Encyclopedia of Mathematics and Its Applications 124

HILBERT TRANSFORMS

Volume 1

Frederick W. King

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Hilbert Transforms

The Hilbert transform arises widely in a variety of applications, including problems in aerodynamics, condensed matter physics, optics, fluids, and engineering. This work, written in an easy-to-use style, is destined to become the definitive reference on the subject. It contains a thorough discussion of all the common Hilbert transforms, mathematical techniques for evaluating them, and a detailed discussion of their application. Especially valuable features are the tabulation of analytically evaluated Hilbert transforms, and an atlas that immediately illustrates how the Hilbert transform alters a function. These will provide useful and convenient resources for researchers.

A collection of exercises is provided for the reader to test comprehension of the material in each chapter. The bibliography is an extensive collection of references to both the classical mathematical papers, and to a diverse array of applications.

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Hilbert transforms

Volume 1

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521887625

First published 2009

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
King, Frederick W., 1947–
Hilbert transforms / Frederick W. King.
p. cm.
Includes bibliographical references and index.
ISBN 978-0-521-88762-5 (hardback)
1. Hilbert transform. I. Title.
QA432.K56 2008
515′.723–dc22 2008013534

ISBN 978-0-521-88762-5

To the memory of my mother

Contents

Volume II

Preface

My objective in this book is to present an elementary introduction to the theory of the Hilbert transform and a selection of applications where this transform is applied. The treatment is directed primarily at mathematically well prepared upper division undergraduates in physics and related sciences, as well as engineering, and first-year graduate students in these areas. Undergraduate students with a major in applied mathematics will find material of interest in this work.

I have attempted to make the treatment self-contained. To that end, I have collected a number of topics for review in Chapter 2. A reader with a good undergraduate mathematics background could possibly skip over much of this chapter. For others, it might serve as a highly condensed review of material used later in the text. The principal background mathematics assumed of the reader is a solid foundation in basic calculus, including introductory differential equations, a course in linear and abstract algebra, some exposure to operator theory basics, and an introductory knowledge of complex variables. Readers with a few deficiencies in these areas will find a number of recommendations for further reading at the end of Chapter 2. Some of the applications discussed require the reader to be familiar with basic electrodynamics.

A focus of the book is on problem solving rather than on proving theorems. Theorems are, for the most part, not stated or proved in the most general form possible. The end-notes will typically provide additional reference sources of more detailed discussions about the various theorems presented. I have not attempted to sketch the proof of every theorem stated, but for the key results connected to the Hilbert transform, at minimum an outline of the essential elements is usually presented. Consistent with the problem-solving emphasis is that all the different techniques that I know for evaluating Hilbert transforms are displayed in the book.

I take the opportunity to introduce special functions in a number of settings. I do this for two reasons. Special functions occur widely in problems of great importance in many areas of physics and engineering, and, accordingly, it is essential that students gain exposure to this important area of mathematics. Since many Hilbert transforms evaluate to special functions, it is imperative that the reader know when to stop doing algebraic manipulations. I have incorporated several mathematical topics for which few or no applications are known to the writer. The selection process was governed in part by the potential that I thought a particular area might have in problem solving, and I have done this with the full knowledge that crystal-ball reading is an art rather than a science!

The exercises are intended as a means for the reader to test his/her comprehension of the material in each chapter. The vast majority of the problems are by design routine applications of ideas discussed in the text. A small percentage of the problems are likely to be fairly challenging for an undergraduate reader, and a few problems could be labeled rather difficult. Most readers will have no trouble deciding when they have encountered an example of this latter group.

I have compiled an extensive table of Hilbert transforms of common mathematical functions. I hope this table will be useful in three ways. First, it serves as the answer key for a number of exercises that are placed throughout the text. Since many additional Hilbert transform pairs can be established by differentiation, or by appropriate multiplicative operations, etc., this table can be used to generate a great number of exercises, much to the delight of the reader. Second, I hope it will provide a useful reference source for those looking for the Hilbert transform of a particular function. Finally, for those searching for a particular Hilbert transform not present in the table, finding related transforms may give an idea on how to approach the evaluation, and give some clues as to whether a closed form expression in terms of standard functions is likely to be possible. In several sections the table includes a few specific cases followed by the general formula. This has been done to allow the reader to access the Hilbert transform of some of the simpler special cases as quickly as possible, rather than reducing a more complicated general formula.

The mini atlas of functions and the associated Hilbert transforms given in Appendix 2 is intended to provide a visual representation for a selection of Hilbert transform pairs. I hope this will be valuable for students in the applied sciences and engineering.

The reference list is rather extensive, but is not intended to be exhaustive. There are far too many published articles on Hilbert transforms to provide a complete set of references. I have attempted to give a generous number of references to applications. Many citations are given to the classical mathematical papers on the topics of the book, and for the serious student these works can be read with great profit. The Notes section at the end of each chapter gives a guide as to where to start reading for further information on topics discussed in the chapter. Elaborations and further details on the proofs of different theorems will often be located in the references cited in the end-notes.

My final task is to thank those who have helped. Logan Ausman, Dr. Matt Feldmann, Geir Helleloid, Dr. Kai-Erik Peiponen, Dr. Ignacio Porras, Dr. Jarkko Saarinen, and Corey Schuster read various chapters and made a number of useful suggestions to improve the presentation. Dr. Walter Reid and Dr. Jim Walker gave me some helpful comments on a preliminary draft of the first three chapters. Julia Boryskina and Hristina Ninova assisted with the translation of a number of technical papers. Several other students did translations and I offer a collective thanks to them.

Julia also provided assistance in the construction of the atlas of Hilbert transforms and with a number of the figures. Ali Elgindi did some numerical checking on the table of Hilbert transforms and Julia also did a few preliminary tests. Thanks are extended to Irene Pizzie for her efforts to improve the presentation.

The author would greatly appreciate if readers would bring to his attention any errors that escaped detection. The URL http://www.chem.uwec.edu/king/ is the web address where corrections will be posted. It is the author's intention to maintain this site actively.

Symbols

The first occurrence or a definition is indicated by a section reference or an equation number. HT is an abbreviation for the tables of Hilbert transforms given in the Appendixes (Table 1.1).

$\|a\|$	sum of the components of the multi-index a; §15.5
$\arg z$	argument of a complex number; Eq. (2.69)
A_p	the A_p condition for $1 \le p \le \infty$; Eq. (7.377), §7.12
$b\Omega$	boundary of a bounded domain Ω; §3.1
B	Boas transform operator; §16.4
B_n	generalization of the Boas transform operator; Eq. (16.84)
$\mathbf{B}(t)$	magnetic induction; §17.9
$\mathcal{B}$	generalization of the Boas transform operator; Eq. (16.80)
$B(a,b)$	beta function (Euler's integral of the first kind); Eq. (5.112), HT-01
$BV([a,b])$	class of functions that have bounded variation on the interval $[a,b]$; §4.25
C	designation for a contour (usually closed); §2.8.1.
C	a positive (often unspecified) constant; in derivations such a constant need not be the same at each occurrence, even though the same symbol is employed.
C	SI unit for charge, the coulomb, §19.1
$\mathbb{C}$	the set of complex numbers; §2.10
C_n	symmetry operation such that rotation by $2\pi/n$ leaves the system invariant; §21.3
C^∞	infinitely differentiable function for all points of $\mathbb{R}$; §2.15.2
C_0^∞	class of functions that are infinitely differentiable with compact support; §2.15.2
C^k	class of functions that are continuously differentiable up to order k; §2.15.2

C_0^k	class of functions that are continuously differentiable up to order k and have compact support; §2.15.2.
C_p	positive constant depending on the parameter p; often not the same at each occurrence in the sequence of steps of a proof
cas	Hartley cas function; Eq. (5.59)
$\mathcal{C}f$	Cauchy transform of the function f; Eq. (3.19)
$C(z)$	Fresnel cosine integral; Eq. (14.171), HT-01
$\mathrm{chirp}(x)$	chirp function, Exercise 18.13
$\mathrm{Ci}(x)$	cosine integral; Eq. (8.78), HT-01
$\mathrm{ci}(x)$	cosine integral; HT-01
$\mathrm{cie}(\alpha, \beta)$	cosine-exponential integral; HT-01
$\mathrm{Cie}(\alpha, \beta)$	cosine-exponential integral; HT-01
$\mathrm{Cl}_2(x)$	Clausen function; HT-01
$C_n^\lambda(x)$	Gegenbauer polynomials (ultraspherical polynomials); §9.1, Eq. (11.298), HT-01

$\mathbf{D}$	electric displacement; §19.1
$\mathcal{D}$	space of all C^∞ functions with compact support; §2.15.2, §10.2
$\mathcal{D}'$	space of all distributions on $\mathcal{D}$; §10.2
$\mathcal{D}'_+$	space of distributions with support on the right of some point; §10.2
$\mathcal{D}_{L^p}$	space of test functions; §10.2
$\mathcal{D}'_{L^q}$	space of distributions; §10.2
$\mathcal{D}'_{\mathscr{L}}$	space of distributions; Eq. (17.240)
$D(x)$	Dawson's integral; Eq. (5.32)
$D_n(\theta)$	Dirichlet kernel; Eq. (6.56)
$D_n^\lambda(x)$	ultraspherical function of the second kind; Eq. (11.299)

$-e$	electronic charge
E	identity element; §2.10, Eq. (2.150)
E	energy of a signal; Eq. (18.1)
E	one-dimensional Euclidean space
E^1	one-dimensional Euclidean space; §2.11.1
E^n	n-dimensional Euclidean space; §2.11.1
E^σ	class of entire functions of exponential type; §2.8.7, §7.4
$\mathcal{E}$	space of all C^∞ functions with arbitrary support on $\mathbb{R}$; §10.2
$\mathcal{E}(t)$	envelope function; Eq. (18.76)
$\mathcal{E}'$	space of distributions having compact support; §10.2
$E_1(x)$	exponential integral; Eq. (5.98)
E_n	eigenvalues of the unperturbed Hamiltonian; §22.4, Eq. (22.57)
$E_n(z)$	exponential integral; Eq. (14.200), HT-01
$\mathbf{E}(t)$	electric field; §17.9
$\mathbf{E_i}(\omega)$	incident electric field; Eq. (20.3)
$\mathbf{E_r}(\omega)$	reflected electric field; Eq. (20.3)

$\mathbf{E_L}$	left circularly polarized electric wave; Eq. (21.16)
$\mathbf{E_R}$	right circularly polarized electric wave; Eq. (21.17)
$\mathrm{Ei}(x)$	exponential integral function; Eq. (5.101), HT-01
$\mathrm{erf}(z)$	error function; Eq. (5.27), HT-01
$\mathrm{erfc}(z)$	complementary error function; Eq. (5.141)
$\mathbf{E}_v(z)$	Weber's function; HT-01
$\mathbf{F}$	Lorentz force; Eq. (21.50).
$f()$ or f	function (at no particular specified point); §1.2
$f(x)$	function f evaluated at the point x; §1.2
f_j	oscillator strength; §19.2
$f[n]$	element of a discrete sequence; §13.2, §13.6
$\{f[n]\}$	discrete sequence; §13.6
$f_e(x)$	even function; Eq. (4.8)
$f_o(x)$	odd function; Eq. (4.9)
$f_\downarrow(c)$	limit approaching c from $c+0$; Eq. (2.22)
$f_\uparrow(c)$	limit approaching c from $c-0$; Eq. (2.23)
$\mathcal{F}f$	fourier transform of the function f; §2.6, Eq. (2.46)
$\mathcal{F}_n f$	n-dimensional Fourier transform of the function f; §15.6
$\mathcal{F}^{-1}f$	inverse Fourier transform of the function f; §2.6, Eq. (2.47)
$\mathcal{F}_c f$	Fourier cosine transform of the function f; Eq. (5.41)
$\mathcal{F}_s f$	Fourier sine transform of the function f; Eq. (5.40)
$\mathcal{F}_N$	N-point DFT operator; §13.4
$\mathcal{F}_Q$	fractional Fourier transform; §18.10, Eq. (18.147)
$\mathfrak{F}_\alpha$	discrete fractional Fourier transform; §18.13, Eq. (18.240)
$\hat{f}$	Fourier transform of the function f; §2.6
$\tilde{f}$	conjugate series of f; Eq. (6.118); alternative notation for $\mathcal{H}f$; §6.1
f'	derivative of the function f
$f^+(z)$	function f evaluated at an interior point to a contour; Eq. (3.152)
$f^-(z)$	function f evaluated at an exterior point to a contour; Eq. (3.153)
$F_n(\theta)$	Fejér kernel; Eq. (6.63)
${}_1F_1(\alpha;\beta;x)$	Kummer's confluent hypergeometric function; Eq. (5.30), HT-01
${}_2F_1(a,b;c;z)$	hypergeometric function (or Gauss' hypergeometric function); HT-01
$\boldsymbol{F}_{\mathrm{ext}}$	external force; §17.9
$\boldsymbol{F}_{\mathrm{rad}}$	radiative reaction force; §17.9
$f(\omega, 0)$	scattering amplitude at $\theta = 0$; Eq. (19.309)
$F_{\mathbf{h}}$	scattering factor; §23.4
$\mathrm{floor}[x]$	the greatest integer $\leq x$
$G(a,x)$	Hilbert transform of the Gaussian function; §4.7. The abbreviation $G(1,x) \equiv G(x)$ is employed; §9.3
$G^{(n)}_{kl_1\cdots l_2}(t_1,t_2,\ldots,t_n)$	tensor components of the nth-order response function; §22.1

$\hbar$	Planck's constant divided by 2π
H	Hilbert transform operator on $\mathbb{R}$; Eqs. (1.2) and (1.4)
$\mathcal{H}$	Hilbert transform operator for the disc; Eq. (3.202)
$\mathbf{H}$	magnetic field; Eq. (19.7)
$\mathcal{H}_\tau$	Hilbert transform operator for period 2τ; Eq. (3.286)
Hf	Hilbert transform of the function f; §1.2
$(Hf)(x)$	Hilbert transform of the function f on the real line evaluated at the point x; Eq. (1.2)
$H_e f$	Hilbert transform of the even function f on $\mathbb{R}^+$; Eq. (4.11)
$H_o f$	Hilbert transform of the odd function f on $\mathbb{R}^+$; Eq. (4.12)
$H_1 f$	one-sided Hilbert transform of the function f; Eq. (8.18)
$H_I f$	Hilbert's integral of the function f; Eq. (7.33)
$H_n f$	n-dimensional Hilbert transform of the function f; Eq. (15.26)
$\mathscr{H}_n f$	general n-dimensional Hilbert transform of the function f in E^n; Eq. (15.2)
$\mathscr{H}_{n,\varepsilon} f$	general n-dimensional truncated Hilbert transform of the function f in E^n; Eq. (15.7)
$H_{(k)} f$	Hilbert transform of the function $f(x_1, x_2, \ldots, x_k, \ldots, x_n)$ in the variable x_k; Eq. (15.36)
H^{-1}	inverse Hilbert transform operator; Eq. (4.26)
H^+	adjoint of the Hilbert transform operator; Eq. (4.194)
H_α	fractional Hilbert transform operator; Eqs. (18.209) and (18.216)
$\mathscr{H}$	Hamiltonian for an electronic system; §22.4, Eq. (22.54)
$\mathscr{H}_0$	unperturbed Hamiltonian for an electronic system; §22.4, Eq. (22.57)
$\mathscr{H}$	space of test functions; §10.14
$\mathcal{H}$	inner product space; §2.10.1
$\mathcal{H}$	Hilbert space; §2.10
$H_n f$	n-dimensional Hilbert transform of the function f, for $n \geq 2$; Eq. (15.26)
$H_1(f, g)(x)$	bilinear Hilbert transform; §16.5
$H_a(f, g)(x)$	bilinear singular integral operator; Eq. (16.85)
$H_n(x)$	Hermite polynomials; §9.3, Eq. (9.39), HT-01
$\mathrm{H}(x)$	Heaviside step function; Eqs. (10.54) and (18.116)
$\mathsf{H}(\omega)$	response function for a linear system; Eq. (13.1), §18.2
$\mathsf{H}_p(\omega)$	fractional Hilbert filter; §18.9, Eq. (18.142)
$H^p(D)$	Hardy space for the unit disc; §2.10.2
H^p	Hardy space for the upper half complex plane; §2.10.2
H_ν	response function at the frequency ν; Eq. (13.3)
$H_\varepsilon f$	truncated Hilbert transform; Eq. (3.3)
$H_E f$	truncated Hilbert transform; Eq. (4.507)

$H_{\mathrm{M}} f$	maximal Hilbert transform function; Eq. (7.280)
$\mathcal{H}_{\mathrm{M}} f$	maximal Hilbert transform function; Eq. (7.282)
$H_{\mathrm{S}} F$	Hilbert–Stieltjes transform of the function F; Eq. (4.551)
H_{K}	Kober's extension of the Hilbert transform operator; §16.3
H_{R_m}	Redheffer's extension of the Hilbert transform operator; §16.2
$\mathbf{H}_j$	vectorial Hilbert transform operator; Eq. (16.100)
$H_{\theta,\varepsilon}$	truncated directional Hilbert transform operator; Eq. (16.103)
H_θ	directional Hilbert transform operator; Eq. (16.104)
H_θ	helical Hilbert transform operator; Eq. (16.131)
H_{M_θ}	directional maximal Hilbert transform operator; Eq. (16.105)
H_{M_θ}	maximal helical Hilbert transform operator; Eq. (16.132)
$H_{\mathrm{M}_{n\theta}}$	double maximal helical Hilbert transform operator; Eq. (16.136)
$H_\Gamma f$	Hilbert transform of f along the curve Γ; Eq. (16.109)
$\bar{H}_\Gamma f$	modified Hilbert transform of f along the curve Γ; Eq. (16.113)
$\mathrm{H}_{\mathrm{A}} f$	Hartley transform of a function f; Eq. (5.58)
$\mathrm{H}_{\mathrm{A}}^{-1}$	inverse Hartley transform operator; Eq. (5.60)
$H_{\pm v}^{(1)}(z),\ H_{\pm v}^{(2)}(z)$	Bessel functions of the third kind (Hankel functions of the first kind and second kind, respectively); §9.9
$h_n^{(1)}(z)$	spherical Bessel functions of the third kind; Eq. (9.131) (spherical Hankel functions of the first kind)
$h_n^{(2)}(z)$	spherical Bessel functions of the third kind; Eq. (9.132) (spherical Hankel functions of the second kind)
$h_n(x)$	Hermite–Gaussian functions; Eq. (18.179)
$\mathbf{h}_k$	discrete Hermite–Gaussian vector functions; Eqs. (18.254) and (18.255)
$\mathbf{H}_v(z)$	Struve's function; Eq. (9.77), HT-01
$H_{\mathrm{D}}\{f[n]\}$	discrete Hilbert transform of the sequence $\{f[n]\}$; Eq. (13.127)
$\{H_{\mathrm{sD}} f\}(x)$	semi-discrete Hilbert transform of the sequence $\{f[]\}$; Eq. (13.133)
$\mathcal{H}_{\mathrm{D}}\{f[n]\}$	alternative definition of the discrete Hilbert transform; Eq. (13.158)
$(\mathcal{H}_{\mathrm{D}_{pq}} x)[n]$	discrete fractional Hilbert transform; Eq. (18.269)
i	imaginary unit (engineers typically use j); §2.8
I	identity operator; §4.4
I	interval; §7.9
$\|I\|$	length of an interval; §7.9
$I_n(x)$	modified Bessel function of the first kind; HT-01
$i(t)$	input (time-dependent in general) to a system; §17.1–17.2
iff	if and only if
Im	imaginary part of a complex function
inf	infimum, the greatest lower bound of a set; §2.8

