# Non-Perturbative Field Theory 

From Two-Dimensional Conformal Field Theory to QCD in Four Dimensions

YITZHAK FRISHMAN<br>AND JACOB SONNENSCHEIN

CAMBRIDGE MONOGRAPHS<br>ON MATHEMATICAL PHYSICS

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# NON-PERTURBATIVE FIELD THEORY <br> From Two-Dimensional Conformal Field Theory to QCD in Four Dimensions 

Providing a new perspective on quantum field theory, this book gives a pedagogical and up-to-date exposition of non-perturbative methods in relativistic quantum field theory and introduces the reader to modern research work in theoretical physics.

It describes in detail non-perturbative methods in quantum field theory, and explores two-dimensional and four-dimensional gauge dynamics using those methods. The book concludes with a summary emphasizing the interplay between two- and four-dimensional gauge theories.

Aimed at graduate students and researchers, this book covers topics from twodimensional conformal symmetry, affine Lie algebras, solitons, integrable models, bosonization and 't Hooft model, to four-dimensional conformal invariance, integrability, large N expansion, Skyrme model, monopoles and instantons. Applications, first to simple field theories and gauge dynamics in two dimensions, and then to gauge theories in four dimensions and quantum chromodynamics (QCD) in particular, are thoroughly described.

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# To my wife Yehudith, mother Faiga <br> and daughter Einat 

Yitzhak Frishman

To my mother Hilda, wife Nava
and children Nir, Ori and Tal
Jacob Sonnenschein

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## Preface

Field theory is the framework with which one describes the theory of the standard model of elementary particles and their interactions. The electromagnetic sector (QED) of the standard model is understood extremely well using perturbation theory, but the color interaction (QCD) which is responsible for hadron physics can only be accounted for perturbatively for a limited set of observational data. Due to the fact that at long distances the color interaction is strongly coupled, one cannot reliably apply perturbative methods to extract, for instance, the spectrum of the hadrons. The arsenal of tools to handle strongly coupled systems is obviously much more limited than the one used for weakly coupled ones. Nevertheless, several methods to handle non-perturbative field theories have been developed. The main goal of this book is to expose the reader to those techniques and to describe their applications in two-dimensional and four-dimensional field theories and finally in QCD in four dimensions.

The topic of non-perturbative field theory is by itself very rich and it is clear that one cannot cover it in a non superficial manner in one book. Thus we had to make certain decisions about the flow of the book and about the topics that should be addressed. As for the former issue we have decided to present the book in three parts. In the first part we describe, in detail, the most important non-perturbative techniques of two-dimensional field theory. The reason for this is obvious since physical systems with one space dimension and one time dimension are the simplest and hence it is easier to grasp the non-perturbative tools when applied to these systems. In the second part of the book we study two-dimensional gauge theories with the emphasis on employing the techniques developed in the first part. The third part is devoted to the non-perturbative aspects of gauge dynamics in four dimensions. In this part we elevate the techniques of the first part to four dimensions and we examine to what extent gauge theories in four dimensions behave like their two-dimensional simplified analogs.

There are several books on the shelves discussing non-perturbative methods in general such as [66] and [182], there are books describing one particular method, like conformal field theory in two dimensions for instance [77], there are books that describe two-dimensional QCD, [2] and books that study various aspects of four-dimensional QCD, for example [151] and of course there are books on the basics of field theory, for example [37], [130], [173] and [215]. The aim of this book is three-fold, to review a package of non-perturbative methods, to present a picture which is close to the state-of-the-art in the topics described and to
demonstrate application of the methods in addressing several questions of gauge dynamics.

The particular methods we explore in Part 1 of the book associate with conformal field theory, with affine Lie algebras, with topological properties of fields, solitons and integrable models, with bosonization and with the large $N$ approximation.

In Part 2 we first present the basics of gauge field theories in two dimensions and in particular the bosonized version of them, we then describe the seminal large $N$ solution of 't Hooft of the mesonic spectrum of two-dimensional QCD; we address the mesonic spectrum using current algebra methods, we describe the discrete light-cone quantization of QCD with quarks in the fundamental representation and also adjoint quarks, we compute the spectrum of baryons and their properties in the strong coupling limit, we discuss the issue of confinement versus screening behavior, we analyze $Q C D_{2}$ using coset model and BRST techniques, and finally we digress and devote a chapter to generalized Yang-Mills theory on Riemann surfaces and their stringy nature.

