill's conjecture" by Patashnik [1980], predicted that:

Conjecture A ("Gammill") *The n^d game is a draw if and only if there are more points than winning lines.*

For example, the 3^2 and 4^3 games both support this conjecture. Indeed, the 3^2 game is a draw and number-of-lines $= 8 < 9 =$ number-of-points; on the other hand, the 4^3 game is a first player win and number-of-lines $= 76 > 64 =$ number-of-points.

In the 5^3 game, which is believed to be a draw, there are $(7^3 - 5^3)/2 = 109$ lines and $5^3 = 125$ points. On the other hand, in the 5^4 game, which is believed to be a first player win, there are $(7^4 - 5^4)/2 = 938$ lines and $5^4 = 625$ points.

A modification of Citrenbaum's conjecture (see Patashnik [1980]) predicted that:

Conjecture B ("Citrenbaum") *If $d > n$, then the first player has a winning strategy in the n^d game.*

Of course, we have to be very critical about conjectures like these two: it is difficult to make any reasonable prediction based on such a small number of solved cases. And indeed, both Conjectures A and B turned out to be false; in Section 34, we prove that both have infinitely many counterexamples.

Unfortunately, our method doesn't work in lower dimensions: an explicit relatively low-dimensional counter-example to Conjecture A that we could come up with is the 144^{80}-game (it has more lines than points), and an explicit counter-example to Conjecture B is the 214^{215}-game which is a draw. These are pretty large dimensions; we have no idea what's going on in low dimensions.

The failure of the at-first-sight-reasonable Conjectures A and B illustrates the difficulty of coming up with a "simple" conjecture about the "phase transition" from draw to win for hypercube Tic-Tac-Toe games. We don't feel confident enough to formulate a conjecture ourselves. We challenge the reader to come up with something that makes sense. Of course, to formulate a conjecture is one thing (usually the "easy part"), and to prove it is a totally different thing (the "hard part").

Before discussing more games, let me stop here, and use the opportunity to emphasize the traditional *viewpoint* of Game Theory. First of all, we assume that the reader is familiar with the concept of **strategy** (the basic concept of Game Theory!). Of course, strategy is such a natural/intuitive "common sense" notion that we are tempted to skip the formal definition; but just in case, if there is any doubt, the reader can always consult Appendix C for a formal treatment. Now the traditional viewpoint: Game Theory is about **optimal strategies**, which is shortly expressed in the vague term: "the players play rationally." We certainly share the traditional viewpoint: we always assume that either player knows an optimal strategy, even if finding one requires "superhuman powers" such as performing a case study of size (say) $10^{1000!}$.

A pithy way to emphasize the traditional viewpoint is to name the two players after some *gods*. Let us take, for example, the most famous war in Greek Mythology: the *Trojan War* between the Greeks and Troy, see *The Iliad* of Homer. Motivated by the Trojan War we may call the first player *Xena* (or *Xenia*) and the second player *Apollo*. Of course, Xena uses mark X and Apollo uses mark O. Xena is an epithet of Pallas Athena, meaning "hospitable." Xena (alias Pallas Athena), goddess of wisdom, sided with the Greeks, and Apollo, god of arts and learning, sided with Troy.

It is most natural to expect a god/goddess to know his/her optimal strategy; carrying out a case study of size (say) $10^{1000!}$ shouldn't be a problem for them (but it is a BIG problem for us humans!).

The only reason why we don't follow the advice, and don't use a name-pair such as Xena/Apollo (or something similar), is to avoid the awkwardness of he/she, him/her, and his/her (the gender problem of the English language).

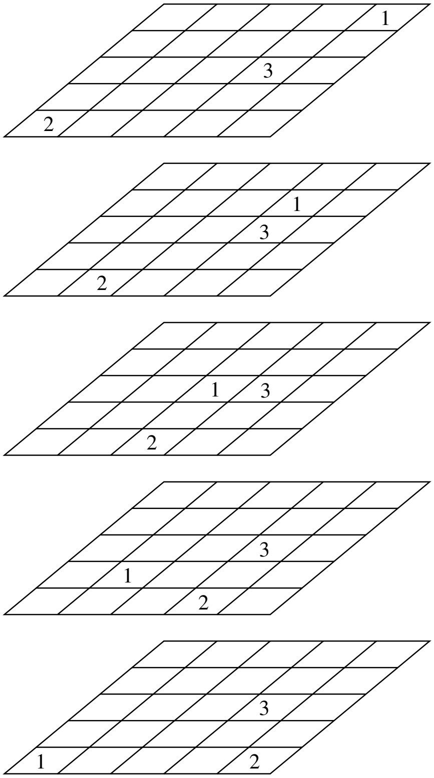
We conclude Section 3 with an entertaining observation, and a picture.

Consider the following number game: two players alternately select integers from 1 to 9 and no number may be used twice. A player wins by getting three numbers the sum of which is 15; the first to do that is declared to be the winner. Who wins?

Well, this is a mathematical joke! The number game is just Tic-Tac-Toe in disguise! Indeed, there are exactly 8 solutions (a, b, c) of the equation $a+b+c=15$, $1 \leq a < b < c \leq 9$, and the 8 solutions are represented by the 8 winning lines of Tic-Tac-Toe (3 horizontals, 3 verticals, and 2 diagonals). Therefore, this number game is a draw.

2	7	6
9	5	1
4	3	8

Finally, a figure illustrating the hopeless Open Problem 3.2.



A hopeless problem:

5^3 Tic-Tac-Toe

Is this a draw game?

$5^3 = 125$ points ("cells")

$6^3 - 5^3 = 91$ combinatorial lines like (1) and (3), and 18 geometric lines which are not combinatorial lines like (2)

(1) xxx $\quad x = 1, 2, 3, 4, 5$

(2) $x1x'$ $\quad x' = 6 - x$

(3) $43x$

4

More examples: Tic-Tac-Toe like games

Tic-Tac-Toe itself is a simple game, but some natural changes in the rules quickly lead to very difficult or even hopelessly difficult games. We have already mentioned the 3-dimensional $4 \times 4 \times 4$ version ("Qubic"), which was solved by a huge computer-assisted case study (it is a first player win). The next case, the $5 \times 5 \times 5$ version, is expected to be a draw, but there is no hope of proving it (brute force is intractable). A perhaps more promising direction is to go back to the plane, and study 2-dimensional variants of Tic-Tac-Toe. We will discuss several 2-dimensional variants: (1) unrestricted n-in-a-row, (2) Harary's Animal Tic-Tac-Toe, (3) Kaplansky's n-in-a-line, (4) Hex, and (5) Gale's Bridge-it game. They are all "who does it first" games.

1. Unrestricted n-in-a-row. The 5^2 Tic-Tac-Toe, that is, the "5-in-a-row on a 5×5 board" is a very easy draw game, but if the 5×5 board is extended to the whole plane, we get a very interesting and still unsolved game called *unrestricted* 5-in-a-row. *Unrestricted* means that the game is played on an *infinite* chessboard, infinite in every direction. In the *unrestricted* 5-in-a-row game the players alternately occupy little squares of an infinite chessboard; the first player marks his squares by X, and the second player marks his squares by O. The person who first gets 5 consecutive marks of his own in a row horizontally, or vertically, or diagonally (of slope 1 or -1) is the winner; if no one succeeds, the play ends in a draw. *Unrestricted* n-in-a-row differs in only one aspect: the winning size is n instead of 5.