Ind f	index of a function; Eq. (11.179)
J	SI unit of energy, the joule; §19.1
$J_{\pm v}(z)$	Bessel function of the first kind; §9.6, HT-01
$\mathbf{J}_v(z)$	Anger's function; §9.12, HT-01
$j_n(z)$	spherical Bessel function of the first kind; Eq. (9.115)
k	wave number; Eq. (19.87)
$\mathbf{k}$	wave vector; §20.7
$k(x, y)$	Kernel function; §1.2, Eq. (1.3)
$K(x)$	Calderón–Zygmund kernel function; §15.1
$K_n(x)$	modified Bessel function of the third kind; HT-01
$l(I)$	length of an interval I; §2.11.1
$\mathscr{L} f$	Laplace transform of the function f; Eq. (5.91)
$\mathscr{L}_2 f$	bilateral (or two-sided) Laplace transform of the function f; Eq. (5.92)
L	class of functions that are Lebesgue integrable on a given interval; §2.11.1
$L(a, b)$	class of functions that are Lebesgue integrable on the interval (a, b); 2.11.1
L^1_{loc}	class of functions that are Lebesgue integrable on every subinterval of a given interval; Eq. (4.121)
L^2	class of functions that are Lebesgue square integrable on a given interval; §2.11.1
L^p	class of functions f such that $\|f\|^p$ is Lebesgue integrable on a given interval; §2.11.1
$L^p(\mathbb{R})$	class of functions f such that $\|f\|^p$ is Lebesgue integrable on the real line; §2.11.1
l^p	§13.11
$l^p(\mathbb{Z})$	§13.11
L^∞	class of essentially bounded functions; §2.11.1
$L^p_{2\pi}$	class of periodic functions f such that $\|f\|^p$ is Lebesgue integrable on the interval $(0, 2\pi)$. $L^p_{2\tau}$ has a similar meaning for periodic functions with period 2τ.
$L^{\alpha,p}$	class of functions f such that $\|x\|^\alpha \|f(x)\|^p$ is Lebesgue integrable on a particular interval; Eq. (7.186)
$L^p(\mu)$	class of μ-measurable functions; §7.12
$L_n(x)$	Laguerre polynomials; §9.4, Eq. (9.60)
$\mathbf{L}_v(z)$	modified Struve function; HT-01
$\text{Li}_n(z)$	polylogarithm function; HT-01
$\text{Li}_2(z)$	dilogarithm function; HT-01
Lip m	Lipschitz condition of order m; §2.3
log	logarithm to the base e; the alternative notation ln is also common usage
$\log^+ f$	maximum of $\{\log\|f\|, 0\}$; Eq. (7.74)

$\mathbf{M}$	magnetization; Eq. (20.111)
$\operatorname{mod} z$	modulus of a complex number; Eq. (2.68)
$m(E)$	measure of the set E; §2.11.1
Mf	Hardy–Littlewood maximal function; §7.9
Mf	Mellin transform of f; Eq. (5.102)
M^{-1}	inverse Mellin transform operator; Eq. (5.107)
m	SI unit for length, the meter; §19.1
$m\{g(\lambda)\}$	distribution function of g; §4.25, §7.2, Eqs. (4.556), (7.55)
$m_{X,Y}(\omega)$	relative multiplier connecting $X(\omega)$ and $Y(\omega)$; Eq. (18.60)
$\mathbb{N}$	set of positive integers; $1, 2, 3, \ldots$
N	complex refractive index; Eq. (19.90)
N^{NL}	nonlinear complex refractive index; §22.13
$\mathcal{N}$	number of molecules per unit volume; §19.2
$n(\omega)$	angular frequency-dependent refractive index; Eq. (19.91)
$n^{\mathrm{NL}}(\omega, E)$	nonlinear refractive index; §22.13, Eq. (22.238)
$N_{\pm}(\omega)$	complex refractive indices for circularly polarized modes; §21.3, Eq. (21.47)
$n_{\pm}(\omega)$	real parts of $N_{\pm}(\omega)$; Eqs. (21.79) and (21.80)
$\mathcal{O}$	linear operator on a vector space; §2.10
$\mathcal{O}^{+}$	*adjoint* operator to $\mathcal{O}$; §2.10
$\mathcal{O}^{-1}$	inverse of an operator $\mathcal{O}$; §2.10
$O(\,)$	Bachmann order notation, of the order of; Eq. (2.1)
$o(\,)$	Landau order notation, of the order of; Eq. (2.6)
$\mathcal{O}'_C$	space of distributions that decrease rapidly at infinity; §10.2
$P\int$	Cauchy principal value; §2.4, Eq. (2.18)
$P(r,\theta)$	Poisson kernel for the disc; Eq. (3.49)
$P(x,y)$	Poisson kernel for the half plane; Eq. (3.31)
P_{ε}	Poisson operator; §7.10, Eq. (7.290)
P_{+}	projection operator; Eq. (4.352)
P_{-}	projection operator; Eq. (4.353)
$Pf(x^{-1})$	pseudofunction; §10.1
$\mathbf{P}(\mathbf{x})$	electric polarization of a medium; Eq. (19.1)
$\mathrm{P}_n(x)$	one of the orthogonal polynomials; §9.1
$P_n(x)$	Legendre polynomials; §9.2, Eqs. (9.10) and (9.27)
$P_{\nu}^{m}(x)$	associated Legendre function of the first kind; HT-01
$P_n^{(\alpha,\beta)}(x)$	Jacobi polynomials; §9.1
$\mathcal{P}_{\tau}$	space of periodic testing functions of period τ; §10.2
$\mathcal{P}'_{\tau}$	space of periodic distributions of period T; §10.2
$p.v.\dfrac{1}{x}$	distribution; §10.1

$\mathcal{P}\frac{1}{x}$	distribution; §10.1
$Q(r,\theta)$	conjugate Poisson kernel for the disc; Eq. (3.50)
$Q(x,y)$	conjugate Poisson kernel for the half plane; Eq. (3.32)
Q_ε	conjugate Poisson operator; §7.10, Eq. (7.291)
$Q_n(x)$	Legendre function of the second kind; Eq. (11.263)
$Q_\nu^m(x)$	associated Legendre function of the second kind; HT-01
$Q_n^{(\alpha,\beta)}(x)$	Jacobi function of the second kind; HT-01
R	reflection operator; Eq. (4.73)
R	radius for a semicircular contour
A	Radon transform; §5.10, Eqs. (5.152) and (5.155)
$R_i(z_i)$	residue corresponding to the pole at $z = z_i$; §2.8.5
$R_j f$	Riesz transform of the function f; §15.12
$\mathbb{R}$	real line; the set of real numbers
$\mathbb{R}^+$	positive real axis interval; §3.4
$\mathbb{R}\times\mathbb{R}$	Euclidean plane
$\mathbb{R}^n$	n-dimensional Euclidean space; §2.15.2
$\mathcal{R}$	simply connected region; §2.8.1
$\mathcal{R}$	radius for a semicircular contour
$\Re_p$	Riesz constant; §4.20, Eqs. (4.382) and (4.384)
$\tilde{r}(\omega)$	generalized or complex reflectivity; Eq. (20.1)
$\tilde{r}_\pm(\omega)$	generalized reflectivity for circularly polarized modes; Eq. (21.132)
$r(\omega)$	reflectivity amplitude; Eq. (20.1)
$r(t)$	response (time-dependent) from a system; §17.1, §17.2
R_{n0}	rotational strength; Eq. (21.233)
$R(\omega)$	reflectivity; Eq. (20.2)
$re^{i\theta}$	polar form of the complex number z
Re	real part of a complex number
rect(x)	reactangular pulse function; §18.7.3
$\text{Res}\{g(z)\}_{z=z_0}$	residue at the pole $z = z_0$ of the function g; §2.8.5, Eq. (2.93)
$\mathsf{S}f$	Stieltjes transform of the function f; Eqs. (5.77), and (8.6)
S_a	dilation operator (homothetic operator); Eqs. (4.70) and (15.68)
sgn	signum function (sign function); Eqs. (1.14) and (18.120)
S^{n-1}	locus of points $x \in \mathbb{R}^n$ for which $\|x\| = 1$; §16.6
$S(z)$	Fresnel sine integral; Eq. (14.172), HT-01
$S(E)$	S-function (S-matrix); §17.12
$S(\omega)$	Fourier transform of a signal $s(t)$ in the frequency domain; §18.1, Eq. (18.2)
$s(t)$	signal in the time domain; §18.1
Shi(z)	hyperbolic sine integral function; Eq. (14.201), HT-01
Si(x)	sine integral; Eq. (8.79), HT-01

$\mathrm{si}(x)$	integral; Eq. (9.170), HT-01
$\mathrm{sie}\,(\alpha,\ \beta)$	sine-exponential integral; HT-01
$\mathrm{sinc}\,x$	sinc function; Eq. (4.260), HT-01
sup	supremum, the least upper bound
supp	support of the function; §2.15.2
T	finite Hilbert transform operator; chap. 11, Eq. (11.2)
T	used to denote a distribution; §2.15.2
T_{ab}	finite Hilbert transform operator on the interval (a, b); Eq. (12.98)
$T_n(x)$	Chebyshev polynomials of the first kind; §9.1, HT-01
$\mathbb{T}$	circle group; §3.10
Tr	trace; §22.4, Eq. (22.63)
$U_n(x)$	Chebyshev polynomials of the second kind; §9.1, HT-01
$u[n]$	unit step sequence; Eq. (13.91)
V	total variation of a function; Eq. (4.554)
V	SI unit for potential, the volt; §19.1
$w(x)$	weight function; §9.1
$W(x)$	weight function; §14.4
w_i	weight points in a quadrature scheme; Eq. (14.15)
$W^{p,m}$	Sobolev space; §10.2
$\bar{x}_j$	any value in the interval $[x_{j-1}, x_j]$; §2.11
x_i	sampling points in a quadrature scheme; Eq. (14.15)
$\lvert x\rvert$	norm of x in E^n; §15.1
$\mathbf{\times}$	vector cross product
$\times$	direct product; §10.6. Also used for Cartesian product of Euclidean spaces; §15.13
$\boldsymbol{x}(t)$	time-dependent particle displacement; §17.2, §17.9
$X(z)$	Z transform (one-sided or two-sided); Eqs. (13.38) and (13.39)
$Y_v(z)$	Bessel function of the second kind (Weber's function, Neumann's function); §9.6, 9.8, HT-01
$y_n(z)$	spherical Bessel function of the second kind; Eq. (9.116)
z	complex variable, $z = x + \mathrm{i}y$; Eq. (2.67)
$\bar{z}$	complex conjugate of z
z^*	complex conjugate of z
z_I	inverse point (or image point) of z; Eq. (3.35)
$\mathbb{Z}$	set of integers $0, \pm 1, \pm 2, \ldots$
$\mathbb{Z}^+$	set of non-negative integers $0, 1, 2, \ldots$
$Z\{x_n\}$	Z transform of the sequence $\{x_n\}$; §13.6, Eq. (13.38)

$\mathcal{Z}$ space of test functions whose Fourier transforms belong to $\mathcal{D}$; §10.2
$\mathcal{Z}'$ space of ultradistributions; §10.2
$\mathcal{Z}_1$ space of Fourier transforms of test functions belonging to $\mathscr{S}_1$; §10.14
$Z_s(\omega)$ surface impedance function; Eq. (20.129)

Greek letters

α polarizability § 19.1, Eq. (19.6)
$\alpha(\omega)$ absorption coefficient of a medium; Eq. (19.92)
$\beta(2)$ Catalan's constant (0.915 965 594 177 219 015 1 . . .); HT-01
γ Euler's constant (0.577 215 664 9 . . .)
Γ contour in the complex plane (frequently used to signify a non-closed contour)
$\Gamma(z)$ gamma function; Eq. (4.118), HT-01
$\Gamma(a,z)$ incomplete gamma function; Eq. (8.38), HT-01
Γ_{mn} damping constant for a transition between the levels m and n; §22.4, Eq. (22.71)
$\delta(x)$ Dirac delta distribution; §2.15, §10.3, Eq. (10.1)
$\delta[n]$ unit sample sequence; Eq. (13.92)
$\delta^+(x)$ Heisenberg delta function; Eq. (10.9)
$\delta^-(x)$ Heisenberg delta function; Eq. (10.10)
δ_{nm} Kronecker delta; Eq. (2.38). When one of the subscript indices appears with a negative sign, the notation $\delta_{n,-m}$ is employed
Δ difference operator; Eq. (2.305)
Δ Laplacian operator; Eq. (15.161)
Δ_τ Dirac comb distribution; Eq. (10.212)
$\Delta R(\omega)$ magnetoreflection; Eq. (21.125)
Δx length of an interval; §2.11, Eq. (2.166)
ε permittivity of the medium; Eq. (19.5)
ε^{NL} nonlinear dielectric permittivity; §22.13
ε_0 vacuum permittivity; §19.1
ϵ_{ijk} Levi-Civita pseudotensor; Eq. (21.212)
$\boldsymbol{\varepsilon}(\mathbf{k}, \omega)$ spatial-dependent electric permittivity; Eq. (20.177)
$\varsigma(n)$ Riemann zeta function; Eq. (2.288)
$\theta(\omega)$ phase; Eq. (20.1)
$\theta(\omega)$ ellipticity per unit length; Eq. (21.195)
$\theta_F(\omega)$ ellipticity function; Eq. (21.95)
$\kappa(\omega)$ measure of the absorption of a propagating wave in a medium; Eq. (19.92)
$\kappa_\pm(\omega)$ imaginary parts of $N_\pm(\omega)$; Eqs. (21.79) and (21.80)
$\kappa^{NL}(\omega, E)$ imaginary part of the nonlinear complex refractive index; §22.13, Eq. (22.238)
$\Lambda(x)$ unit triangular function; Eq. (4.265)

Λ_α	space of Lipschitz continuous functions; §6.16, §15.1
μ	continuous Borel measure; §7.12
μ	magnetic permeability; §19.1
$\mu_o(A)$	Lebesgue outer measure of a set A; Eq. (2.183)
μ_0	permeability of the vacuum; §19.1
$\boldsymbol{\mu}$	electric dipole operator; Eq. (22.56)
$\Pi_{2a}(x)$	unit rectangular step function; Eqs. (9.19) and (18.122). For $a = 1/2$, the abbreviation $\Pi(x) \equiv \Pi_1(x)$ is employed
ρ	density operator; §22.4, Eq. (22.51)
$\rho(\mathbf{r})$	electronic density; §23.4, Eq. (23.75)
ρ_{mn}	matrix element of the density operator; §22.4
$\rho_s(t)$	auto-convolution function for a signal; Eq. (18.16)
$\rho_f(t)$	auto-correlation function; Eq. (18.23)
$\rho_{fg}(t)$	cross-correlation function; Eq. (18.21)
σ	type of an entire function; Eq. (2.110)
$\sigma(H)$	symbol of H; Eq. (5.37)
$\sigma(0)$	conductivity at $\omega = 0$; §19.8, Eq. (19.176)
$\sigma(\omega)$	complex conductivity; Eq. (20.84)
$\sigma_t(\omega)$	total scattering cross-section; Eq. (19.316)
Σ	unit sphere; §15.1
τ_a	translation operator; Eqs. (4.64) and Eq. (15.63)
φ_n	eigenfunctions of the unperturbed Hamiltonian; §22.4, Eq. 22.57
ϕ	test function in a particular space; §2.15.2, §10.1
$\phi(\omega)$	optical rotatory dispersion; Eq. (21.192)
$\phi(t)$	instantaneous phase; Eq. (18.78)
$\phi_F(\omega)$	magneto-rotatory dispersion function; Eq. (21.94)
$\Phi(z, s, v)$	Lerch function; HT-01
$\Phi(\omega)$	complex optical rotation function; Eq. (21.197)
χ	(linear) electric susceptibility; §19.1
$\boldsymbol{\chi}^{(n)}$	nth-order electric susceptibility tensor; §22.1
χ_m	magnetic susceptibility; Eq. (20.112)
$\chi_S(x)$	characteristic function associated with the set S; Eq. (2.191)
$\chi_{[x_1,x_2]}$	characteristic function with the interval where the function is non-zero specified by a subscript; §2.11.1
$\{\psi_n\}$	sequence of step functions; §2.14
$\psi(x)$	step function; Eq. (2.190)
$\psi(z)$	psi (or digamma) function; Eq. (4.222), HT-01
$\psi^{(n)}(z)$	Polygamma function; HT-01
$\Psi(\omega)$	ellipticity function, Eq. (21.193)
ω	angular frequency
$\omega(t)$	instantaneous frequency; Eq. (18.79)
ω_z	complex angular frequency; §17.7, Eq. (17.53)
ω_r	real part of a complex angular frequency; §17.7, Eq. (17.53)

ω_i	imaginary part of a complex angular frequency; §17.7, Eq. (17.53)
ω_p	plasma frequency of the medium; Eq. (19.13)
ω_c	cyclotron frequency; Eq. (21.56)
ω_{mn}	energy separation (in frequency units) between the levels m and n; §22.4
Ω	part of the Calderón–Zygmund kernel; §15.1

Miscellaneous notations

$\sum_{k=-\infty}^{\infty\prime}$	summation with a particular value of k excluded (usually $k = 0$)
$\dfrac{\partial}{\partial x}$	partial derivative operator (with respect to x)
∂^m	shorthand for the mth derivative (with respect to the variable under discussion), for $m \geq 1$.
$(a)_k$	Pochhammer symbol; Eq. (5.31). The notation a_k is also employed
$\forall$	for all
∇^2	Laplacian (del-squared) operator; Eq. (7.3)
$\boldsymbol{\nabla}$	gradient (del) operator
$\Rightarrow$	implies;
$\sim$	same order as; §2.2
$\sim$	correspondence; Eq. (3.269)
$\sim$	twiddle sign, employed to indicate asymptotic equivalence between functions in a particular limit; §8.1
$f[\,]$	functional notation, e.g. $f[g(x)]$; §2.15.2, Eq. (2.283)
$\mathcal{O}^+$	adjoint of a linear operator $\mathcal{O}$; §2.10
$[\alpha, \beta]$	closed interval, that is $\alpha \leq x \leq \beta$; §2.3
$[\mathcal{O}_1, \mathcal{O}_2]$	commutator of two operators; §2.10, Eq. (2.146)
$\{\mathcal{O}_1, \mathcal{O}_2\}$	anticommutator of two operators; Eq. (4.67)
(α, β)	open interval, that is $\alpha < x < \beta$; §2.3
(f, g)	scalar product for two functions $f(x)$ and $g(x)$; Eq. (14.52)
$\langle f, g\rangle$	inner product for two functions f and g in Dirac bra–ket notation; §2.10
$\begin{pmatrix} m \\ n \end{pmatrix}$	binomial coefficient; Eq. (2.309)
$*$	convolution operator, e.g. $f * g$; Eq. (2.53)
*	complex conjugate of a function (e.g. z^*)
$\star$	pentagram symbol for the cross-correlation operation; Eq. (18.21)
$\int_a^b f(x)\mathrm{d}x$	Riemann or Lebegue integral (depending on context); §2.11
$\int_C$	integral along the specified contour C; §2.8.1. $\int_\Gamma$ is sometimes used when the contour is not closed