In Part 3 we demonstrate the applications in four-dimensional gauge dynamics of conformal invariance, techniques of integrable models, of large $N$ expansion and of topology. In particular we devote chapters to Skyrmions, magnetic monopoles and gauge theory instantons.

As we have mentioned above we had to take decisions about what topics related to non-perturbative field theory we should not include. We decided not to address string theories, supersymmetric field theories and the holographic string (gravity)/gauge duality. The main reason for this decision was that to cover each of these topics requires a book in itself, or even more than one book. In fact certain subjects that we do cover in the book, like conformal field theory, magnetic monopoles or instantons would require a full book to cover properly. What we have tried to achieve is to describe the basic ideas of each topic and to demonstrate its application. We have also not treated subjects like anomalies, lattice formulations, sigma models, chiral Lagrangians and other non-perturbative topics.

Some topics described in the book are "fully established topics", in the sense that presumably the most important developments in those have been already achieved, for instance conformal field theory in two dimensions and bosonization in two dimensions. On the other hand some other topics of the book are under current intensive investigation and are certainly still not fully established. An example of the latter is integrability in four-dimensional gauge dynamics. The reason we have decided to include topics of the latter kind is that we wanted the book to be fairly up to date and useful to researchers investigating "modern" topics.

In the more basic issues we have made an effort to present the material in a pedagogical manner and to be self contained. For instance our discussion started from a free massless scalar field theory in two dimensions and gradually evolved
into general conformal field theories. In dealing with more advanced topics, like for instance instantons in four dimensions, the reader will need to consult with specialized references to obtain a more complete and wider picture of the topic.

Some of the content of the book, mainly in Part 2, is based on the research work of the authors, but most of the material is a review of the work of many researchers in the field.

The book is aimed for advanced Ph.D. students, post-docs and other newcomers to the arena of non-perturbative methods in field theory. The reader should definitely be equipped with a basic knowledge of field theory, group theory and algebra, differential equations, geometry and topology.

Throughout the book we refer to only a limited list of references. The number of scientific contributions to the topics discussed in this book is enormous and since we could not cover all of them we have referred to papers that initiated the various topics, and to review papers and books where a much more exhaustive list of references can be found.

We have made an attempt to keep the same notations throughout the book. However in certain instances we have changed notations during the course of the book, mainly to be in accordance with relevant literature. In these cases we specified explicitly the change in notation made.

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## PART I

Non-perturbative methods in two-dimensional field theory

## 1

## From massless free scalar field to conformal field theories

In this chapter we analyze the simplest field theory, which is the theory of a free massless scalar field in two space-time dimensions, one space and one time. ${ }^{1}$ The rich symmetry and algebraic structure of this theory encapsulates the basic concepts of two-dimensional conformal field theory, which will be the topic of the next chapter.

### 1.1 Complex geometry

It is convenient for the discussion of two-dimensional free scalar theory and later conformal field theories to introduce complex coordinates as follows: ${ }^{2}$

$$
\begin{equation*}
\xi=x^{0}+i x^{1} \quad \bar{\xi}=x^{0}-i x^{1} . \tag{1.1}
\end{equation*}
$$

We now take $x^{0}$ and $x^{1}$ to be in Euclidean space. Correspondingly we define the derivatives

$$
\begin{equation*}
\partial_{\xi}=\frac{1}{2}\left(\partial_{0}-i \partial_{1}\right) \quad \partial_{\bar{\xi}}=\frac{1}{2}\left(\partial_{0}+i \partial_{1}\right), \tag{1.2}
\end{equation*}
$$

which is a special case of the decomposition to components of vectors, namely

$$
\begin{array}{ll}
A_{\xi}=\frac{1}{2}\left(A^{0}-i A^{1}\right) & A_{\bar{\xi}}=\frac{1}{2}\left(A^{0}+i A^{0}\right) \\
A^{\xi}=\left(A^{0}+i A^{1}\right) & A^{\bar{\xi}}=\left(A^{0}-i A^{1}\right) \tag{1.3}
\end{array}
$$

The metric of the flat Euclidean space-time $\mathrm{d} s^{2}=\mathrm{d} x^{0^{2}}+\mathrm{d} x^{1^{2}}$ translates into $\mathrm{d} s^{2}=\mathrm{d} \xi d \bar{\xi}$, namely

$$
\begin{equation*}
g_{\xi \bar{\xi}}=g_{\bar{\xi} \xi}=\frac{1}{2}, \quad g^{\xi \bar{\xi}}=g^{\bar{\xi} \xi}=2, \quad g_{\xi \xi}=g_{\bar{\xi} \bar{\xi}}=g^{\xi \xi}=g^{\bar{\xi} \bar{\xi}}=0 \tag{1.4}
\end{equation*}
$$

With this metric at hand the scalar product of two vectors takes the form

$$
\begin{equation*}
A^{\mu} B_{\mu}=A^{\xi} B_{\xi}+A^{\bar{\xi}} B_{\bar{\xi}}=\frac{1}{2}\left(A^{\xi} B^{\bar{\xi}}+A^{\bar{\xi}} B^{\xi}\right) \tag{1.5}
\end{equation*}
$$

Complex components of higher-order tensors relate in a similar manner to the real components, in particular for a symmetric two-tensor (like the

[^0]

Fig. 1.1. The map between $\xi$ and $z$.
energy-momentum tensor),

$$
\begin{array}{r}
T \equiv T_{\xi \xi}=\frac{1}{4}\left(T_{00}-2 i T_{10}-T_{11}\right) \\
\bar{T} \equiv T_{\bar{\xi} \bar{\xi}}=\frac{1}{4}\left(T_{00}+2 i T_{10}-T_{11}\right) \\
T_{\bar{\xi} \xi}=T_{\xi \bar{\xi}}=\frac{1}{4}\left(T_{00}+T_{11}\right) . \tag{1.6}
\end{array}
$$

Often, especially in the context of string theory, the space direction is no longer $\mathcal{R}$, but rather is compactified on $S^{1}$ so that $x^{1} \equiv x^{1}+2 \pi$. For such a geometry it is convenient to introduce the following conformal map:

$$
\xi \rightarrow z=\mathrm{e}^{\xi}=\mathrm{e}^{x^{0}+i x^{1}}
$$

which maps the cylinder to the complex plane (see Fig. 1.1).
In particular the past $x^{0}=-\infty$ is mapped into the origin and the future $x^{0}=\infty$ into a circle with an infinite radius. It is clear that the relations between $(\xi, \bar{\xi})$ and $\left(x^{0}, x^{1}\right)$ derived above hold also between $(z, \bar{z})$ and $(\operatorname{Real}(z), \operatorname{Im}(z))$. The holomorphic and anti-holomorphic derivatives with respect to $z$ will be denoted by $\partial \equiv \partial_{z}$ and $\bar{\partial} \equiv \partial_{\bar{z}}$.

### 1.2 Free massless scalar field

The action $S$ of the free massless scalar field $\hat{X}(z, \bar{z})$ is

$$
\begin{align*}
S & =\int \mathrm{d}^{2} x \mathcal{L}=\frac{1}{8 \pi} \int \mathrm{~d}^{2} x \partial_{\nu} \hat{X} \bar{\partial}^{\nu} \hat{X} \\
& =\frac{1}{4 \pi} \int \mathrm{~d}^{2} \xi \partial_{\xi} \hat{X} \partial_{\bar{\xi}} \hat{X}=\frac{1}{4 \pi} \int \mathrm{~d}^{2} z \partial \hat{X} \bar{\partial} \hat{X} \tag{1.7}
\end{align*}
$$

where $\mathcal{L}$ is the Lagrangian density. The factor $\frac{1}{4 \pi}$ is used to match the normalization of the bosonic string theory (with $\alpha^{\prime}=2$ ). In the complex coordinate notation $(\xi, \bar{\xi})$ and $(z, \bar{z})$ the measure of the integral is $\mathrm{d}^{2} \xi=(i / 2) d \xi \wedge \mathrm{~d} \bar{\xi}$ and $\mathrm{d}^{2} z=(i / 2) \mathrm{d} z \wedge \mathrm{~d} \bar{z}$, respectively. Note that $\mathcal{L}$ is a local expression and thus is the same for the Euclidean plane or for any compact two-surface.