$\int_{\mathbb{R}} f(x)\mathrm{d}x$	integral over the real line; §2.11.1
$\int_{\mathbb{R}^2} f(x,y)\mathrm{d}x\,\mathrm{d}y$	integral over the xy-plane; Eq. (2.213)
$\int_{\mathbb{T}} f(\theta)\mathrm{d}\theta$	integral over a 2π period
$\oint_C$	integral along the closed contour C taken in a specified orientation; §2.8.1
$\int_E f(x)\mathrm{d}x$	Lebesgue integral of f on E; Eq. (2.198)
$\int\limits_{\|x-t\|>\varepsilon} f(x,t)\mathrm{d}t$	integral for which a segment $(x-\varepsilon, x+\varepsilon)$ is excluded; Eq. (3.3)
$\exists$	there exists
$\in$	belongs to
$\notin$	does not belong to
$\mathscr{S}$	space of test functions that have rapid decay; §10.2
$\mathscr{S}_1$	space of test functions that belong to $\mathscr{S}$ and vanish on the interval function $(-a,\ a)$ for some $a>0$; §10.14
$\mathscr{S}'$	space of all tempered distributions; §10.2
$\langle\ \|$	Dirac bra; §2.10
$\|\ \rangle$	Dirac ket; §2.10
$\|f\|$	absolute value of the function f
$\|x\|$	norm of x in E^n; §15.1
$\|a\|$	sum of the components of the multi-index a; §15.5
$\|\theta\in[-\pi,\pi] : \|g(\theta)\|\geq\lambda\|$	distribution function of g, §7.2, Eq. (7.55)
$\\|\phi\\|$	norm of a vector ϕ; §2.10, Eq. (2.136)
$\\|f\\|_p$	pth-power norm of f; Eq. (2.202)
$\\|f\\|_\infty$	essential supremum of $\|f\|$; Eq. (2.203)
$\\|f(\theta)\\|_{\alpha,p}$	weighted norm; Eq. (7.186)
$\\|f(\theta)\\|_{\alpha,\infty}$	weighted norm; Eq. (7.187)
$\\|f\\|_{p,\mu}$	norm $(\int\|f\|^p\,\mathrm{d}\mu)^{1/p}<\infty$, $1<p<\infty$; Eq. (7.376)
$\\|f\\|_{W^{p,m}}$	Sobolev norm; §10.2, Eq. (10.40)
$\\|p\\|$	longest subinterval; Eq. (2.167)
$\{x_i : a\leq x\leq b\}$	set of points $\{x_i\}$ such that $a\leq x\leq b$; §2.10
$\varnothing$	empty set; §2.10
$\subset$	subset of, as in $A\subset B$, A is a (proper) subset of B; §2.10
$\subseteq$	included in, as in $A\subseteq B$, A is included in B; §2.10
$\cap$	intersection of sets, as in $A\cap B$; §2.10
$\cup$	union of sets, as in $A\cup B$; §2.10
$\bigcup\limits_k A_k$	union of the collection of sets A_k
$\backslash$	relative complement of a set, that is, the relative complement of B with respect to A (the difference of A and B) is denoted by $A\backslash B$; §2.10
$(m)!!$	double factorial; Eqs. (4.119) and (4.120)
$\lfloor m\rfloor$	floor function, the greatest integer less than or equal to m; Eq. (9.28)

$[m/2]$	value $m/2$ if m is an even integer or $(m-1)/2$ if m is an odd integer
$[\arg f(z)]_C$	change in $\arg f(z)$ as the contour is traversed; Eq. (11.174)
$\{x_n\}$	sequence of numbers
$\otimes$	tensor product (direct product); §10.6, Eq. (10.88)

Abbreviations

a.e.	almost everywhere: §2.11.1
CD	circular dichroism; §21.1
DFT	discrete Fourier transform; §13.2
DFHT	discrete fractional Hilbert transform; §18.14
DFRFT	discrete fractional Fourier transform; §18.13
EMD	empirical mode decomposition; §18.16
FFT	fast Fourier transform; §14.9, §14.10
FHT	fractional Hilbert transform; §18.9
FRFT	fractional Fourier transform; §18.9
FTNMR	Fourier transform nuclear magnetic resonance (spectroscopy); §1.1
FTIR	Fourier transform infrared (spectroscopy); §1.1
IDFT	inverse DFT; §13.4
IMF	intrinsic mode function; §18.16
MCD	magnetic circular dichroism; §21.1
MOR	magnetic optical rotation; §21.3
MRS	magnetic rotation spectra; §21.3
ORD	optical rotatory dispersion; §21.1
SHG	second-harmonic generation; §22.2
THG	third-harmonic generation; §22.2

1

Introduction

1.1 Some common integral transforms

Transform techniques have become familiar to recent generations of undergraduates in various areas of mathematics, science, and engineering. The principal integral transform that is perhaps best known is the Fourier transform. The jump from the time domain to the frequency domain is a characteristic feature of a number of important instrumental methods that are routinely employed in many university science departments and industrial laboratories. Fourier transform nuclear magnetic resonance spectroscopy (acronym FTNMR) and Fourier transform infrared spectroscopy (FTIR) are two extremely significant techniques where the Fourier transform methodology finds important application. Two transforms derived from the Fourier transform, the Fourier sine and Fourier cosine transforms, also find wide application. The Laplace transform is often encountered fairly early in the undergraduate mathematics curriculum, because of its utility in aiding the solution of certain types of elementary differential equations. The transforms that bear the names of Abel, Cauchy, Mellin, Hankel, Hartley, Hilbert, Radon, Stieltjes, and some more modern inventions, such as the wavelet transform, are much less well known, tending to be the working tools of specialists in various areas. The focus of this work is about the Hilbert transform. In the course of discussing the Hilbert transform, connections with some of the other transforms will be encountered, including the Fourier transform, the Fourier sine and Fourier cosine offspring, and the Hartley, Laplace, Stieltjes, Mellin, and Cauchy transforms. The Z-transform is studied as a prelude to a discussion of the discrete Hilbert transform.

In this chapter the principal objective is to provide a non-rigorous introduction to the Hilbert transform, and to establish the idea of the Hilbert transform operator. Some brief historical comments are presented on the emergence of the Hilbert transform. Finally, some areas are given where the Hilbert transform finds application.

1.2 Definition of the Hilbert transform

Many of the common integral transforms can be written in the following form:

$$g(x) = \int_a^b k(x,y)f(y)\mathrm{d}y, \tag{1.1}$$

where $k(x,y)$ is called the *kernel function*, or just the kernel of the equation. Equation (1.1) can also be thought of as an example of an *integral equation*, if one desires to determine the function f in terms of g. More specifically, it is termed a Fredholm equation of the first kind. The limits on the integral can be finite or infinite. When the kernel function has a singularity in the integration range, it is possible in a number of cases to extend the definition of the integral in Eq. (1.1) to accommodate these cases. Such equations are referred to as singular integral equations.

The Hilbert transform on $\mathbb{R}$, the real line, is defined by

$$Hf(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(y)\mathrm{d}y}{x-y}, \quad \text{for } x \in \mathbb{R}. \tag{1.2}$$

The kernel function in this definition is given by

$$k(x,y) = \frac{1}{\pi(x-y)}, \tag{1.3}$$

which is singular when $y = x$. The symbol $P\int$ denotes an extension of the normal definition of the integral called the *Cauchy principal value*. This is discussed in detail in Chapter 2. The integral becomes well behaved for many common functions if an infinitesimally small section of the integration interval centered at the singularity $y = x$ is deleted, as part of the definition of the integral. This is the essential effect of evaluating the integral as a principal value integral.

A word on notation may be useful at this juncture. Commonly, f is used to denote a function of a single variable and $f(x)$ is the value of the function evaluated at the point x. It is prevalent in the sciences to use the notation $f(x)$ to denote the function and also the value of the function evaluated at the point x. Usually the context makes it clear which of the two meanings is intended, although the use of f or $f(\ )$ for the function, and $f(x)$ for the value of the function evaluated at the point x, makes the meaning much clearer. The interpretation of Eq. (1.2) is that Hf signifies a new function and $Hf(x)$ is the value of this function evaluated at the point x. The notation Hf is used when there is no need to specify the point at which the transform is evaluated, which is convenient in a number of cases, particularly where additional operators such as the Fourier or inverse Fourier transform operator are also being applied to the function f. Occasionally the notation $H[\ ]$ or $H\{\ \}$ is employed; this is expedient when the Hilbert transform of a product of functions is taken, but the notation is not used exclusively for this purpose. In this book the notation $H[f(x)]$ or $H\{f(x)\}$ is employed with some frequency as a shorthand for $H[f(t)](x)$. In the latter form, t is the dummy integration variable for the Hilbert transform, and the function Hf is evaluated at the point x. Occasionally the notation $H[f,x]$ is used in the literature to denote the Hilbert transform of the function f evaluated at the point x. Sometimes, mostly by mathematicians, the Hilbert transform of the function

f is denoted by $\tilde{f}$. In the literature, the symbol T is also employed to denote the Hilbert transform. In this book, T is used to denote the finite Hilbert transform. When no confusion is likely, operator identities involving H are written with no function specified, and it is assumed that functions can be found for which the operator equality holds.

Historically, Eq. (1.2) was not the definition given by David Hilbert. Working in the area of integral equations, he arrived at a pair of integral equations connecting the real and imaginary parts of a function analytic in the unit disc, leading to the definition of the Hilbert transform for the circle (Hilbert, 1904, 1912). The transform appearing in Eq. (1.2) seems to have been first discussed with some level of rigor by the cricket loving English mathematician G. H. Hardy (1902, 1908), and named by him in 1924 the Hilbert transform, in honor of Hilbert's contribution. It is perhaps interesting to speculate how this transform might have been named by later workers had Hardy not graciously named the transform as he did. In a sense, Alfred Tauber's contribution (Tauber, 1891) appears to have been overlooked. In hindsight, perhaps the transform should bear the names of the three aforementioned authors. Most of the early developments on the Hilbert transform were not performed by David Hilbert, but by Hardy (1924a, 1924b, 1932) and Titchmarsh (1925a, 1929, 1930a, 1930b). A related form was given by Young (1912). Variants of the Hilbert transform on $\mathbb{R}$ are presented in later chapters; these include the Hilbert transform for the circle, the finite Hilbert transform, the multi-dimensional Hilbert transform, the discrete Hilbert transform, and others.

The reader is alerted to the existence of an alternative definition of the Hilbert transform for the real line, one where the kernel $k(x, y) = \{\pi(y - x)\}^{-1}$ is employed. Unfortunately, a consensus agreement on the definition has not been reached, and both forms occur rather commonly in the literature, though the definition given in Eq. (1.2) appears to be increasingly favored. For a number of purposes this difference in sign is not important, but obviously is significant for the evaluation of the Hilbert transform of a particular function, which means that the reader needs to be alert to the sign choice when pulling entries from tables of Hilbert transforms. Occasionally the Hilbert transform is defined with the factor π^{-1} omitted. Employing the definition given in Eq. (1.2) does have the advantage that factors of π that would frequently appear are incorporated into the definition of the Hilbert transform. A few authors define the Hilbert transform with the imaginary unit factor included, that is, π^{-1} is replaced by $(\pi \mathrm{i})^{-1}$.

Note that nothing has been said about what conditions must be specified for the function f in order that the integral in Eq. (1.2) exists. Different levels of rigor can be brought to bear on this question. For almost all applications in the physical sciences, the existence of the Riemann integral of the function $|f|^2$ over the interval $(-\infty, \infty)$ is all that is required to guarantee that the Hilbert transform of f is bounded. The Hilbert transform can be defined for a wider class of functions than the aforementioned, and this is addressed in Chapter 3.

1.3 The Hilbert transform as an operator

The key idea in the application of any of the simple integral transforms is that the function f is acted on by an "integral operator," to yield a new function, g, which is referred to as the "name" transform of f. In the case of the Hilbert transform, the integral operation is given by

$$H \equiv \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(\;) - s}, \tag{1.4}$$

where the identity of the function and the point at which the Hilbert transform is evaluated are left unspecified. The Hilbert transform of f can be thought of as the application of the integral operator in Eq. (1.4) on the function $f(\;)$, to yield

$$Hf(\;) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(\;) - s} f(s), \tag{1.5}$$

and the left-hand side of Eq. (1.5) is frequently denoted by the function $g(\;)$. Clearly, the function g depends on the entire shape of f. In other words, g at some point x, $g(x)$, is not determined simply by the value of the function f evaluated at the same point. That is, g has a *non-local* dependence on f. The situation where $g(x)$ is determined directly by the value $f(x)$ arises when there is a simple functional connection between f and g; for example, suppose $g(x) = \sin[f(x)]$, then the value of g at the point x depends only on the value of f evaluated at x. This notion has important consequences. A function f could be zero over a large region of the real axis and finite for a small region, but its Hilbert transform could be everywhere non-zero. Applications will be encountered later that reflect this type of behavior.

To visualize the changes that take place when the Hilbert transform of a function is evaluated, consider the following choice:

$$f(x) = \frac{a}{a^2 + x^2}, \tag{1.6}$$

where a is a real positive constant. This functional form appears in several diverse applications, and is sometimes referred to as a Cauchy pulse, and in other applications is closely related to the Lorentzian profile. The Hilbert transform of this function is given by

$$g(x) = Hf(x) = \frac{x}{a^2 + x^2}. \tag{1.7}$$

Figure 1.1 shows a plot of $f(x)$ and its Hilbert transform for the value $a = 1$.

The particular methods that are most effective for evaluating this relatively straightforward Hilbert transform are discussed in Chapter 2 and illustrated with examples in Chapters 3 and 4.

The function f of the preceding example can be recovered from g using the expression $f(x) = -Hg(x)$. In fact, this is a rather general result. The two formulas

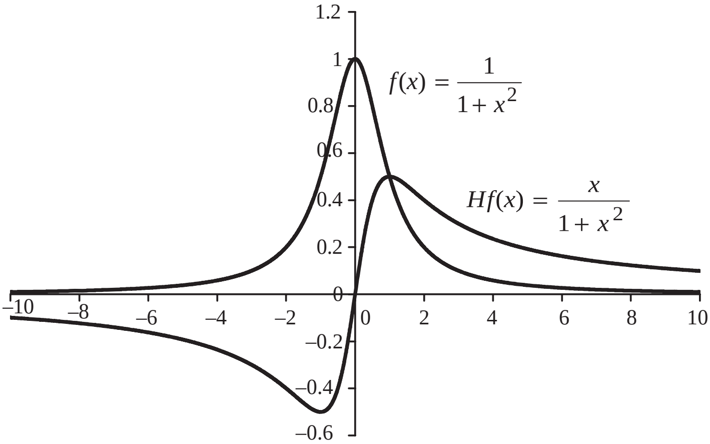

Figure 1.1. Plot of the Cauchy pulse and its Hilbert transform.

$g(x) = Hf(x)$ and $f(x) = -Hg(x)$ constitute a *Hilbert transform pair*. This Hilbert transform pair is explored in detail in later chapters, and it is shown that there is a very close connection with the theory of analytic functions. Pairs of functions that satisfy this type of skew-reciprocal character have been known for a considerable time. For example, the results (for $a > 0$)

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\sin as\,\mathrm{d}s}{x-s} = -\cos ax \tag{1.8}$$

and

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\cos as\,\mathrm{d}s}{x-s} = \sin ax \tag{1.9}$$

were given well over one hundred years ago (Schlömilch, 1848 p. 153; Bierens de Haan, 1867). The sine and cosine functions thus form a Hilbert transform pair.

Hardy (1908, 1924a, 1924b, 1928a, 1932) was one of those who pioneered the study of the mathematical foundations of the Hilbert transform. Prior to Hilbert's publications, Hardy (1902) had investigated the properties of Cauchy principal value integrals, and, in particular, he derived the preceding two formulas. Let $I(x, a)$ denote the following integral:

$$I(x,a) = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\sin as\,\mathrm{d}s}{x-s}, \tag{1.10}$$

where a is a constant. From the preceding formula, Hardy obtained the following differential equation:

$$\frac{\mathrm{d}^2 I}{\mathrm{d}x^2} + a^2 I = 0. \tag{1.11}$$

The topic of differentiation of the Hilbert transform is discussed in detail later. The solution of Eq. (1.11) is

$$I(x,a) = \alpha \cos ax + \beta \sin ax, \tag{1.12}$$

where α and β are arbitrary constants. Setting $x = 0$ and using the result

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as\, \mathrm{d}s}{s} = \operatorname{sgn} a, \tag{1.13}$$

where

$$\operatorname{sgn} a = \begin{cases} 1, & a > 0 \\ 0, & a = 0 \\ -1, & a < 0 \end{cases} \tag{1.14}$$

gives $\alpha = -\operatorname{sgn} a$. It is straightforward to show that $I(x,a) = -I(-x,-a)$, from which it follows that $\beta = 0$, and hence

$$I(x,a) = -\operatorname{sgn} a \cos ax. \tag{1.15}$$

Hardy (1902) gave this result for the case $a > 0$, and he also gave a result equivalent to Eq. (1.9).

1.4 Diversity of applications of the Hilbert transform

Historically, work on Hilbert transforms developed on three main fronts. Mathematicians made the seminal developments in the first quarter of the twentieth century by putting the Hilbert transform into various useful forms, and established a number of key theorems that turned out to be of critical importance for future developments in the physical sciences. Hilbert transforms arose first in potential theory. Around the time of the dawn of modern quantum theory, Kramers (1926, 1927) and, working independently, Kronig (1926) obtained the reciprocal relations between the frequency dependent refractive index and the absorption coefficient of a medium. The resulting equations involved principal value integrals over the frequency interval $[0, \infty)$, which can be recast as a pair of standard Hilbert transforms. These equations became known in the physics and chemistry literature as the Kramers–Kronig relations. In parallel with this development, electrical engineers applied the same and some closely related mathematical ideas in circuit analysis (Carson, 1926). The real and imaginary parts of the general complex impedance were found to be connected to each other by Hilbert transforms. These relations are sometimes referred to as the Bode relations (Bode, 1945). In branches of engineering the Hilbert transforms are sometimes referred to as Wiener–Lee transforms (Papoulis, 1962, p. 192). In modern signal processing the terms 90° phase shift filter or quadrature filter are also employed to describe a Hilbert transform. The former of these two designations comes from the fact that the Hilbert transform of a sine function yields a cosine function, and this can be recast as a sine function with a shift of the argument by 90°. Somewhat later, with activity rising significantly in the early 1960s, Hilbert transforms found important applications in the study of various scattering processes in elementary particle physics and some other branches of physics. The key equations developed to describe the scattering processes are called *dispersion relations*, which are, in many cases, Hilbert transform relations or relatively minor extensions of the Hilbert transform concept. The Hilbert transform technique has clearly acquired multiple names as it has been employed in different applications. This multiplicity of names makes it more difficult to assess the true impact of Hilbert's contribution to transform calculus in the physical sciences. In addition to Hilbert, perhaps it is not inappropriate to give due credit to the nineteenth century mathematicians Poisson and, in particular, Cauchy, whose contributions laid the foundations for the work of Hilbert and others on the transform that finds such a diverse number of applications.

The question of why one should be interested in studying the theory of Hilbert transforms can be best answered in the following manner. There are numerous practical applications of Hilbert transforms, such as those mentioned in the preceding paragraph. To that list of applications can be added problems in aerofoil theory, crack formation in materials, aspects of the theory of elasticity, applications in wave propagation theory, problems in potential theory, and the study of dispersion forces. Further applications arise in certain areas in digital signal processing, and problems in the reconstruction of images. Readers with an interest in the stock market might be fascinated to see how a discrete version of the Hilbert transform has been used as a modeling tool (Ehlers, 2001). For some of these topics, the Hilbert transform or some variant of the standard form occurs as part of an integral equation or of an integro-differential equation. An example that is discussed later is the study of solitary waves. Because of the rich and diverse array of applications, the study of Hilbert transform theory can be a rewarding exercise.

Hilbert transform theory of course finds a number of applications in pure mathematics. The theory of the conventional Hilbert transform can be viewed as a paradigm for the mathematical investigation of singular integrals in general. This opens up a whole area of study in singular integral equations. Hilbert transform theory has served as a springboard to the study of singular integrals in n-dimensional Euclidean space. The Hilbert transform has played an important role in addressing some fundamental questions in the theory of Fourier series. This transform has a very close connection to some areas of complex analysis, and it plays an essential role in the theory of Fourier transforms of causal functions. The Hilbert transform is the key ingredient in characterizing operators that commute with the translation and dilation operators. Parts of all of the aforementioned topics are discussed in an introductory fashion in the following chapters.

Notes

The end-notes for each chapter provide sources, both books and journal articles, where additional reading on various topics may be pursued. The books that are recommended on standard topics reflect in large part the contents of the author's personal library. On many standard topics, particularly the background material covered in Chapter 2, the reader should be able to find a large number of additional reference texts beyond the ones cited. For a delightful account of the life and times of David Hilbert, intended for a general audience, see Reid (1996).

§1.1 For further reading on integral equations, consult Gakhov (1966), Hochstadt (1973), Tricomi (1985), Mikhlin and Prössdorf (1986), Pipkin (1991), Muskhelishvili (1992), and Kress (1999). Good sources on integral transforms with an applied emphasis include Sneddon (1972) and Davies (1978). The books by Zayed (1996) and Debnath and Bhatta (2007), and the individual accounts in Poularikas (1996a), are highly recommended reading.

§1.2 Hardy's work referenced in this book can be found in the seven volumes of his collected papers, Hardy (1966).

§1.3 Additional Hilbert transform pairs can be found in the nineteenth century literature; see, for example, Schlömilch (1848) or Bierens de Haan (1867). For some more recent collections of Hilbert transforms, see the following: Erdélyi *et al*. (1954, Vol. II), MacDonald and Brachman (1956), Smith and Lyness (1969), Alavi-Sereshki and Prabhakar (1972), and Hahn (1996a, 1996b). Hilbert transform relations of the type given in Eqs. (1.8) and (1.9) are due to the great French mathematician Augustin-Louis Cauchy.

§1.4 Some further reading on various applications of the Hilbert transform can be found in: Tricomi (1985, p. 173) and Zayed (1996, p. 287) for aerofoil theory; Wright and Hutchinson (1999) for the determination of oscillator phases for atomic motions; Ferry (1970), Booij and Thoone (1982), Madych (1990), and Herdman and Turi (1991), for elasticity theory; Aki and Richards (1980, p. 852) for crack propagation; Hinojosa and Mickus (2002) for the study of gravity gradient profiles; Červený and Zahradník (1975) for a review of geophysical applications; Weaver and Pao (1981), Beltzer (1983), and Bampi and Zordan (1992), for wave propagation theory; Duffin (1972), Nabighian (1984), and Sugiyama (1992), for aspects of potential theory; Sakai and Vanasse (1966) for an application in Fourier spectroscopy; Karl (1989), Hahn (1996a, 1996b), Oppenheim, Schafer, and Buck, (1999), for signal processing; and Lowenthal and Belvaux (1967), Herman (1980), Kohlmann (1996), Arnison *et al*. (2000), Davis, McNamara, and Cottrell (2000), and Shaik and Iftekharuddin (2003), for image reconstruction theory. A study of dispersion forces using dispersion theoretic techniques can be found in Feinberg, Sucher, and Au (1989). A number of applications have been made in Raman spectroscopy; see Chinsky *et al*. (1982), Stallard *et al*. (1983), Patapoff, Turpin, and Peticolis (1986), and Lee and Yeo (1994). For the development of a dispersion-type relation for the ground-state energy of two-electron atomic systems as a function of nuclear charge, see Ivanov and Dubau (1998). For further general reading on matters mathematical, see Butzer and Trebels (1968), Butzer and Nessel (1971), and Pandey (1996). A concise introductory account on the Hilbert transform can be found in Peters (1995).

Exercises

The table of Hilbert transforms in Appendix 1 should prove to be of value to you, both for checking the answers to a number of exercises throughout the book, and for solving some of the exercises.

1.1 Given

$$I(x,a) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\cos as \, \mathrm{d}s}{x-s},$$

where a is a constant, set up a differential equation by differentiation with respect to x, and hence determine the value for $I(x,a)$. Justify the differentiation step.

1.2 Given

$$P\int_{-\infty}^{\infty} f(s)\mathrm{d}s = \lim_{\varepsilon\to 0+}\left\{\int_{-\infty}^{x-\varepsilon} f(s)\mathrm{d}s + \int_{x+\varepsilon}^{\infty} f(s)\mathrm{d}s\right\},$$

show for

$$f(s) = (x-s)^{-1} \text{ that } P\int_{-\infty}^{\infty} f(s)\mathrm{d}s = 0.$$

1.3 What is the value of $\int_{-\infty}^{\infty} \mathrm{d}s/(x-s)$?

1.4 Show that $Hf(x)$ equals $-a/(x^2+a^2)$ for $f(s) = s/(a^2+s^2)$, where a is a positive constant. Hint: The identity

$$\frac{s}{(s^2+a^2)(x-s)} = \frac{1}{x^2+a^2}\left\{\frac{x}{x-s} + \frac{xs}{s^2+a^2} - \frac{a^2}{s^2+a^2}\right\}$$

leads to a straightforward calculation.

1.5 Show that $Hf(x)$ equals $x/a(x^2+a^2)$ for $f(s) = 1/a^2+s^2$, where a is a positive constant. Hint: The identity

$$\frac{1}{(s^2+a^2)(x-s)} = \frac{1}{x^2+a^2}\left\{\frac{x}{s^2+a^2} + \frac{1}{x-s} + \frac{s}{s^2+a^2}\right\}$$

simplifies the calculation.

1.6 If c is a constant, evaluate $H[c]$.

1.7 Evaluate $H[\sin(ax+b)]$, where a and b are real constants.

1.8 Evaluate $H[\cos(ax+b)]$, where a and b are real constants.

1.9 Evaluate $H[\sin^2(\alpha x)]$, where α is a real constant.

1.10 If α is a real constant, does $H[x^{-1}\sin(\alpha x)]$ converge?

1.11 For α a real constant, determine whether $H[x^{-1}\cos(\alpha x)]$ converges.

1.12 Prove Eq. (1.13).

1.13 For $f(x) = x(x^2 + \alpha^2)^{-1}$ with α a real constant greater than zero, how does Hf behave as $\alpha \to 0+$?

1.14 If

$$f(x) = \begin{cases} 0, & x < 0 \\ e^{-\alpha x}, & x \geq 0 \end{cases} \quad \text{with } \alpha > 0,$$

evaluate $Hf(x)$.

1.15 If $f(x) = \begin{cases} 0, & |x| > 1 \\ 1, & |x| \leq 1 \end{cases}$, evaluate $Hf(x)$.

1.16 For $a > 0$, is the statement $H[x^2(a^2 + x^2)^{-1}] = -ax\,(a^2 + x^2)^{-1}$, true or false?

2

Review of some background mathematics

2.1 Introduction

The principal objective of this chapter is to present some of the essential basic mathematical background that is employed in later sections. A good deal of this material should be straightforward for a well trained undergraduate mathematics or physics major; however, there are a few slightly more advanced topics that are treated concisely. For these topics, collateral reading in a standard text is highly recommended. Some suggestions of where to start additional reading are provided in the end-notes. The mathematically talented reader could bypass most of this chapter and skip to the derivations in Chapter 3.

Almost all the mathematical notation employed can be found in the List of symbols; please consult that list for the definition or for the first use of a particular symbol. Some common notational devices are reviewed first, and this is followed by a concise description of some of the more important mathematical tools, such as Fourier analysis, complex variable theory, and the basics of integration theory, i.e. topics that are central to later developments. Further extensions of some of these tools are given later as needed.

2.2 Order symbols $O(\,)$ and $o(\,)$

There are a number of situations in later sections where it is necessary to address the asymptotic behavior of $f(x)$, for x approaching some limit, which may be finite or infinite. In a number of circumstances, $f(x)$ may be so complicated that it may be an advantage to replace $f(x)$ by a simpler choice, $g(x)$, as x approaches the limit under consideration. Three different notations are employed to express the asymptotic behavior of $f(x)$.

If the value of the ratio $f(x)/g(x)$ is bounded by a constant as $x \to \alpha$, then $f(x)$ is at most *of the order of magnitude* of $g(x)$ as $x \to \alpha$. Symbolically this is written using the Bachmann order notation:

$$f(x) = O(g(x)). \tag{2.1}$$

Commonly, this notation is also used to imply, for a positive constant C, that $|f(x)| \leq C\,|g(x)|$ as x approaches some given limit. Some examples of the use of this notation are as follows. If $P_k(x)$ denotes a polynomial of degree k, then

$$P_k(x) = O(x^k), \quad \text{as } x \to \infty. \tag{2.2}$$

A second example is

$$\sin x = O(x), \quad \text{as } x \to 0. \tag{2.3}$$

If $f(x)$ is bounded for all x as $x \to$ some limit, then

$$f(x) = O(1), \quad \text{as } x \to \text{some limit}. \tag{2.4}$$

For example,

$$\exp(-|x|) = O(1), \quad \text{as } x \to \infty. \tag{2.5}$$

If $f(x)/g(x) \to 1$ as $x \to$ some limit, then $f(x)$ has the same order as $g(x)$, which is expressed symbolically as $f(x) \sim g(x)$ as x tends to its limit. This is clearly a more restrictive condition than Eq. (2.1).

If $f(x)/g(x) \to 0$ as x tends to its limit, then $f(x)$ has a smaller order of magnitude than $g(x)$, which is denoted symbolically by

$$f(x) = o(g(x)). \tag{2.6}$$

Some examples are as follows:

$$x^m = o(\mathrm{e}^x), \quad \text{as } x \to \infty, \quad \text{for } m > 0; \tag{2.7}$$

$$x^{-m} = o(1), \quad \text{as } x \to \infty, \quad \text{for } m > 0; \tag{2.8}$$

$$P_k(x) = o(x^{k+1}), \quad \text{as } x \to \infty; \tag{2.9}$$

and

$$\cos x - 1 = o(x), \quad \text{as } x \to 0. \tag{2.10}$$

If $f(x) = o(g(x))$ as x tends to its limit, this implies $f(x) = O(g(x))$.

2.3 Lipschitz and Hölder conditions

A function f satisfies the Lipschitz condition at a point x_0 if there exists a positive constant C such that

$$|f(x) - f(x_0)| \leq C|x - x_0|, \tag{2.11}$$

for all values of x in some neighborhood of x_0.

A function f satisfies the Hölder condition at a point x_0 if

$$|f(x) - f(x_0)| \leq C|x - x_0|^m, \tag{2.12}$$

for all values of x in some neighborhood of x_0, and C and m are positive constants. The parameter C is called the Hölder constant, and m is termed the order of the Hölder condition. The designations *index* and *exponent* are also employed in place of order. Equation (2.12) is also referred to as a Lipschitz condition of order m. This is written as

$$f \in \mathrm{Lip}\, m. \tag{2.13}$$

A function f satisfies a Hölder condition on an interval $[\alpha, \beta]$ if

$$|f(x_2) - f(x_1)| \leq C|x_2 - x_1|^m, \tag{2.14}$$

for all x_1 and x_2 on $[\alpha, \beta]$. A function satisfying a Hölder condition on an interval is continuous on that interval. The reader is reminded that square brackets, as in $[\alpha, \beta]$, denote the closed interval $\alpha \leq x \leq \beta$, and that parentheses (,) as in (α, β) designate the open interval $\alpha < x < \beta$.

2.4 Cauchy principal value

Consider an integral with a singularity present in the integration interval. As an example, suppose

$$f(t) = \int_\alpha^\beta \frac{\mathrm{d}x}{(t - x)}, \tag{2.15}$$

where t lies in the interval (α, β). If the evaluation of this integral is approached as an improper Riemann integral, then

$$\begin{aligned} f(t) &= \lim_{\varepsilon \to 0} \int_\alpha^{t-\varepsilon} \frac{\mathrm{d}x}{(t - x)} + \lim_{\rho \to 0} \int_{t+\rho}^\beta \frac{\mathrm{d}x}{(t - x)} \\ &= -\lim_{\varepsilon \to 0} \{\log|t - x|\}_\alpha^{t-\varepsilon} - \lim_{\rho \to 0} \{\log|x - t|\}_{t+\rho}^\beta \\ &= \lim_{\varepsilon \to 0} \lim_{\rho \to 0} \log\left|\frac{t - \alpha}{\beta - t} \frac{\rho}{\varepsilon}\right|. \end{aligned} \tag{2.16}$$

In this book the notation log always refers to a logarithm to the base e. The result in Eq. (2.16) can take on any value, depending on the value of the ratio ρ/ε. Since the integral in Eq. (2.15) is ill defined as it stands, a simple modification of the approach indicated in Eq. (2.16), first utilized by Cauchy in 1822 (Smithies, 1997), leads to the following:

$$
\begin{aligned}
f(t) &= \lim_{\varepsilon \to 0}\left[\int_{\alpha}^{t-\varepsilon} \frac{\mathrm{d}x}{(t-x)} + \int_{t+\varepsilon}^{\beta} \frac{\mathrm{d}x}{(t-x)}\right] \\
&= \log\left|\frac{t-\alpha}{\beta - t}\right|,
\end{aligned} \tag{2.17}
$$

which is clearly well defined for $t \neq \alpha$ and $t \neq \beta$. The limiting operation given in Eq. (2.17) is called the principal value of the integral, or, more appropriately, the Cauchy principal value of the integral. The common notations employed to symbolize the limiting process displayed in Eq. (2.17) are

$$
P\int f(x)\mathrm{d}x, \quad PV\int f(x)\mathrm{d}x, \quad VP\int f(x)\mathrm{d}x, \quad \int^{*} f(x)\mathrm{d}x, \quad ⨍ f(x)\mathrm{d}x,
$$

and $f(x)$ has a singularity in the interval over which the integral is evaluated. That is,

$$
P\int_{\alpha}^{\beta} f(x)\mathrm{d}x = \lim_{\varepsilon \to 0}\left[\int_{\alpha}^{t-\varepsilon} f(x)\mathrm{d}x + \int_{t+\varepsilon}^{\beta} f(x)\mathrm{d}x\right], \tag{2.18}
$$

where $f(x)$ has a singularity at $x = t$. The third of the five notational devices given, *VP*, (valeur principale) is seen in European writings. In a number of works, the standard integral sign is used to denote a principal value integral, with an explicit statement given to the reader that a principal value integral is under discussion. In the remainder of this work, the first of the five symbols just listed is employed.

It is not difficult to find examples of functions with singularities such that the limit in Eq. (2.18) is not finite. For the case $f(x) = (x - t)^{-2}$ with $t \in (\alpha, \beta)$, the Cauchy principal value integral in Eq. (2.18) diverges.

2.5 Fourier series

Some of the basics of Fourier series that find application in other chapters are reviewed in this section. It is possible in a number of cases to represent various experimental data sets by a Fourier series, from which important information can be extracted on taking the Hilbert transform. Fourier series arise in a natural way when the Hilbert transforms of periodic functions are considered.

2.5.1 Periodic property

Let the function f satisfy

$$
f(x \pm P) = f(x), \tag{2.19}
$$

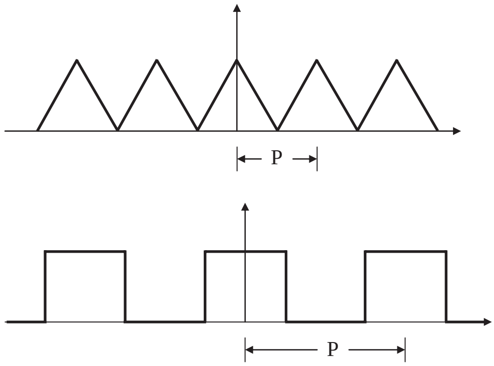

Figure 2.1. Two examples of periodic functions. The period for each example is P.

for all values of x, and $P > 0$. If Eq. (2.19) holds, f is called a *periodic* function, and the smallest P for which the equation holds is termed the period of f. Figure 2.1 illustrates two examples.

Some elementary examples of periodic functions are $\cos x$ and $\sin x$, since

$$\cos x = \cos(x + 2\pi) = \cos(x + 4\pi) = \cos(x + 6\pi) = \cdots, \tag{2.20}$$

$$\sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = \sin(x + 6\pi) = \cdots. \tag{2.21}$$

The period in both examples is 2π. For the functions $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$ with n a positive integer, the period is $2L$.

2.5.2 Piecewise continuous functions

A function f is *piecewise continuous* in an interval (a, b) if it satisfies the following requirements: the interval (a, b) can be dissected into a finite number of subintervals in which f is continuous, and that the limits as x approaches the endpoints of each subinterval are finite. The term *jump discontinuity* is employed in the following way. If the limits

$$f_{\downarrow}(c) \equiv f(c + 0) = \lim_{x \to c+0} f(x) \tag{2.22}$$

and

$$f_{\uparrow}(c) \equiv f(c - 0) = \lim_{x \to c-0} f(x) \tag{2.23}$$

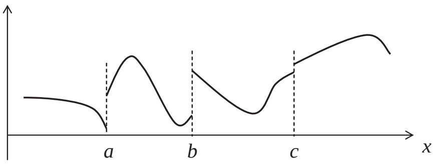

Figure 2.2. The discontinuities in the function occur at $x = a$, $x = b$, and $x = c$. The discontinuities are non-infinite.

exist, but are different, then the function has a jump discontinuity at the point c. A piecewise continuous function in an interval has a finite number of jump discontinuities in that interval. An example of a piecewise continuous function is shown in Figure 2.2.

2.5.3 Definition of Fourier series

Suppose that the function f is defined on an interval $(-L, L)$ and is periodic outside the interval with period $2L$, that is $f(x) = f(x + 2L)$, then the Fourier series expansion of f is defined by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right). \tag{2.24}$$

The coefficients a_n and b_n are called the Fourier coefficients. The coefficients can be determined in the following manner. Multiply both sides of Eq. (2.24) by $\cos(m\pi x/L)$ and integrate over the interval $(-L, L)$ to obtain

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx. \tag{2.25}$$

Multiplying both sides of Eq. (2.24) by $\sin(m\pi x/L)$ and integrating over the interval $(-L, L)$ yields

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{2.26}$$

To obtain these expressions, the following elementary integrals are employed:

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, m \neq n \\ L, m = n \end{cases}, \tag{2.27}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0, \tag{2.28}$$

where m and n take on integer values.

The conditions that are necessary for f to be expanded in the form of a Fourier series are examined next. If the following requirements (due to Dirichlet) hold:

(1) f is defined and single-valued on the interval $(-L, L)$, with the possible exception of a finite number of points;
(2) f is periodic with period $2L$;
(3) f and its derivative are piecewise continuous in $(-L; L)$,

then it is possible to write the following:

$$\frac{1}{2}\{f(x+0)+f(x-0)\} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right). \quad (2.29)$$

If x is not a point of discontinuity, then

$$f(x) = \frac{1}{2}\{f(x+0)+f(x-0)\}. \quad (2.30)$$

Two examples of well known Fourier series expansions are:

$$x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\sin nx, \quad \text{for } -\pi < x < \pi, \quad (2.31)$$

and

$$\log\left\{2\cos\left(\frac{x}{2}\right)\right\} = \sum_{n=1}^{\infty}(-1)^{n+1}\cos nx, \quad \text{for } -\pi < x < \pi. \quad (2.32)$$

The three conditions on f are known to be sufficient conditions to guarantee convergence. The conditions are not necessary; that is, if the three conditions as stated are not satisfied, then the series in Eq. (2.29) may or may not converge. From a practical standpoint, with a view to the type of applications to be discussed later, the Dirichlet conditions will almost always be satisfied.

An alternative Fourier series expansion is employed in later sections. With the same conditions stated previously, it is possible to write the Fourier series for f in complex form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}. \quad (2.33)$$

The coefficients c_n can be determined by multiplying both sides of Eq. (2.33) by $e^{im\pi x/L}$ and integrating over the interval $(-L, L)$, to obtain

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx. \quad (2.34)$$

If f is not continuous at the point x, then $1/2\{f(x+0)+f(x-0)\}$ replaces $f(x)$ in Eq. (2.33).