Varying the scalar field $\hat{X}(z, \bar{z}) \rightarrow \hat{X}(z, \bar{z})+\delta \hat{X}(z, \bar{z})$ induces a variation in the action of the form

$$
\begin{equation*}
\delta S=-\frac{1}{2 \pi} \int \mathrm{~d}^{2} z(\partial \bar{\partial} \hat{X}) \delta \hat{X} \tag{1.8}
\end{equation*}
$$

The action is thus extremized by configurations that solve the corresponding equation of motion

$$
\begin{equation*}
\partial \bar{\partial} \hat{X}=0 \tag{1.9}
\end{equation*}
$$

It is thus clear that $\partial \hat{X}$ is a holomorphic function and $\bar{\partial} \hat{X}$ is an anti-holomorphic function, and the most general solution takes the form

$$
\begin{equation*}
\hat{X}(z, \bar{z})=[X(z)+\bar{X}(\bar{z})] . \tag{1.10}
\end{equation*}
$$

### 1.3 Symmetries of the classical action

By construction the action is invariant under translations and $S O(2)$ rotations. Translations in $x^{0}$ and $x^{1}$ translate in complex coordinates to

$$
\begin{equation*}
z \rightarrow z+a ; \quad \bar{z} \rightarrow \bar{z}+\bar{a} \tag{1.11}
\end{equation*}
$$

where $a$ is a constant complex number, and the $S O(2)$ rotations, in infinitesimal form, to

$$
\begin{equation*}
\delta z=-i \epsilon z ; \quad \delta \bar{z}=i \epsilon \bar{z} \tag{1.12}
\end{equation*}
$$

where $\epsilon$ is an infinitesimal real parameter.
When we go back to Minkowski space, the $S O(2)$ rotations turn into $S O(1,1)$ transformations. In addition it is easy to realize that a shift of the field by a constant $A$,

$$
\begin{equation*}
\hat{X}(z, \bar{z}) \rightarrow \hat{X}(z, \bar{z})+A \tag{1.13}
\end{equation*}
$$

leaves the Lagrangian invariant. It is a special feature of two dimensions that the symmetry group of the action is in fact much richer since one can replace the constant $A$ with $A(z)$ and the constant $\bar{A}$ with $\bar{A}(\bar{z})$, which are arbitrary holomorphic and anti-holomorphic functions, respectively,

$$
\begin{equation*}
\hat{X}(z, \bar{z}) \rightarrow \hat{X}(z, \bar{z})+A(z) ; \quad \hat{X}(z, \bar{z}) \rightarrow \hat{X}(z, \bar{z})+\bar{A}(\bar{z}) . \tag{1.14}
\end{equation*}
$$

These are the affine current algebra transformations. ${ }^{3}$

[^1]In a similar manner the space-time translations (1.11) can also be elevated to holomorphic and anti-holomorphic transformations,

$$
\begin{equation*}
z \rightarrow f(z) ; \quad \bar{z} \rightarrow \bar{f}(\bar{z}) \tag{1.15}
\end{equation*}
$$

referred to as two-dimensional conformal transformations. Affine current algebra transformations and conformal transformations will be further discussed in Sections 1.10 and 1.11

### 1.4 Mode expansion

The mode expansion of the classical solution depends on the boundary conditions. For the case where the underlying two-dimensional manifold is the infinite plane, a standard Fourier transform is used:

$$
\begin{equation*}
\hat{X}\left(x^{0}, x^{1}\right)=\int \frac{\mathrm{d} k^{1}}{\sqrt{2 \pi} \sqrt{k^{0}}}\left[a\left(k^{1}\right) \mathrm{e}^{-i k \cdot x}+a^{\dagger}\left(k^{1}\right) \mathrm{e}^{i k \cdot x}\right] . \tag{1.16}
\end{equation*}
$$

If the range of the space coordinate is bounded, one may impose two types of boundary conditions, associated with closed and open strings. In the case of closed strings the boundary conditions

$$
\begin{equation*}
\hat{X}\left(x^{0}, x^{1}\right)=\hat{X}\left(x^{0}, x^{1}+2 \pi\right) \tag{1.17}
\end{equation*}
$$

are automatically obeyed by $\hat{X}(z, \bar{z})$. For this case the mode expansion is expressed in terms of a Laurent series,

$$
\begin{equation*}
\partial X=-i \sum_{n=-\infty}^{\infty} \frac{\alpha_{n}}{z^{n+1}} \quad \bar{\partial} \bar{X}=-i \sum_{n=-\infty}^{\infty} \frac{\bar{\alpha}_{n}}{\bar{z}^{n+1}} . \tag{1.18}
\end{equation*}
$$