In later chapters it will be necessary to expand a function in terms of a complete set of orthonormal functions on a particular interval $[a, b]$. A set of real-valued functions $\phi_n(x), n = 0, 1, 2, \ldots$, is *orthogonal* on the interval $[a, b]$ if

$$\int_a^b \phi_n(x)\phi_m(x)\mathrm{d}x = 0, \quad \text{for } n \neq m. \tag{2.35}$$

The preceding formula can be generalized to include a weight function $w(x)$ in the integrand, for which case the $\phi_n(x)$ are termed orthogonal on the interval $[a, b]$ with weight $w(x)$. The set of functions is termed *orthonormal* if Eq. (2.35) is satisfied and

$$\int_a^b \phi_n^2(x)\mathrm{d}x = 1 \tag{2.36}$$

holds for all n. The last two results are often combined into the following single result:

$$\int_a^b \phi_n(x)\phi_m(x)\mathrm{d}x = \delta_{nm}. \tag{2.37}$$

The symbol δ_{nm} is called the Kronecker delta, and is defined by

$$\delta_{mn} = \begin{cases} 1, & \text{for } n = m \\ 0, & \text{for } n \neq m \end{cases} \quad \text{with } n, m \in \mathbb{Z}, \tag{2.38}$$

where $\mathbb{Z}$ denotes the set of integers, $0, \pm 1, \pm 2, \ldots$. On the interval $[0, \pi]$, the set of functions $\{(\surd(2/\pi)) \sin nx\}_{n=1}^{\infty}$ forms a complete orthonormal set. If a function f and its derivative f' are piecewise continuous in an interval $[a, b]$, then the function can be expanded in terms of a complete orthonormal set:

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \quad \text{for } a \leq x \leq b, \tag{2.39}$$

where the c_n are termed generalized Fourier coefficients. The partial sum of the first N terms of the series is defined as follows:

$$S_N(x) = \sum_{n=1}^{N} c_n\phi_n(x). \tag{2.40}$$

The set of functions ϕ_n is termed *complete* if

$$\lim_{N\to\infty} \int_a^b \{f(x) - S_N(x)\}^2 \, \mathrm{d}x = 0. \tag{2.41}$$

The generalized Fourier series for f converges in the mean square sense, also called converges in norm, if Eq. (2.41) holds.

2.5.4 Bessel's inequality

If $\{\phi_n\}$ form an orthonormal set on the interval $[a, b]$, then the coefficients in the expansion in Eq. (2.39) can be determined from

$$c_n = \int_a^b f(x)\phi_n(x)\mathrm{d}x. \tag{2.42}$$

By a straightforward calculation, it follows that

$$\begin{aligned}\int_a^b \{f(x) - S_N(x)\}^2\,\mathrm{d}x &= \int_a^b f^2(x)\mathrm{d}x - 2\int_a^b f(x)S_N(x)\mathrm{d}x + \int_a^b S_N^2(x)\mathrm{d}x \\ &= \int_a^b f^2(x)\mathrm{d}x - 2\sum_{n=0}^{\infty} c_n \sum_{m=0}^{N} c_m \int_a^b \phi_n(x)\phi_m(x)\mathrm{d}x \\ &\quad + \sum_{n=0}^{N} c_n \sum_{m=0}^{N} c_m \int_a^b \phi_n(x)\phi_m(x)\mathrm{d}x \\ &= \int_a^b f^2(x)\mathrm{d}x - \sum_{n=0}^{N} c_n^2.\end{aligned} \tag{2.43}$$

It follows on taking the limit $N \to \infty$ that

$$\sum_{n=0}^{\infty} c_n^2 \le \int_a^b f^2(x)\mathrm{d}x, \tag{2.44}$$

which is Bessel's inequality.

2.6 Fourier transforms

In this section a few properties of Fourier transforms that are employed later are concisely reviewed. Ideas developed from Fourier analysis provide one route to the derivation of the conjugate function relationships, and these ideas have an important bearing on issues connected with the numerical evaluation of Hilbert transforms. They also play a central role in the connection of causal arguments with analytic behavior, and hence to the Hilbert transform relations.

2.6.1 Definition of the Fourier transform

Suppose f is an absolutely integrable function on $\mathbb{R}$, that is

$$\int_{-\infty}^{\infty} |f(x)|\mathrm{d}x < \infty; \tag{2.45}$$

then the Fourier transform of f, denoted by $\mathcal{F}f$, is defined by

$$\mathcal{F}f(x) = \int_{-\infty}^{\infty} f(s)\mathrm{e}^{-\mathrm{i}xs}\, \mathrm{d}s. \tag{2.46}$$

The notation $\hat{f}$ is also widely used in the mathematics literature to denote $\mathcal{F}f$, and this is employed occasionally in this book when the use of $\mathcal{F}$ would make the expression rather cluttered. In the engineering disciplines, the notation $\hat{f}$ is sometimes used to denote the Hilbert transform, a convention not employed in this book. Condition (2.45) is sufficient, but not necessary, for the existence of the Fourier transform. There are functions which do not satisfy Eq. (2.45), but still have a well defined Fourier transform. An example is the function $x^{-1}\sin x$. Weaker conditions on f can be given to ensure that $\mathcal{F}f$ is defined (Titchmarsh, 1948, chap. 3; Champeney, 1987, p. 47; Papoulis, 1962, p. 9).

Let $g(x) = \mathcal{F}f(x)$, then the inverse transform, written symbolically as $\mathcal{F}^{-1}g(x)$, is given by

$$f(s) = \mathcal{F}^{-1}g(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\mathrm{e}^{\mathrm{i}xs}\, \mathrm{d}x. \tag{2.47}$$

The Fourier transform is also commonly defined in the following ways:

$$\mathcal{F}f(x) = \int_{-\infty}^{\infty} f(s)\mathrm{e}^{-\mathrm{i}2\pi xs}\, \mathrm{d}s, \quad \mathcal{F}^{-1}f(x) = \int_{-\infty}^{\infty} f(s)\mathrm{e}^{\mathrm{i}2\pi xs}\, \mathrm{d}s, \tag{2.48}$$

$$\mathcal{F}f(x) = \frac{1}{\surd(2\pi)}\int_{-\infty}^{\infty} f(s)\mathrm{e}^{-\mathrm{i}xs}\, \mathrm{d}s, \quad \mathcal{F}^{-1}f(x) = \frac{1}{\surd(2\pi)}\int_{-\infty}^{\infty} f(s)\mathrm{e}^{\mathrm{i}xs}\, \mathrm{d}s, \tag{2.49}$$

$$\mathcal{F}f(x) = \int_{-\infty}^{\infty} f(s)\mathrm{e}^{\mathrm{i}xs}\, \mathrm{d}s, \quad \mathcal{F}^{-1}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\mathrm{e}^{-\mathrm{i}xs}\, \mathrm{d}s. \tag{2.50}$$

The alternative definitions given in Eqs. (2.48) and (2.49) reflect that the placement of the 2π factor is arbitrary. Equations (2.48) and (2.49) have a more symmetric appearance than the pair given by Eqs. (2.46) and (2.47). The definitions of the Fourier transform given in Eqs. (2.46) and (2.50) differ only in the sign of the exponent term. In formulas involving both the Fourier and Hilbert transforms, this difference in sign can lead to some formulas differing by a minus sign. This occurs widely in the scientific and engineering literature. Since the Hilbert transform can also be defined with one of two possible sign choices, the reader needs to be alert when reading literature sources, where unexpected signs may be traceable either to the choice of definition of the Hilbert transform or the Fourier transform employed. An important formula where this sign issue arises is given in Section 5.2 (see Eq. (5.3)).

Using the definition given in Eq. (2.46), two examples are as follows:

$$f(x) = \mathrm{e}^{-ax^2}, \text{ with } a > 0, \quad \mathcal{F}f(x) = \surd\left(\frac{\pi}{a}\right)\mathrm{e}^{-x^2/(4a)}. \tag{2.51}$$

For

$$f(x) = \frac{1}{a^2 + x^2}, \text{ with } a > 0, \quad \mathcal{F}f(x) = \pi a^{-1} e^{-a|x|}. \tag{2.52}$$

2.6.2 Convolution theorem

The convolution of two functions f and g defined on $\mathbb{R}$ is given by the following definition:

$$\{f * g\}(x) = \int_{-\infty}^{\infty} f(s)g(x - s)\,ds, \tag{2.53}$$

provided the integral exists. The notation $f(x) * g(x)$ is also commonly employed in place of $\{f * g\}(x)$. A result that will turn out to be extremely useful at a later juncture is the convolution theorem for Fourier transforms. This result states that the Fourier transform of the convolution of f and g is the product of the Fourier transforms of f and g; thus,

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\}\mathcal{F}\{g\}. \tag{2.54}$$

The convolution operation defined in Eq. (2.53) obeys the commutative and distributive laws. The associative property is obeyed for functions in certain classes; see, for example, Howell (2001, p. 376).

2.6.3 The Parseval and Plancherel formulas

If the functions f and $g \in L^2(\mathbb{R})$ and $\hat{f}$ and $\hat{g}$ denote the corresponding Fourier transforms, then

$$\int_{-\infty}^{\infty} \left|\hat{f}(x)\right|^2 dx = \int_{-\infty}^{\infty} |f(x)|^2\,dx \tag{2.55}$$

and

$$\int_{-\infty}^{\infty} \hat{f}(x)\hat{g}^*(x)dx = \int_{-\infty}^{\infty} f(x)g^*(x)dx. \tag{2.56}$$

The notation L denotes the class of Lebesgue integrable functions, a topic that is discussed in some detail in Section 2.11, and the superscript $*$ denotes the complex conjugate. Mathematicians call the first of these two relations Parseval's formula (sometimes the Parseval–Plancherel formula), and physicists associate the formula with Rayleigh. The second equation is designated as Plancherel's formula, although a number of authors refer to both equations as Parseval's formula. These results have a number of practical applications, and there are analogs of these formulas involving the Hilbert transform. Each formula can be obtained rather quickly from the other. Equation (2.56) can be established as follows:

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{f}(x)\hat{g}^*(x)\mathrm{d}x &= \int_{-\infty}^{\infty} \hat{f}(x)\left(\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} g^*(y)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} g^*(y)\left(\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \hat{f}(x)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}x\right)\mathrm{d}y \\ &= \int_{-\infty}^{\infty} g^*(y)f(y)\mathrm{d}y. \end{aligned} \tag{2.57}$$

The symmetric distribution of the 2π term has been used to avoid these factors in the final formula. If the definition in Eq. (2.46) is employed, then the right-hand side of Eq. (2.57) must be multiplied by a factor of 2π. The interchange of order of integration can be justified using Fubini's theorem (see Section 2.12). Setting $g = f$ in Eq. (2.57) recovers Eq. (2.55).

2.7 The Fourier integral

Combining Eqs. (2.24), (2.25), and (2.26) leads to

$$\begin{aligned} f(x) =& \frac{1}{2L}\int_{-L}^{L} f(s)\,\mathrm{d}s + \frac{1}{L}\sum_{n=1}^{\infty}\left(\int_{-L}^{L} f(s)\cos\left(\frac{n\pi s}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}s\right. \\ &\left. + \int_{-L}^{L} f(s)\sin\left(\frac{n\pi s}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}s\right) \\ =& \frac{1}{2L}\int_{-L}^{L} f(s)\mathrm{d}s + \frac{1}{L}\sum_{n=1}^{\infty}\left(\int_{-L}^{L} f(s)\cos\left[\frac{n\pi(s-x)}{L}\right]\mathrm{d}s\right), \end{aligned} \tag{2.58}$$

where the trigonometric identity

$$\cos(A - B) = \sin A \sin B + \cos A \cos B \tag{2.59}$$

has been used. If the substitution $n\pi/L = t_n$ is employed, and $\delta t = t_{n+1} - t_n$, then

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(s)\mathrm{d}s + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-L}^{L} f(s)\cos[t_n(s-x)]\mathrm{d}s\right)\delta t. \tag{2.60}$$

If f is integrable on $\mathbb{R}$, then in the limit as $L \to \infty$ the first term on the right-hand side of Eq. (2.60) vanishes, and the sum in Eq. (2.60) can be replaced by an integral, so that

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\mathrm{d}t\int_{-\infty}^{\infty} f(s)\cos[t(s-x)]\mathrm{d}s, \tag{2.61}$$

which can be rewritten as follows:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} f(s)\mathrm{e}^{\mathrm{i}t(s-x)}\, \mathrm{d}s. \tag{2.62}$$

Both of the preceding two results are referred to as the Fourier integral formula. Equation (2.61) is occasionally useful for the situation where $f(x)$ has particular symmetry properties. The reader is reminded of the following definitions of even and odd functions. An even function satisfies the condition $f(-x) = f(x)$, and an odd function satisfies $f(-x) = -f(x)$. An immediate consequence is

$$\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x, \quad \text{for } f(x) \text{ even}, \tag{2.63}$$

and

$$\int_{-a}^{a} f(x)\mathrm{d}x = 0, \quad \text{for } f(x) \text{ odd}. \tag{2.64}$$

For $f(x)$ even, Eq. (2.61) simplifies, with the aid of Eq. (2.59), to

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \cos xt\, \mathrm{d}t \int_{0}^{\infty} \cos ts\, f(s)\mathrm{d}s; \tag{2.65}$$

and for $f(x)$ an odd function,

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \sin xt\, \mathrm{d}t \int_{0}^{\infty} \sin ts\, f(s)\mathrm{d}s. \tag{2.66}$$

Equations (2.65) and (2.66) are the Fourier cosine and Fourier sine integral formulas, respectively.

2.8 Some basic results from complex variable theory

In this section, some of the essential basic ideas on functions of a single complex variable are summarized. More involved results are developed elsewhere as required. The theory of analytic functions plays a central role in several developments in Hilbert transform theory. This branch of mathematics is the key link between the experimental notion of causality and the reciprocal relations that bear the names of Kramers, Kronig, and Bode, and also dispersion theoretic methods. On the mathematical side, analytic function theory provides a connecting link between conjugate functions expressed as Fourier series, Fourier integrals, or as Hilbert transforms. Analytic function theory represents a powerful set of techniques for solving problems in many branches of pure and applied mathematics.

A complex variable z is defined by

$$z = x + \mathrm{i}y, \tag{2.67}$$

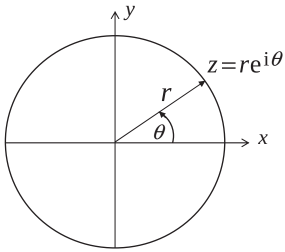

Figure 2.3. Argand diagram displaying the point z in terms of the polar variables r and θ.

where i is the imaginary unit, with the property that $\mathrm{i}^2 = -1$, and x and y are real numbers. Parenthetically it is noted that the symbol j is used in various branches of engineering in place of i. Since z is determined by an ordered pair (x, y), it can be conveniently represented diagrammatically by points in the xy-plane, called the complex plane, or, as it is also termed, an Argand diagram. In many applications, a polar representation of a complex number is of considerable value. The point z in Figure 2.3 can be characterized by (x, y) or by (r, θ), where r, the modulus, and θ, the argument, are respectively given by

$$\operatorname{mod} z = |z| = |x + \mathrm{i}y| = r = \sqrt{(x^2 + y^2)} \tag{2.68}$$

and

$$\arg z = \theta = \tan^{-1}\left(\frac{x}{y}\right). \tag{2.69}$$

An idea that will find later use is the ability to express z in a polar representation about some point z_0, which is not the origin of the x–y coordinates. In this case,

$$z = z_0 + R\mathrm{e}^{\mathrm{i}\theta}, \tag{2.70}$$

where the circle on which z lies has a radius R and origin z_0, and the angle θ is measured from the new origin.

A function f is continuous at a point z_0 if, for a given $\varepsilon > 0$, there exists a number δ such that

$$|f(z) - f(z_0)| < \varepsilon, \tag{2.71}$$

for all points z in a domain satisfying the condition $|z - z_0| < \delta$. If f is a single-valued function in a domain of the complex plane, then f is differentiable at a point z_0 of that domain, if

$$\frac{f(z) - f(z_0)}{z - z_0}$$

tends to a unique limit as $z \to z_0$, and this limit is denoted $f'(z_0)$, that is

$$f'(z_0) = \left.\frac{\mathrm{d}f(z)}{\mathrm{d}z}\right|_{z=z_0}. \tag{2.72}$$

If a function f is single-valued and differentiable at every point z in a domain of the complex plane, then the function is called *analytic* in that domain. The terms *holomorphic* and *regular* are often used synonymously with analytic. If a function f is differentiable in a domain, except for a finite number of points, those points are called the *singular* points (or singularities) of the function.

There are several types of singular points that will be encountered later. The key types are as follows. A singular point z_0 is termed *isolated* if there is a neighborhood of sufficiently small radius ε surrounding the point containing no other singular points. If the following condition holds for positive integer m,

$$\lim_{z\to z_0} (z - z_0)^m f(z) = a \neq 0, \tag{2.73}$$

then the point z_0 of $f(z)$ is termed a *pole of order m*. The case where $m = 1$ is called a *simple pole*. If a function f can be defined at a singular point z_0, that is a meaning can be attached to $\lim_{z\to z_0} f(z)$, then the singularity is called a *removable* singularity. A singularity that is neither an isolated singularity nor a pole, is termed an *essential* singularity. The latter is defined more rigorously later in Section 2.8.5. The preceding discussion of singularities can be recast in terms of the Laurent series expansion for an analytic function, and that topic is also treated in Section 2.8.5. Here are some examples of the types of singularities that have just been discussed. The function $f(z) = (z^2 - \alpha^2)^{-1}$ has simple poles at $z = \alpha$ and $z = -\alpha$, while the function $f(z) = (z - \beta)^{-4}$ has a pole of order four at $z = \beta$; in both cases, the singularities are isolated. A function with a removable singularity is $f(z) = z^{-2}(1 - \cos z)$, which is finite in the limit $z \to 0$. The function $f(z) = \sin z^{-1}$ has an essential singularity at $z = 0$.

Let $f(z)$ be written in the form

$$f(z) = u(x, y) + \mathrm{i}v(x, y), \tag{2.74}$$

where $u(x, y)$ and $v(x, y)$ are real-valued functions. Then the necessary and sufficient conditions for f to be analytic in a domain of the complex plane is that the four partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ exist and satisfy in that domain the following:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{2.75}$$

These are called the Cauchy–Riemann equations.

Functions that are not single-valued will be encountered in later chapters. Consider the case $f(z) = z^{1/n}$ with $n = 2, 3, 4, \ldots$. This is an example of a multiple-valued function. The case $n = 2$ is examined in the sequel in some detail. If z is expressed in polar notation, that is $z = re^{i\theta}$, then

$$f(z) = \sqrt{(r)}\, e^{i\theta/2}, \tag{2.76}$$

where θ denotes the angle shown in Figure 2.4. If one complete cycle around the contour labeled Γ is made, starting and returning to the point P, then $\theta \to \theta + 2\pi$, and

$$f(z) = \sqrt{(r)}\, e^{i(\theta+2\pi)/2} = -\sqrt{(r)}\, e^{i\theta/2}, \tag{2.77}$$

which results in a different value for the function. By making another complete circuit of the contour, the original value is obtained. A point in the complex plane is called a *branch point* if the value of a function is altered when one complete cycle around the point is completed. The point $z = 0$ is a branch point of the function $\sqrt{z}$. The function $f(z) = \sqrt{z}$ can be converted to a single-valued function by agreeing to restrict θ, so that

$$\begin{aligned} f(z) &= \sqrt{(r)}\, e^{i\theta/2}, \quad \text{for } 0 \leq \theta < 2\pi, \\ f(z) &= \sqrt{(r)}\, e^{i(\theta+2\pi)/2}, \quad \text{for } 0 \leq \theta < 2\pi. \end{aligned} \tag{2.78}$$

The function f can be regarded as being defined on two separate sheets, the collection of which is called the Riemann surface for the function, as in Figure 2.5.