Integrating this expansion we get

$$
\begin{equation*}
\hat{X}(z, \bar{z})=\mathcal{X}-i \mathcal{P} \ln (z \bar{z})+i \sum_{m=-\infty, m \neq 0}^{\infty}\left(\frac{\alpha_{m}}{m} z^{-m}+\frac{\bar{\alpha}_{m}}{m} \bar{z}^{-m}\right) \tag{1.19}
\end{equation*}
$$

with $\mathcal{X}$ a constant and

$$
\begin{equation*}
\mathcal{P}=\alpha_{0}=\bar{\alpha}_{0} . \tag{1.20}
\end{equation*}
$$

For open strings the boundary conditions are of Neumann type, namely

$$
\begin{equation*}
\partial_{1} \hat{X}\left(x^{0}, x^{1}=0\right)=\partial_{1} \hat{X}\left(x^{0}, x^{1}=\pi\right)=0 \Longrightarrow \partial \hat{X}(z, \bar{z}=z)=\bar{\partial} \hat{X}(z, \bar{z}=z) \tag{1.21}
\end{equation*}
$$

The corresponding mode expansion takes the form

$$
\begin{equation*}
\hat{X}(z, \bar{z})=\mathcal{X}-i \mathcal{P} \ln (z \bar{z})+i \sum_{m=-\infty, m \neq 0}^{\infty} \frac{\alpha_{m}}{m}\left(z^{-m}+\bar{z}^{-m}\right) \tag{1.22}
\end{equation*}
$$

### 1.5 Noether currents and charges

Associated with the symmetries (1.14) and (1.15) are conserved Noether currents and charges. In the Noether procedure one is instructed to elevate the global parameters of transformations into local ones and extract the associated currents from the variation of the action, namely $\delta S \sim \int \mathrm{~d}^{2} x J_{\mu} \partial^{\mu} \epsilon$. Let us apply this procedure first to the affine current algebra transformations so that we vary the action with respect to $\delta \hat{X}(z, \bar{z})=\epsilon(z, \bar{z})$ yielding

$$
\begin{equation*}
\delta S=\frac{1}{4 \pi} \int \mathrm{~d}^{2} z[\partial \epsilon(z, \bar{z}) \bar{\partial} \hat{X}(z, \bar{z})+\bar{\partial} \epsilon(z, \bar{z}) \partial \hat{X}(z, \bar{z})] \tag{1.23}
\end{equation*}
$$

Unlike the situation in more than two dimensions, and due to the fact that the symmetries (1.14) are in fact not only global ones but rather "half local", the currents

$$
\begin{equation*}
J \equiv \partial X ; \quad \bar{J} \equiv \bar{\partial} \bar{X} \tag{1.24}
\end{equation*}
$$

are holomorphic and anti-holomorphic conserved,

$$
\begin{equation*}
\bar{\partial} J \equiv \bar{\partial} \partial X=0 ; \quad \partial \bar{J} \equiv \partial \bar{\partial} \bar{X}=0 \tag{1.25}
\end{equation*}
$$

The classical currents are determined up to an overall constant.
A similar situation occurs with respect to the conformal transformation. Replacing in the infinitesimal version of (1.15) $\delta z \rightarrow \epsilon(z, \bar{z})$ and $\delta \bar{z} \rightarrow \bar{\epsilon}(z, \bar{z})$ one finds,

$$
\begin{equation*}
\delta S=\frac{1}{2 \pi} \int \mathrm{~d}^{2} z[\partial \bar{\epsilon}(z, \bar{z}) \bar{\partial} \hat{X}(z, \bar{z}) \bar{\partial} \hat{X}(z, \bar{z})+\bar{\partial} \epsilon(z, \bar{z}) \partial \hat{X}(z, \bar{z}) \partial \hat{X}(z, \bar{z})] \tag{1.26}
\end{equation*}
$$

The associated holomorphic and anti-holomorphic conserved energymomentum tensor components are

$$
\begin{equation*}
T=-\frac{1}{2} \partial X \partial X ; \quad \bar{T}=-\frac{1}{2} \bar{\partial} \bar{X} \bar{\partial} \bar{X} \tag{1.27}
\end{equation*}
$$

where the coefficients were chosen in a way that will turn out to be convenient when discussing the corresponding quantum generators.