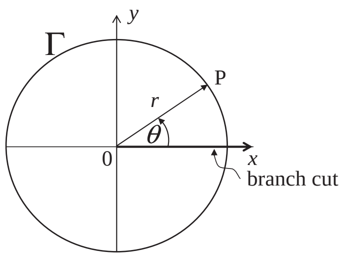

Figure 2.4. Branch cut and polar notation for the function $f(z) = \sqrt{z}$.

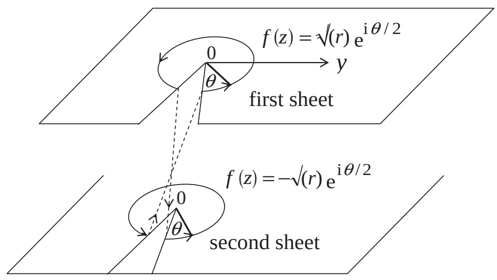

Figure 2.5. Riemann surfaces for the function $\sqrt{z}$.

To achieve a connection between the two sheets a *branch line* (or *branch cut*) is made from $z = 0$ to $z = \infty$, often taken along the positive real axis, although it can be set at any angle from the real axis, and all points that lie on this line are removed from the definition of the function. The outcome is that, on each sheet, the function is single-valued and continuous.

2.8.1 Integration of analytic functions

To carry out an integration of a function of a complex variable, it is necessary to specify the path of integration. Consider the curve shown in Figure 2.6. If the integral from (z_1, z_2) is to be evaluated, it is necessary to specify the path of the contour, that is

$$\int_{z_1}^{z_2} f(z)\mathrm{d}z \to \int_C f(z)\mathrm{d}z, \tag{2.79}$$

where the notation $\int_C$ symbolizes not only the endpoints, but also the pathway between them. A contour, or piecewise smooth arc, is said to be closed if the starting and ending points are the same. A contour described parametrically by $z(t)$ is called *smooth* if $z(t)$ has a continuous derivative along the entire contour; it is called *piecewise smooth* if the contour can be represented by a finite sum of smooth arcs. Figure 2.7 indicates a *simple* closed contour and Figure 2.8 illustrates a non-simple closed contour.

An integral taken around a closed contour C is often indicated by $\oint_C$. The sense of direction for the integral needs to be specified. Unless stated to the contrary, all contour integrals will assume a *counter-clockwise* sense. This is sometimes indicated with an arrow pointing in the counter-clockwise sense on the circle for the integral symbol just given.

In the following discussion, and for the rest of this book, the focus is on regions (designated $\mathcal{R}$) that are *simply connected*. A simply connected region $\mathcal{R}$ is one where

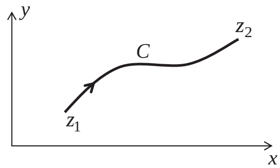

Figure 2.6. Contour showing the path between z_1 and z_2.

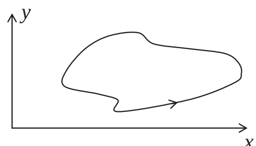

Figure 2.7. Simple closed contour.

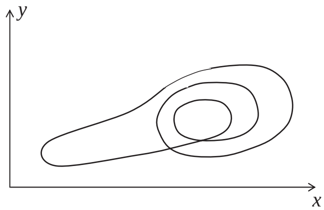

Figure 2.8. Non-simple closed contour.

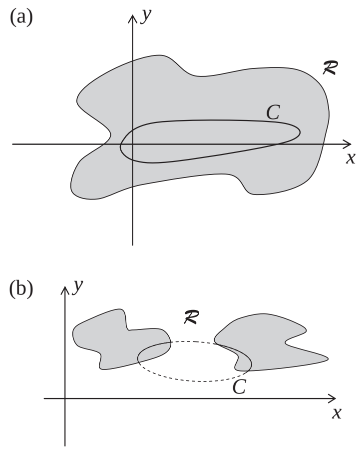

Figure 2.9. (a) A simply connected region. (b) A region that is not simply connected.

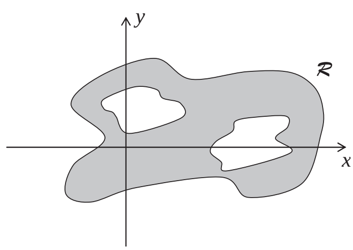

Figure 2.10. A multiply connected region.

any closed contour in $\mathcal{R}$ contains only points belonging to $\mathcal{R}$. Figure 2.9(a) shows a simply connected region, and Figure 2.9(b) illustrates a region that is not simply connected, that is, it is disconnected.

The dashed portion of the curve C in Figure 2.9(b) does not lie entirely in $\mathcal{R}$. Figure 2.10 illustrates a multiply connected region. In this case, it is possible to connect points within $\mathcal{R}$ by different contours which cannot be continuously deformed into one another without passing through points not in the region $\mathcal{R}$.

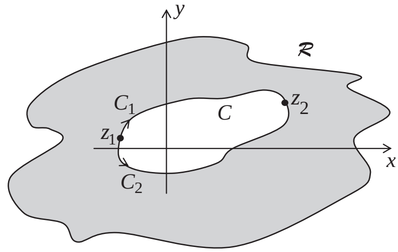

Figure 2.11. C_1 is the contour from z_1 to z_2 in the clockwise sense and C_2 is the contour from z_1 to z_2 in the counter-clockwise sense. C denotes the entire contour.

2.8.2 Cauchy integral theorem

One of the two results that will prove to be central in later developments is the Cauchy integral theorem (the other is the Cauchy integral formula, which is discussed in Section 2.8.3). The key result of this section is also referred to as the Cauchy–Goursat theorem. If $f(z)$ is analytic and if $f'(z)$ is continuous at each point of a domain, including the boundary C, then

$$\oint_C f(z)\mathrm{d}z = 0. \tag{2.80}$$

As a simple example, consider the choice $f(z) = z^2$, with the contour C taken to be a circle of radius r and center at the origin, then it is straightforward calculation (using $z = re^{i\theta}$) to show that Eq. (2.80) holds true. There are a number of proofs of this theorem available (see the end-notes), with different starting assumptions. Equation (2.80) with the conditions specified is due to Cauchy. The hypothesis requiring $f'(z)$ to be continuous was discovered by Goursat to be unnecessary. That leads to the statement of the Cauchy–Goursat theorem as: let $f(z)$ be analytic in a domain and on its boundary C. Then Eq. (2.80) holds.

A result that has considerable practical value is based on the type of closed contour shown in Figure 2.11. For $f(z)$ analytic in the region $\mathcal{R}$, it follows that

$$\int_C f(z)\mathrm{d}z = \int_{-C_1} f(z)\mathrm{d}z + \int_{C_2} f(z)\mathrm{d}z = 0 \tag{2.81}$$

and hence

$$\int_{C_1} f(z)\mathrm{d}z = \int_{C_2} f(z)\mathrm{d}z. \tag{2.82}$$

In writing Eq. (2.81), it is assumed that the contour is traversed in the counter-clockwise orientation, which is taken to be the positive sense along the curve. In later sections, a number of contours are examined that include segments that, in some appropriate limit, have the same form as in Eq. (2.81). When this occurs, the evaluation of the overall contour integral is simplified. Equation (2.80) has a number of important practical applications in applied fields, and in particular to topics to be discussed later.

There is a converse theorem due to Morera. Let $f(z)$ be continuous throughout a region $\mathcal{R}$; if

$$\oint_C f(z)\mathrm{d}z = 0 \tag{2.83}$$

for every contour C in $\mathcal{R}$, then $f(z)$ is analytic throughout $\mathcal{R}$.

2.8.3 Cauchy integral formula

The Cauchy integral formula is one of the most powerful results in analytic function theory. The formula finds extensive application in the discussion of Hilbert transforms. Let $f(z)$ be analytic within and on a closed contour C, then, if z_0 is an interior point of C,

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{z - z_0}\, \mathrm{d}z. \tag{2.84}$$

This is a rather remarkable mathematical result: the value of $f(z)$ at an interior point z_0 is determined by the values taken by the function on a prescribed contour encircling the point. There is no analog of this result in the theory of functions of a real variable.

As an elementary example, consider the integral

$$\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{z - 2}\, \mathrm{d}z,$$

where C is a circular contour of radius R centered at $z = 2$ and $f(z) = z^2$. Then a straightforward calculation, on writing $z - 2 = R\mathrm{e}^{\mathrm{i}\theta}$, yields the following:

$$\begin{aligned}\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{z - 2}\, \mathrm{d}z &= \frac{1}{2\pi \mathrm{i}} \oint_C \frac{z^2}{z - 2}\, \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_0^{2\pi} \frac{(2 + R\mathrm{e}^{\mathrm{i}\theta})^2 \mathrm{i} R\mathrm{e}^{\mathrm{i}\theta}\, \mathrm{d}\theta}{R\mathrm{e}^{\mathrm{i}\theta}} \\ &= \frac{1}{2\pi} \int_0^{2\pi} (2 + R\mathrm{e}^{\mathrm{i}\theta})^2\, \mathrm{d}\theta \\ &= 4.\end{aligned} \tag{2.85}$$

2.8.4 Jordan's lemma

A function that is analytic everywhere in the finite part of the complex plane, except at a finite number of poles, is said to be *meromorphic*. Jordan's lemma is as follows. If C denotes a semicircular contour with radius R and center the origin, and $f(z)$ is

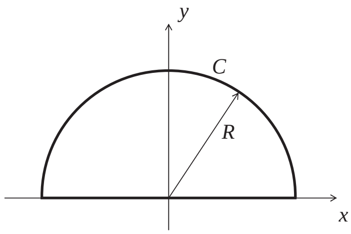

Figure 2.12. Semicircular contour in the upper half plane.

meromorphic in the upper half plane and satisfies the condition $|f(z)| \to 0$ uniformly as $|z| \to \infty$ for $0 \leq \arg z \leq \pi$, then, for $m > 0$,

$$\lim_{R\to\infty} \int_C \mathrm{e}^{\mathrm{i}mz} f(z)\mathrm{d}z = 0. \tag{2.86}$$

The first part of the stated condition means that there exists a radius R_0 chosen sufficiently large to enclose all the singularities of $f(z)$. The uniformity constraint implies that a constant $\varepsilon > 0$ can be selected so that, for some radius $R > R_0$, and for z on C it follows that

$$|f(z)| \leq \varepsilon. \tag{2.87}$$

The contour being employed is shown in Figure 2.12. Semicircular contours of this type find routine application in many problems of scientific interest. They are most frequently converted into closed semicircular contours by incorporating a segment of the real axis from $-R$ to R. The conditions given in the statement of Jordan's lemma for Eq. (2.86) to hold are sufficient. The result actually applies under weaker conditions. These conditions are encountered in later sections.

2.8.5 The Laurent expansion

Let $f(z)$ be analytic within and on the boundary of a circle C, centered at the point $z = z_0$, then the value of the function f at any point within C can be represented by the power series given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{2.88}$$

where

$$a_n = \frac{1}{n!} \frac{\mathrm{d}^n f(z)}{\mathrm{d}z^n}\bigg|_{z=z_0} = \frac{1}{2\pi\mathrm{i}} \int_C \frac{f(z)\mathrm{d}z}{(z - z_0)^{n+1}}. \tag{2.89}$$

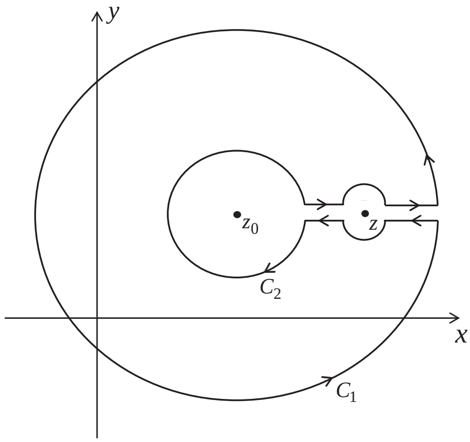

Figure 2.13. The circular contours C_1 and C_2 have a common origin at z_0.

This is the generalization of the well known Taylor series expansion to cover analytic functions. Its proof, which starts with the Cauchy integral formula, is straightforward, and is left as an exercise for the reader.

If the function $f(z)$ is not analytic in C, then the following approach can be employed. Suppose $f(z)$ is analytic in the annular region and on the concentric circular contours C_1 and C_2 (see Figure 2.13), then $f(z)$ can be expanded in the power series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \tag{2.90}$$

where

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(z)\mathrm{d}z}{(z - z_0)^{n+1}} \tag{2.91}$$

and Γ is any contour contained within the annular region and encircling the point z_0. The contributions from the horizontal sections of the contour adjacent to the point z cancel as the radius of the semicircular arcs at this point shrink to zero. The directional orientation of these straight line segments is obviously important to obtain the cancellation. The proof of Eq. (2.90) follows directly upon integrating $(z' - z)^{-1} f(z')$ (with respect to z') around the contour shown in Figure 2.13 and making appropriate expansions of the denominators for the contour integrals on C_1 and C_2. The expansion in Eq. (2.90) is the Laurent series for an analytic function. The contribution to the Laurent series from the terms with $n \geq 0$ is called the *analytic part*, and the remaining terms are called the *principal part*.

The Laurent series facilitates the discussion of singular points. Suppose that $f(z)$ has an isolated singularity at the point $z = z_0$ and is analytic everywhere else within and on the boundary of a circle C centered at $z = z_0$. Then the Laurent series converges for all points z in the circle C except at the point z_0. If

$$f(z) = \cdots \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots, \tag{2.92}$$

then $f(z)$ has a *pole of order* $m(m > 0)$ at $z = z_0$ if $a_n = 0$ for $n = -(m+1), -(m+2), \ldots$. If the principal part is *non-terminating*, that is there is no value m satisfying the aforementioned condition, then $f(z)$ has an *essential singularity* at $z = z_0$. When $f(z)$ has a pole of order m, the coefficient a_{-1} is called the residue of $f(z)$ at $z = z_0$. For a pole of order one, the residue, which is denoted by $\mathrm{Res}(z_0)$ (the notation $R_0(z_0)$ is also sometimes seen in the literature), can be determined from the following formula:

$$a_{-1} \equiv \mathrm{Res}(z_0) = \lim_{z \to z_0} (z - z_0) f(z). \tag{2.93}$$

When the context does not make it clear that the function f is under consideration, then the notation $\mathrm{Res}\{f(z)\}_{z=z_0}$ is employed to designate the residue at $z = z_0$. For a pole of order m, it is easy to show from the Laurent series that

$$a_{-1} \equiv \mathrm{Res}(z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \{(z - z_0)^m f(z)\}. \tag{2.94}$$

2.8.6 The Cauchy residue theorem

The Cauchy residue theorem, or, as it is frequently called, the residue theorem, plays a central role in a number of places in this book. It is an essential bit of mathematical machinery required for the evaluation of many Hilbert transforms. Suppose $f(z)$ is analytic within and on a closed contour C except for the point $z = z_0$, where the function has a pole of order m. With reference to Figure 2.14, application of the

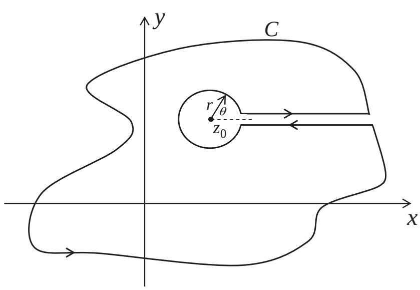

Figure 2.14. Standard modification of a contour to avoid a singularity.

Cauchy–Goursat theorem leads to

$$
\begin{aligned}
\oint_C f(z)\,\mathrm{d}z &= \oint_C \sum_{n=-m}^{\infty} a_n (z-z_0)^n\,\mathrm{d}z \\
&= \sum_{n=-m}^{\infty} a_n \oint_C (z-z_0)^n\,\mathrm{d}z \\
&= \sum_{n=-m}^{\infty} a_n \int_0^{2\pi} r^{n+1}\mathrm{e}^{\mathrm{i}(n+1)\theta}\,\mathrm{i}\,\mathrm{d}\theta \\
&= 2\pi\mathrm{i} \sum_{n=-m}^{\infty} a_n \delta_{n,-1} \\
&= 2\pi\mathrm{i}\, a_{-1}.
\end{aligned} \tag{2.95}
$$

The preceding argument can be generalized in a straightforward fashion to include a finite number of poles. The residue theorem is stated thus: suppose $f(z)$ is analytic within and on the boundary of a closed contour C, except for a finite number of poles, then

$$\oint_C f(z)\mathrm{d}z = 2\pi\mathrm{i} \sum_{j=1} \mathrm{Res}(z_j), \tag{2.96}$$

where the sum runs over the finite number of poles at the points z_j located within C, and $\mathrm{Res}(z_j)$ denotes the residue corresponding to the pole at $z = z_j$. Cauchy's theorem (Eq. (2.80)) obviously corresponds to the special case of Eq. (2.96) when there are no poles in the region enclosed by C.

2.8.7 Entire functions

A function analytic everywhere in the finite complex plane is called an *entire* function. The synonym sometimes employed is *integral* function. Two examples of entire functions are $\sin z$ and e^z. Note that the point at infinity, where the function may have an isolated singularity, is excluded from the definition of an entire function. If an entire function is bounded at infinity, then it can be shown that the function is a constant in the whole complex plane (Liouville's theorem).

Let $f(z)$ be a meromorphic function with poles, assumed for simplicity to be simple, at $z_k, k = 1, \ldots, n$, and arranged in order of increasing absolute value. Denote by $r_k, k = 1, \ldots, n$, the residues of the poles at z_k. Let C be a circular contour of radius R centered at the origin, passing through no singularity of $f(z)$ and enclosing n poles. Suppose that $f(z)$ is bounded above by the constant M assumed independent of n, $|f(z)| < M$, then, in the limit $R \to \infty$, a short calculation using the residue theorem leads to

$$f(z) = f(0) + \sum_{k=1}^{n} r_k \left\{ \frac{1}{z - z_k} + \frac{1}{z_k} \right\}. \tag{2.97}$$

This is referred to as the Mittag–Leffler expansion of $f(z)$.

Suppose $g(z)$ is a meromorphic function whose poles are all simple (of order one) and are located at $\alpha_1, \alpha_2, \alpha_3, \ldots$, then an entire function $f(z)$ can be constructed from

$$f(z) = g(z) - \sum_{n=1}^{\infty} \frac{a_n}{z - \alpha_n}, \tag{2.98}$$

where a_n is the residue of $g(z)$ at $z = \alpha_n$. If it is assumed that $g(z)$ is bounded at infinity, then so is $f(z)$, and hence $f(z)$ is a constant. Therefore from Eq. (2.98) it follows that

$$g(z) = g(0) + \sum_{n=1}^{\infty} \left\{ \frac{a_n}{z - \alpha_n} + \frac{a_n}{\alpha_n} \right\}. \tag{2.99}$$

This result is a particular case of the Mittag–Leffler expansion of a meromorphic function.

Suppose that $f(z)$ is entire and has first-order zeros at $\beta_1, \beta_2, \beta_3, \ldots$, none of which are located at the origin, then from this function the following meromorphic function can be constructed, which has simple poles at $\beta_1, \beta_2, \beta_3, \ldots$:

$$g(z) = \frac{\mathrm{d} \log f(z)}{\mathrm{d}z} = \frac{1}{f(z)} \frac{\mathrm{d}f(z)}{\mathrm{d}z}. \tag{2.100}$$

The logarithm of a complex number being understood in the sense that if $z = \mathrm{e}^w$, then $w = \log z$. The function $\log z$ is multi-valued. If $g(z)$ satisfies the conditions necessary to carry out a Mittag–Leffler expansion, then

$$\frac{\mathrm{d} \log f(z)}{\mathrm{d}z} = \left. \frac{\mathrm{d} \log f(z)}{\mathrm{d}z} \right|_{z=0} + \sum_{n=1}^{\infty} \left\{ \frac{1}{z - \beta_n} + \frac{1}{\beta_n} \right\}. \tag{2.101}$$

From Eq. (2.101) it follows that

$$\int_0^z \frac{\mathrm{d} \log f(z')}{\mathrm{d}z'} \,\mathrm{d}z' = \int_0^z \left[\left. \frac{\mathrm{d} \log f(z')}{\mathrm{d}z'} \right|_{z'=0} + \sum_{n=1}^{\infty} \left\{ \frac{1}{z' - \beta_n} + \frac{1}{\beta_n} \right\} \right] \mathrm{d}z'; \tag{2.102}$$

that is,

$$\log f(z) = \log f(0) + z \left[\frac{\mathrm{d} \log f(z)}{\mathrm{d}z} \right]_{z=0} + \sum_{n=1}^{\infty} \left[\log \left(\frac{\beta_n - z}{\beta_n} \right) + \frac{z}{\beta_n} \right], \tag{2.103}$$

which can be rewritten as follows:

$$f(z) = f(0)\exp\left\{z\left[\frac{\mathrm{d}\log f(z)}{\mathrm{d}z}\right]_{z=0}\right\}\prod_{n=1}^{\infty}\left(1-\frac{z}{\beta_n}\right)\mathrm{e}^{\beta_n^{-1}z}. \tag{2.104}$$

If $f(z)$ is an even function, the summation limits in Eq. (2.101) extend from $-\infty$ to ∞ with $n = 0$ omitted, and the condition $\beta_{-n} = -\beta_n$ applies, so, with the appropriate change to the lower summation limit, Eq. (2.103) simplifies to

$$f(z) = f(0)\prod_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty}\left(1-\frac{z}{\beta_n}\right)\mathrm{e}^{\beta_n^{-1}z} = f(0)\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\beta_n^2}\right). \tag{2.105}$$

Equations (2.104) and (2.105) are particular cases of the Weierstrass factorization formula. A simple example is $f(z) = z^{-1}\sin z$, for which the zeros are located at $\beta_n = -\beta_{-n} = n\pi$, for $n \in \mathbb{Z}$, and hence

$$\sin z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right). \tag{2.106}$$

The rate of growth of an entire function is characterized by its *order*. An entire function f is of order ρ if, for $\theta_1 < \theta < \theta_2$,

$$\lim_{r\to\infty}\sup\frac{\log\left|f(r\mathrm{e}^{\mathrm{i}\theta})\right|}{r^{\rho+\varepsilon}} = 0, \tag{2.107}$$

for each $\varepsilon > 0$, and uniformly for θ in the given range. The reader is reminded of the following set theory notation. A set S that is bounded above has a least upper bound that is called the *supremum*, denoted sup S, and if the set S is bounded below it has a greatest lower bound, called the *infimum*, which is denoted inf S. Equation (2.107) is frequently stated in terms of the maximum modulus, denoted $M(r)$,

$$M(r) = \max_{0\leq\theta\leq 2\pi}\left|f(r\mathrm{e}^{\mathrm{i}\theta})\right| \tag{2.108}$$

so that

$$\rho = \lim_{r\to\infty}\sup\frac{\log\log M(r)}{\log r}. \tag{2.109}$$

By convention, the order of a constant is zero. Another factor used in the characterization of entire functions is the *type*. An entire function $f(z)$ of order ρ has type σ given by

$$\sigma = \lim_{r\to\infty}\sup\frac{\log M(r)}{r^{\rho}}, \tag{2.110}$$

with $0 \leq \sigma \leq \infty$. A function $f(z)$ of order one and type σ $(\sigma < \infty)$ is said to be a function of exponential type σ. If

$$f(z) = O(e^{a|z|}), \quad z \to \infty, \tag{2.111}$$

where a is a positive constant, then σ is the lower bound of a.

2.9 Conformal mapping

Conformal mapping is a powerful problem-solving technique that finds widespread use in complex analysis. A few of the key aspects of conformal mapping that are used in later applications are summarized in this section.

The equations

$$u = u(x, y) \quad \text{and} \quad v = v(x, y) \tag{2.112}$$

set up a correspondence between a domain in the xy-plane and a domain of the uv-plane. If each point in the xy-plane corresponds to only one point in the uv-plane, and conversely, then this is called a one-to-one transformation or mapping. It can be shown that the transformation is one-to-one if u and v are continuously differentiable in a domain D and if the Jacobian of the transformation, $\partial(u, v)/\partial(x, y)$, where

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}, \tag{2.113}$$

does not vanish in D. If a mapping preserves the sense and the magnitude of angles, as in Figure 2.15, then the transformation is called *conformal*. If only the angles are preserved, the transformation is called *isogonal*.

Suppose C denotes a simple closed curve enclosing the region $\mathcal{R}$ in the complex z-plane, as in Figure 2.16, then the Riemann mapping theorem states that there exists a function $w = f(z)$ that is analytic in $\mathcal{R}$ (which is not the entire complex plane), which maps each point of $\mathcal{R}$ one-to-one into a corresponding point in the unit disc.

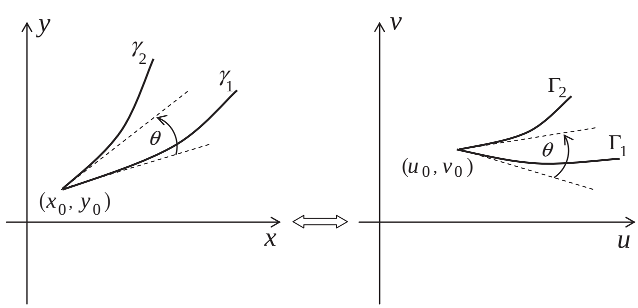

Figure 2.15. Preservation of the angle θ on mapping from the xy-plane to the uv-plane.

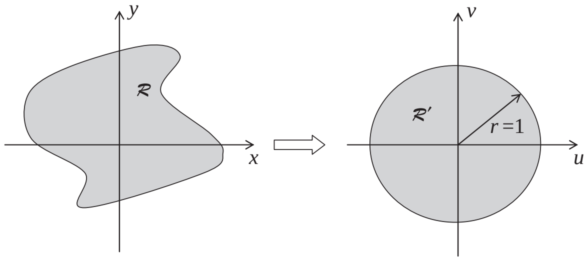

Figure 2.16. Mapping a region $\mathcal{R}$ in the xy-plane to the unit disc in the uv-plane.

Some of the simple and useful transformations are as follows:

$$w = z + \alpha \quad \text{(translation)}; \tag{2.114}$$

$$w = az \quad \text{(contraction for } 0 < a < 1\text{, stretching for } a > 1\text{)}; \tag{2.115}$$

$$w = z\mathrm{e}^{\mathrm{i}\theta} \quad \text{(rotation by } \theta\text{)}; \tag{2.116}$$

$$w = z^{-1} \quad \text{(inversion)}. \tag{2.117}$$

The *linear* transformation takes the form

$$w = \alpha z + \beta, \tag{2.118}$$

where α and β are complex constants. The *bilinear* transformation is given by

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \text{for } \alpha\delta - \beta\gamma \neq 0. \tag{2.119}$$

In later sections, there will be interest in mapping the upper half complex z-plane into the interior of the unit circle in the w-plane. This allows a connection to be made between functions analytic in the upper half plane and conjugate series. The conversion is accomplished using the Cayley transformation,

$$w = -\frac{z - \mathrm{i}}{z + \mathrm{i}}, \tag{2.120}$$

and is illustrated in Figure 2.17.

It is possible to give a generalization of the bilinear transformation given in Eq. (2.119) (see, for example, Miller, 1970, p. 218 and Nehari, 1975, p. 164). Tables of various conformal transformations can be found in a number of sources (see, for example, Kober (1957) or Krantz (1999b)).

A number of more intricate issues involving analytic functions that build off the theorems and definitions given in Sections 2.8 and 2.9 are discussed in subsequent chapters. It will quickly become apparent to the reader that analytic function theory is one of the principal underlying mathematical structures of much of the original work in the area of Hilbert transform theory.

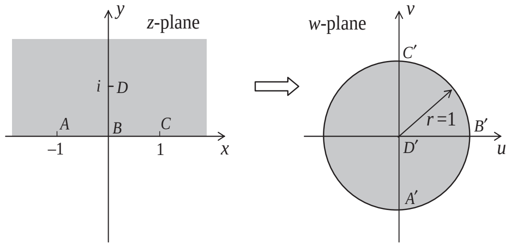

Figure 2.17. Mapping of the upper half of the xy-plane into the unit disc in the uv-plane.

2.10 Some functional analysis basics

The standard notation from set theory that is employed throughout is as follows. The elements x of a set having a property P are written $\{x : P\}$ or as $\{x|P\}$, for example, $A = \{x : a \leq x \leq b\}$ is read as: A is the set of x for which the condition $a \leq x \leq b$ holds. The empty set containing no members is written as $\varnothing$. If α is an element of A, this is written as $\alpha \in A$, and if α is not an element of A, this is denoted by $\alpha \notin A$. If the set A is a subset of the set B, then $A \subset B$ or $B \supset A$. If $A \subset B$ and $A \neq B$ then A is termed a proper subset of B, and the symbol $\subset$ is often used to indicate a proper subset. If the set A is a subset of the set B with the possibility that $A = B$, then $A \subseteq B$. The intersection of sets is written as $A \cap B$, which is the set of members belonging to both A and B, or $A \cap B = \{x : x \in A \text{ and } x \in B\}$. The union of sets is written as $A \cup B$, which is the set of members belonging to either A or B, that is $A \cup B = \{x : x \in A \text{ or } x \in B\}$. The difference (or complement) of two sets, $A - B$, is denoted by $A \backslash B$, and $A \backslash B = \{x : x \in A \text{ and } x \notin B\}$. If every point in a set S has a neighborhood lying in the set, then S is called an *open set*. An open interval (a, b) is an example of an open set. A set S is called a *closed set* if its complement is open. The closed interval $[a, b]$ is a closed set. The complement of $[a, b]$ is $(-\infty, a) \cup (b, \infty)$, which is an open set. If a set S of real numbers x is such that for all members of the set there exists numbers M and m such that $m \leq x \leq M$, then the set is called a *bounded set*. The set is bounded above by the upper bound M and bounded below by the lower bound m.

Let $S \subset \mathbb{R}$. A point p is a limit point (also termed an accumulation point) of a set of points S iff every neighborhood of p contains at least one point of S distinct from p. The abbreviation iff stands for *if and only if*. That is, if A is an open set containing p, $S \cap (A \backslash \{p\}) \neq \varnothing$. Let $S \subset \mathbb{R}$, then the closure of the set S denoted by $\bar{S}$ is the set S together with all the limits points of S.

A few definitions from elementary linear algebra that are employed in later chapters are recounted. The set of complex numbers is denoted by $\mathbb{C}$. Consider the set (designated V_n) of elements $\phi_1, \phi_2, \phi_3, \ldots, \phi_n$ ($\equiv \{\phi_i\}$) for which the following operations hold:

addition,

$$\phi_i = \phi_j + \phi_k; \tag{2.121}$$

multiplication by a scalar α,

$$\phi_i = \alpha\phi_j. \tag{2.122}$$

Suppose that the following conditions apply:

(i) if $\phi_i \in V_n$ and $\phi_j \in V_n$, then $(\phi_i + \phi_j) \in V_n$; (2.123)
(ii) if $\phi_i \in V_n$ and α is a (complex) constant, then $\alpha\phi_i \in V_n$; (2.124)
(iii) $\exists$ a null element 0, such that $\phi_i + 0 = \phi_i$, for any $\phi_i \in V_n$; (2.125)
(iv) for any $\phi_i \in V_n, \exists$ a $\phi_i' \in V_n$, such that $\phi_i' + \phi_i = 0$; (2.126)
(v) $1 \cdot \phi_i = \phi_i$, for all $\phi_i \in V_n$. (2.127)

Suppose also that the commutative and associative laws of addition apply:

(vi) $\phi_i + \phi_j = \phi_j + \phi_i$; (2.128)
(vii) $(\phi_i + \phi_j) + \phi_k = \phi_i + (\phi_j + \phi_k)$; (2.129)

and that for $\alpha, \beta \in \mathbb{C}$:

(viii) associative law of multiplication, $\alpha(\beta\phi_i) = (\alpha\beta)\phi_i$; (2.130)
(ix) distributive law, $(\alpha + \beta)\phi_i = \alpha\phi_i + \beta\phi_i$; (2.131)
(x) distributive law, $\alpha(\phi_i + \phi_j) = \alpha\phi_i + \alpha\phi_j$. (2.132)

Then the set $\{\phi_i\}$ having the properties (i)–(x) is called a *linear vector space* and ϕ_i are called vectors.

A group is a set of elements with a binary (usually multiplication) operation such that the following apply. (i) If a and b are elements of the group, then so is ab. (ii) For all a, b, and c in the group, then $a(bc) = (ab)c$. (iii) The group contains an identity element e such that $ae = ea = a$ for each element a of the group. (iv) For each element a of the group there is a unique inverse element a^{-1} in the group satisfying $aa^{-1} = a^{-1}a = e$. With the additional condition $ab = ba$ for elements a and b in the group, it is called a commutative or Abelian group. A set of elements for which a binary multiplication operation is defined that is associative, and in which the domain of the set includes all ordered pairs of the set, is termed a semigroup. There is no requirement for an identity or inverse element in a semigroup. A field is a set of elements for which the operations of addition and multiplication are defined and the following conditions hold. With addition as the group operation, the set is a commutative group. With multiplication as the group operation, with 0 omitted, the set is a group, with multiplication commutative. The distributive property holds for all elements in the set.

Two linear vector spaces X and Y over the same field are isomorphic if there exists a one-to-one mapping of the vectors of X onto the vectors of Y such that if $T : X \to Y$, then, for $x, y \in X$ and a scalar c, (i) $T(x+y) = Tx + Ty$; (ii) $T(cx) = cTx$.

The *inner product* of two vectors is denoted by (ϕ_i, ϕ_j). The inner product (ϕ_i, ϕ_j) for continuous functions on a suitable space is given by $\int \phi_i^*(t)\phi_j(t)\mathrm{d}t$, where the integral is over the domain for which $\phi_n(t)$ is defined. A notational device that is widely used, particularly in the physics community, is the so-called bra–ket symbolism of Dirac. In this convention, a vector is represented by the ket symbol $|\phi\rangle$ and the bra is $\langle\phi|$. The inner product is written in Dirac notation as $\langle\phi_i|\phi_j\rangle$. The inner product has the following properties:

$$(\phi_i, \phi_j) = (\phi_j, \phi_i)^*, \tag{2.133}$$

$$(\alpha\phi_i + \beta\phi_j, \phi_k) = \alpha(\phi_i, \phi_k) + \beta(\phi_j, \phi_k), \tag{2.134}$$

and

$$(\phi_i, \phi_i) \geq 0, \text{ and } (\phi_i, \phi_i) = 0, \quad \text{iff } \phi_i = 0. \tag{2.135}$$

In Eq. (2.133) the $*$ denotes complex conjugation. The *norm* (length) of a vector ϕ is written as follows:

$$\|\phi\| = \sqrt{(\phi, \phi)}. \tag{2.136}$$

The norm $\|\phi\|$ satisfies the following three properties:

$$\|\phi\| \geq 0, \text{ and } \|\phi\| = 0, \quad \text{iff } \phi = 0, \tag{2.137}$$

$$\|\alpha\phi\| = |\alpha| \, \|\phi\|, \tag{2.138}$$

and

$$\|\phi_i + \phi_j\| \leq \|\phi_i\| + \|\phi_j\|. \tag{2.139}$$

Equation (2.139) is termed the *triangle inequality*. A space equipped with a norm is termed a normed space.

A linear operator $\mathcal{O}$ on a vector space V_n is a procedure for determining, for each $\phi_i \in V_n$, a unique ϕ_j, also in V_n, that is

$$\phi_j = \mathcal{O}\phi_i. \tag{2.140}$$

For a scalar α, and linear operators $\mathcal{O}_1$ and $\mathcal{O}_2$, then the following must hold:

$$\mathcal{O}_1(\phi_i + \phi_j) = \mathcal{O}_1\phi_i + \mathcal{O}_1\phi_j, \tag{2.141}$$

$$(\mathcal{O}_1 + \mathcal{O}_2)\phi_i = \mathcal{O}_1\phi_i + \mathcal{O}_2\phi_i, \tag{2.142}$$

$$(\mathcal{O}_1\mathcal{O}_2)\phi_i = \mathcal{O}_1(\mathcal{O}_2\phi_i), \tag{2.143}$$

and

$$\mathcal{O}_1\alpha\phi_i = \alpha\mathcal{O}_1\phi_i. \tag{2.144}$$

In general, the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ do not commute, that is

$$\mathcal{O}_1\mathcal{O}_2\phi_i \neq \mathcal{O}_2\mathcal{O}_1\phi_i. \tag{2.145}$$

The commutator of two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ is defined by

$$[\mathcal{O}_1, \mathcal{O}_2] = \mathcal{O}_1\mathcal{O}_2 - \mathcal{O}_2\mathcal{O}_1. \tag{2.146}$$

This is a useful notational device for representing the difference in the order of application of a product of operators on some element of a vector space.

A particularly important class of operator equations is of the form

$$\mathcal{O}\phi_i = \alpha_i\phi_i, \tag{2.147}$$

which are termed *eigenvalue* problems. The function ϕ_i in Eq. (2.147) is called an *eigenfunction* (or, less commonly, a characteristic function), and the constant α_i is termed the *eigenvalue* (less frequently, the characteristic value).

The *adjoint* operator of $\mathcal{O}$ is denoted by $\mathcal{O}^+$ (the notation $\mathcal{O}^\dagger$ is also widely employed), and is defined by the requirement that

$$(\phi_i, \mathcal{O}^+\phi_j) = (\phi_j, \mathcal{O}\phi_i)^*, \quad \text{for any } \phi_i, \phi_j \in V_n. \tag{2.148}$$

The inverse of an operator is denoted by $\mathcal{O}^{-1}$, and is defined by the following relationship:

$$\mathcal{O}^{-1}\mathcal{O} = \mathcal{O}\mathcal{O}^{-1} = E, \tag{2.149}$$

where E is the identity element defined by

$$E\phi_i = \phi_i, \quad \text{for any } \phi_i \in V_n. \tag{2.150}$$

Note that the symbol E is also employed to designate the one-dimensional Euclidean space; however, the context should make it completely clear to the reader which meaning is intended. An operator that is *self-adjoint* satisfies the following condition:

$$\mathcal{O}^+ = \mathcal{O}. \tag{2.151}$$

Such an operator is called Hermitian. This group of operators plays a central role in quantum theory.

2.10.1 Hilbert space

An inner product on a linear vector space V assigns to every pair f and g in V a number (in general complex) denoted by (f, g). A space having an inner product defined on it is termed an inner product space, and is denoted by $\mathcal{H}$. A *Cauchy sequence* in $\mathcal{H}$, $\{\phi_n\}$, is defined by the following property: for every $\varepsilon > 0$, there exists an $N(\varepsilon)$ such that

$$\|\phi_n - \phi_m\| < \varepsilon, \text{ for } n, m > N(\varepsilon). \tag{2.152}$$

The sequence $\{\phi_n\}$ is convergent if there exists an element $\phi \in \mathcal{H}$ such that $\lim_{n\to\infty} \|\phi_n - \phi\| = 0$. A vector space is complete if each Cauchy sequence contained therein converges to an element contained in the space:

$$\lim_{n,m\to\infty} \|\phi_n - \phi_m\| = 0. \tag{2.153}$$

An inner product space for which every Cauchy sequence converges to an element of that space is called a *Hilbert space*.

An operator $\mathcal{O}$ acting on a Hilbert space $\mathcal{H}$ is called *isometric* if

$$\|\mathcal{O}\phi\| = \|\phi\|, \text{ for each } \phi \in \mathcal{H}. \tag{2.154}$$

If in addition to Eq. (2.154) the operator satisfies on $\mathcal{H}$ the condition

$$\mathcal{O}^{+} = \mathcal{O}^{-1}, \tag{2.155}$$

it is called a *unitary* operator.

2.10.2 The Hardy space H^p

Analytic functions will prove to be particularly important in the developments of the next chapter. The Hardy spaces deal with these functions and the two cases that arise are the unit disc $|z| < 1$ and the upper half plane $\operatorname{Im} z > 0$. Let f be analytic on the unit disc D and let $0 < p < \infty$, then $f \in H^p(D)$ if

$$\sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right) = \|f\|_{H^p}^p < \infty. \tag{2.156}$$

If $p = \infty$, then

$$\|f\|_{H^\infty} = \sup_{z\in D} |f(z)|. \tag{2.157}$$

For the upper half plane, $f \in H^p$ if $f(z)$ is analytic in the upper half plane complex plane and

$$\int_{-\infty}^{\infty} |f(x + iy)|^p \, dx < \infty, \tag{2.158}$$

for all $y > 0$.

2.10.3 Topological space

Topological notions enter into several aspects of an introductory account of the theory of Hilbert transforms. Some of the basic ideas are given in this subsection. Let X denote a nonempty set and let $\mathscr{T}$ be a collection of subsets of X. The collection $\mathscr{T}$ is called a *topology* on X iff the following three axioms hold:

(i) $\varnothing$ and X belong to $\mathscr{T}$;
(ii) the union of any number of sets in $\mathscr{T}$ belongs to $\mathscr{T}$;
(iii) The intersection of any two sets in $\mathscr{T}$ belongs to $\mathscr{T}$.

Consider the example where X is given by

$$X = \{a, b, c, d, e\}, \tag{2.159}$$

and let

$$\mathscr{T} = \{\varnothing, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}, \tag{2.160}$$

then the conditions (i)–(iii) hold, and $\mathscr{T}$ is a topology on X. However if

$$\mathscr{T} = \{\varnothing, X, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}\}, \tag{2.161}$$

then $\mathscr{T}$ is a not a topology on X, since $\{a, c, d\} \cap \{a, b, d, e\} = \{a, d\} \notin \mathscr{T}$, and so axiom (iii) fails in this example.

If X is a nonempty set and $\mathscr{T}$ is a topology on X, then the pair $(X, \mathscr{T})$ is called a topological space. The sets $\mathscr{T}_i$ contained in $\mathscr{T}$ are called open sets.

Let $(X, \mathscr{T}_1)$ and $(Y, \mathscr{T}_2)$ denote two topological spaces. The topological space $(X, \mathscr{T}_1)$ is a subspace of $(Y, \mathscr{T}_2)$, written $(X, \mathscr{T}_1) \subset (Y, \mathscr{T}_2)$, or, more simply, $\mathscr{T}_1 \subset \mathscr{T}_2$, when $X \subset Y$ and the open sets of $\mathscr{T}_1$ equal the intersections of X with the open sets of $\mathscr{T}_2$.

Given two topological spaces $(X, \mathscr{T}_1)$ and $(X, \mathscr{T}_2)$, then $\mathscr{T}_1 \subset \mathscr{T}_2$ if all the open sets of $\mathscr{T}_1$ are in $\mathscr{T}_2$. The topology $\mathscr{T}_1$ is called a *weaker topology* of X than $\mathscr{T}_2$ and $\mathscr{T}_2$ is termed a stronger topology of X than $\mathscr{T}_1$. Retaining the same X as in the preceding example, let

$$\mathscr{T}_1 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}; \tag{2.162}$$

and

$$\mathscr{T}_2 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\{a, b, c\}, \{a, b, c, d\}\}, \tag{2.163}$$

then clearly $\mathscr{T}_1 \subset \mathscr{T}_2$ and $\mathscr{T}_2$ is a stronger topology on X than $\mathscr{T}_1$.

Two topological spaces $(X, \mathscr{T}_1)$ and $(Y, \mathscr{T}_1)$ are termed topologically equivalent, or homeomorphic, iff there exists a map $f : X \to Y$ such that f is bijective and both f and f^{-1} are continuous. A bijective function is injective (*one-to-one*) and surjective (*onto*). The function f is called a homeomorphism. A function $f : X \to Y$ is invertible iff it is bijective. A continuous linear bijection $f : X \to Y$ is an isomorphism if the inverse f^{-1} is a continuous linear mapping $f^{-1} : Y \to X$. If $X = Y$, then f is called an automorphism on X.

2.10.4 Compact operators

The notion of a compact set is considered first. A set S is called a *compact set* if every infinite sequence of elements in S, $\{X_n\}$ say, has a subsequence that converges to an element of S. Compact sets are bounded. Every closed and bounded interval $[a, b]$ on the real line is an example of a compact set.

Let $\mathcal{O}$ denote an operator mapping the normed space X into the normed space Y. The operator $\mathcal{O}$ is called a *compact operator* if it maps every bounded set in X into a set in Y whose closure is compact. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two compact operators on $\mathcal{H}$, then $\mathcal{O}_1 + \mathcal{O}_2$ is a compact operator, and for a scalar α the operator $\alpha\mathcal{O}_1$ is also compact. Compact operators have important applications in the theory of integral equations. As an example, suppose the kernel function $K(x, y)$ is continuous on the square $0 \le x,\ y \le 1$, then the operator defined by

$$\mathcal{K}f(x) = \int_0^1 K(x, y)f(y)\mathrm{d}y, \tag{2.164}$$

is a compact operator on the space of square integrable functions on $[0, 1]$.

2.11 Lebesgue measure and integration

For most of the applications of the Hilbert transform technique that have been made in the physical sciences the functions of interest represent physically realizable systems and are invariably Hölder continuous over the domain of study. When this is not the case, the function can usually be modified in a straightforward manner so that this condition is true. In such cases, the standard Riemann interpretation of the integral is sufficient. However, it is extremely useful to generalize beyond Riemann integration. The first and obvious reason is that the Hilbert transform can be defined for a wider class of functions. In addition, the jump to reading original papers becomes a lot easier if the reader has had an exposure to Lebesgue integration. This is a vast topic, and only a few of the essential elements of the theory are outlined in this section.

It will help if the reader first recalls the notion of the Riemann integral. Let f be a continuous function defined on the interval $[a, b]$. Suppose the interval $[a, b]$ is divided into a set of subintervals by choosing partition points x_i such that

$$a = x_0 < x_1 < x_2 < x_3 \cdots < x_{m-1} < x_m = b, \tag{2.165}$$

as shown in Figure 2.18.

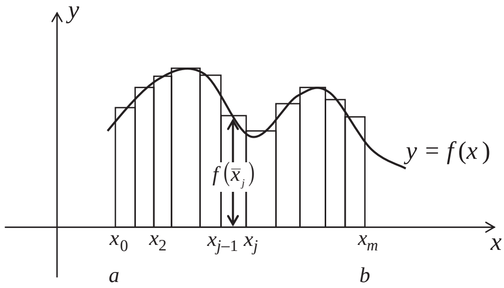

Figure 2.18. Riemann sum used to evaluate an integral.

The length of the jth subinterval is defined by

$$\Delta x_j = x_j - x_{j-1}, \tag{2.166}$$

and $\bar{x}_j$ is used to denote any value in the subinterval $[x_{j-1}, x_j]$. Let $\|p\|$ designate the longest subinterval, that is

$$\|p\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_m\}. \tag{2.167}$$

The Riemann integral of f over the interval $[a, b]$ is given by

$$\int_a^b f(x)\mathrm{d}x = \lim_{\|p\|\to 0} \sum_j f(\bar{x}_j)\Delta x_j, \tag{2.168}$$

assuming the limit exists. If f is positive in the interval $[a, b]$, a useful interpretation is that the integral corresponds to the area under the graph of f from a to b.

Before proceeding, a bit of terminology on types of convergence is reviewed. If the series $\sum_n |a_n|$ converges, then so does the series $\sum_n a_n$, and the series is said to be *absolutely convergent* or the series *converges absolutely*. The converse of the preceding statement is, of course, not true in general. The terminology *square summable* is used for a series that satisfies $\sum_n |a_n|^2 < \infty$. For integrals, $f(x)$ is absolutely integrable on (a, b) if $\int_a^b |f(x)|\mathrm{d}x$ converges. If this integral converges then the integral of $f(x)$ on (a, b) is called absolutely convergent. If $\int_a^b |f(x)|\mathrm{d}x$ is divergent but $\int_a^b f(x)\mathrm{d}x$ converges, then the latter integral is termed *conditionally convergent*. The convergence of $\int_a^b f(x)\mathrm{d}x$ does not, of course, imply that $\int_a^b |f(x)|\mathrm{d}x$ is convergent.

A sequence of functions $f_1(x) + f_2(x) + f_3(x) + \cdots$ converges to a sum $S(x)$ if, for a given $\varepsilon > 0$, there exists a number N (which in general will depend on both ε and x) such that $|S(x) - S_n(x)| < \varepsilon$ for $n > N$, where $S_n(x) = \sum_{k=1}^n f_k(x)$ denotes the nth partial sum. If there exists an N independent of x and depending only on ε, the sequence of functions *converges uniformly* to $S(x)$. In other words, for sufficiently large n the graph of $S_n(x)$ can be bounded above by $S(x) + \varepsilon$ and bounded below by $S(x) - \varepsilon$. If the $f_k(x)$ are continuous in an interval $x \in [a,b]$ and if $\sum_k f_k(x)$ converges uniformly to $S(x)$ for $x \in [a,b]$, then $S(x)$ is continuous for $x \in [a,b]$. Consider the example $f_k(x) = x^2/(1+x^2)^k, k = 1,2,\ldots,$ for $-1/2 \leq x \leq 1/2$. Then

$$S_n(x) = 1 - \frac{1}{(1+x^2)^n} \tag{2.169}$$

and hence, in the limit $n \to \infty$,

$$S(x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0. \end{cases} \tag{2.170}$$

Since $S(x)$ is not continuous in the interval $-1/2 \leq x \leq 1/2$, the sequence is not uniformly convergent for the stated range of x.

A useful test for determining uniform convergence is the Weierstrass M-test. If $|f_k(x)| \leq M_k$ for $k = 1,2,\ldots, x \in (a,b)$, and $\sum_k M_k$ converges, then $\sum_k f_k(x)$ converges uniformly for $x \in (a,b)$. For example, the sequence $x, x^2, x^3, \ldots$ for x in the interval $x \in (0,1/2)$ converges uniformly, since each term is bounded by $1/2$, and the sum $\sum_{k=1}^{\infty} 1/2^k$ converges.

A result that will find considerable application is the fact that a uniformly convergent series of continuous functions can be integrated term-by-term, that is

$$\int_a^b \sum_{k=1}^{\infty} f_k(x)\mathrm{d}x = \sum_{k=1}^{\infty} \int_a^b f_k(x)\mathrm{d}x. \tag{2.171}$$

Note that uniform convergence of a series is not a necessary condition for the term-by-term integration of the series. If the sequence $\{f_k(x)\}$, $k = 1,2,\ldots,$ is uniformly convergent in the interval $[a,b]$, then

$$\lim_{k\to\infty} \int_a^b f_k(x)\mathrm{d}x = \int_a^b \lim_{k\to\infty} f_k(x)\mathrm{d}x. \tag{2.172}$$

For an unbounded interval, for example $\mathbb{R}$, uniform convergence of the sequence $\{f_k(x)\}$ is not sufficient to establish the interchange in Eq. (2.178). If the sequence $\{f_k(x)\}, k = 1,2,\ldots,$ is continuous and the $f_k(x)$ have continuous first derivatives in $[a,b]$, and if $\sum_{k=1}^{\infty} f_k'(x)$ is uniformly convergent in the interval $[a,b]$, then

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} f_k'(x). \tag{2.173}$$

The integral $g(x) = \int_{\alpha}^{\infty} f(t,x)\mathrm{d}t$ is said to be uniformly convergent for $x \in [a,b]$ if, for each $\varepsilon > 0$, a number N can be found depending on $\varepsilon > 0$ but not on x such that

$$\left| g(x) - \int_{\alpha}^{\beta} f(t,x)\mathrm{d}t \right| < \varepsilon, \tag{2.174}$$

for all $\beta > N$ and all $x \in [a,b]$. The Weierstrass M-test can be used to determine uniform convergence of integrals. If a function $M(t) \geq 0$ can be found such that $|f(t,x)| \leq M(t)$ for $x \in [a,b]$, $t > \alpha$, and $\int_{\alpha}^{\infty} M(t)\mathrm{d}t$ converges, then $\int_{\alpha}^{\infty} f(t,x)\mathrm{d}t$ is uniformly convergent for $x \in [a,b]$. If $f(t,x)$ is continuous for $t > \alpha, x \in [a,b]$, and $g(x) = \int_{\alpha}^{\infty} f(t,x)\mathrm{d}t$ is uniformly convergent for $x \in [a,b]$, then $g(x)$ is continuous in the stated interval. For a point x_0,

$$\lim_{x \to x_0} \int_{\alpha}^{\infty} f(t,x)\mathrm{d}t = \int_{\alpha}^{\infty} \lim_{x \to x_0} f(t,x)\mathrm{d}t. \tag{2.175}$$

Also,

$$\int_{a}^{b} \left\{ \int_{\alpha}^{\infty} f(t,x)\mathrm{d}t \right\} \mathrm{d}x = \int_{\alpha}^{\infty} \left\{ \int_{a}^{b} f(t,x)\mathrm{d}x \right\} \mathrm{d}t. \tag{2.176}$$

More care is needed if the function $g(x)$ is also integrated over an infinite interval, even when the convergence of $g(x)$ is uniform. Further discussion on interchanging integration order is given in Section 2.12. If the previously stated conditions on $f(t,x)$ apply, and if $\partial f(t,x)/\partial x$ is continuous for $t > \alpha, x \in [a,b]$, and $\int_{\alpha}^{\infty} \partial f(t,x)/\partial x \, \mathrm{d}t$ converges uniformly in $[a,b]$, then, assuming α is independent of x, it follows that

$$\int_{\alpha}^{\infty} \frac{\partial f(t,x)}{\partial x} \mathrm{d}t = \frac{\partial}{\partial x} \int_{\alpha}^{\infty} f(t,x)\mathrm{d}t. \tag{2.177}$$

2.11.1 The notion of measure

To proceed further, the notion of the *measure* of a set is introduced. This is a generalization of the notion of length in one dimension, of area in two dimensions, and so on. Consider the set E of points $\{x : a \leq x \leq b\}$, then the following geometric interpretation of measure for the one-dimensional case can be given. The measure of the subset E_i satisfies the following conditions:

(i) $m(E_i) \geq 0$, for $E_i \subset E$; (2.178)
(ii) if $E_i \subseteq E_j$, then $m(E_i) \leq m(E_j)$; (2.179)
(iii) for an empty set $\varnothing, m(\varnothing) = 0$; (2.180)
(iv) the measure of sets is additive. For two disjoint sets E_i and E_j,

$$m(E_i + E_j) = m(E_i) + m(E_j), \text{ with } E_i \cap E_j = \varnothing. \tag{2.181}$$

If E_i denotes the set of points in the interval $[a, \alpha]$ and E_j denotes the set of points in the interval $[\beta, b]$, then the measure of these two sets if they are disjoint is the sum

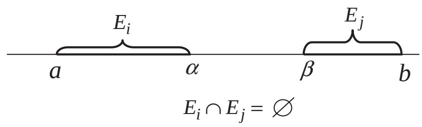

Figure 2.19. Disjoint intervals.

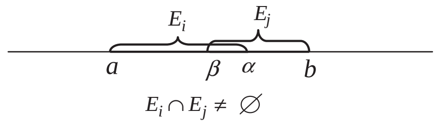

Figure 2.20. Overlapping intervals.

of the two line segments shown in Figure 2.19. If the sets E_i and E_j are not disjoint, as in Figure 2.20, then

$$m(E_i + E_j) = m(E_i) + m(E_j) - m(E_i \cap E_j). \tag{2.182}$$

In this manner the interval (α, β) is counted only once. The preceding appeals to the intuitive notion of the measure of a one-dimensional interval as the length of the interval. In a similar manner, an interpretation of the measure of sets in the Euclidean plane $\mathbb{R} \times \mathbb{R}$ can be given as the area of the associated domain over which the set is defined. The notation E^1 is used to denote the one-dimensional Euclidean space and E^n designates the n-dimensional Euclidean space.

Two sets A and B are termed *equivalent* if there is a one-to-one mapping from the set A onto B. A set S that is finite or equivalent to the set of all positive integers is called *countable*. A set S of real numbers is termed *open* if each point of S is at the center of an open interval totally contained in S. An *open cover* of the set A is the collection of open sets $\{S_\alpha\}$ such that $A \subset \bigcup S_\alpha$.

The Lebesgue outer measure of a set A, denoted $\mu_o(A)$, is defined by

$$\mu_o(A) = \inf \left\{ \sum_k m(A_k) \right\}, \quad \text{with } A \subset \bigcup_k A_k, \tag{2.183}$$

where the inf is taken over the class of all countable open coverings of A. Some properties of μ_o are as follows:

$$\mu_o(\varnothing) = 0; \tag{2.184}$$

$$0 \leq \mu_o(I) \leq \infty; \tag{2.185}$$

$$\mu_o(I) = l(I), \tag{2.186}$$

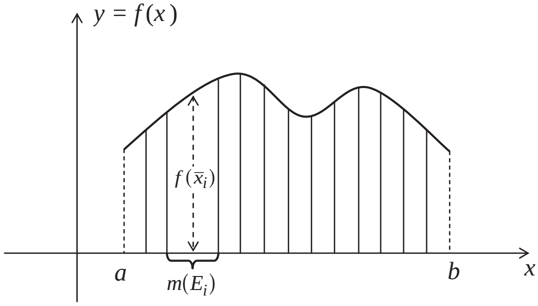

Figure 2.21. Riemann sum in terms of the measure of the intervals.

where $l(I)$ denotes the length of the interval I; and

$$\mu_o(\{\alpha\}) = 0, \tag{2.187}$$

where $\{\alpha\}$ denotes a particular point. This last result reflects the fact that points have zero length, that is, points are dimensionless.

The result for the Riemann integral given in Eq. (2.168) can be recast in terms of measure theoretic language. In Figure 2.21 the continuous function f is shown on the interval $[a, b]$, which is represented by the set of points E, and the interval $[a, b]$ is partitioned such that

$$E = \sum_{i=1}^{m} E_i, \quad \text{with } E_i \cap E_j = 0, \qquad \text{for any } i, j \text{ pair}, \tag{2.188}$$

where the measure of a subdivision E_i is denoted by $m(E_i)$. If $\bar{x}_i$ is any point belonging to the set E_i, then the sum $\sum_{i=1}^{m} f(\bar{x}_i)m(E_i)$ can be formed. By increasing the number of subsets indefinitely, so that for any E_i, $m(E_i) \to 0$, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{m(E_i)\to 0} \sum_i f(\bar{x}_i)m(E_i) \tag{2.189}$$

provided the limit exists and is independent of the subdivision process. The connection with the definition of the Riemann integral given previously should be apparent.

For what follows, it is useful to introduce the idea of a step function, defined on the interval $[a, b]$ by

$$\psi(x) = \begin{cases} c_j, & \text{for } x_{j-1} < x < x_j,\ j = 1, 2, \ldots, n \\ d_j, & \text{for } x = x_j,\ j = 0, 1, 2, \ldots, n. \end{cases} \tag{2.190}$$