### 1.6 Canonical quantization

Prior to imposing the canonical quantization condition one has to identify the time direction. There are several options. Using $x^{0}$ as the time direction, the
corresponding conjugate momentum of $\hat{X}(z, \bar{z})$ is

$$
\Pi=\frac{\delta \mathcal{L}}{\delta x_{0} \hat{X}}=\frac{1}{4 \pi} \partial_{0} \hat{X}
$$

and the standard quantization conditions are

$$
\begin{align*}
{\left[\hat{X}\left(x^{0}, x^{1}\right), \Pi\left(y^{0}, y^{1}\right)\right]_{x^{0}=y^{0}} } & =i \delta\left(x^{1}-y^{1}\right) \\
{\left[\hat{X}\left(x^{0}, x^{1}\right), \hat{X}\left(y^{0}, y^{1}\right)\right]_{x^{0}=y^{0}} } & =0 \\
{\left[\Pi\left(x^{0}, x^{1}\right), \Pi\left(y^{0}, y^{1}\right)\right]_{x^{0}=y^{0}} } & =0 \tag{1.28}
\end{align*}
$$

These conditions yield the standard algebra of the creation and annihilation operators for (1.16),

$$
\begin{equation*}
\left[a\left(k^{1}\right), a^{\dagger}\left(p^{1}\right)\right]=\delta\left(k^{1}-p^{1}\right) ; \quad\left[a\left(k^{1}\right), a\left(p^{1}\right)\right]=\left[a^{\dagger}\left(k^{1}\right), a^{\dagger}\left(p^{1}\right)\right]=0 \tag{1.29}
\end{equation*}
$$

Substituting the mode expansion (1.16) into the expressions of the Noether charges associated with the symmetries of the action (1.7) one finds that the energy-momentum operators are proportional to $a^{\dagger}(k) a(k)+a(k) a^{\dagger}(k)$ and hence their vacuum expectation values are proportional to $\delta(0) \sim L$, where $L$ is the size of the space direction. It is thus clear that for the infinite Euclidean plane (or a Minkowski space-time with space $\mathcal{R}$ ) these expectation values diverge. One then defines the normal ordered operators:

$$
\begin{equation*}
: \mathcal{O}: \equiv \mathcal{O}-<0|\mathcal{O}| 0> \tag{1.30}
\end{equation*}
$$

For free fields this is equivalent to ordering annihilation operators to the right of creation operators, and sufficient to make : $\mathcal{O}$ : finite.

Using the algebra of the creation and annihilation operators and the normal ordered Hamiltonian, the construction of the Fock space is standard. One defines the vacuum state $\mid 0>$ such that

$$
\begin{equation*}
a\left(k^{1}\right) \mid 0>=0 \tag{1.31}
\end{equation*}
$$

The states in the Fock space are

$$
\begin{equation*}
\prod_{i} a^{\dagger}\left(k_{i}\right)^{n i} \mid 0> \tag{1.32}
\end{equation*}
$$

and their energies, by applying the Hamiltonian,

$$
\begin{equation*}
H\left|\prod a^{\dagger}\left(k_{i}\right)^{n i}\right| 0>=\sum_{i}\left(k_{j}^{0}\right) n_{i}\left(k_{i}\right) \prod a^{\dagger}\left(k_{i}\right)^{n i} \mid 0> \tag{1.33}
\end{equation*}
$$

The canonical quantization for the scalar field on a compact space direction, with the boundary conditions of open or closed string, (1.21) and (1.17), respectively, follows very similar steps. Imposing the quantization conditions (1.28) above implies the following algebra for the $\alpha_{n}$ operators of the open string and
for the $\alpha_{n}$ and $\bar{\alpha}_{n}$ operators for the closed string:

$$
\begin{align*}
& {\left[\alpha_{m}, \alpha_{n}\right]=m \delta_{m+n}} \\
& {\left[\bar{\alpha}_{m}, \bar{\alpha}_{n}\right]=m \delta_{m+n}} \\
& {\left[\alpha_{m}, \bar{\alpha}_{n}\right]=0 .} \tag{1.34}
\end{align*}
$$

It is thus clear that $\alpha_{n}$ operators are related to the $a(k)$ operators as

$$
\begin{equation*}
\alpha_{m}=\sqrt{m} a(m), m>0 ; \quad \alpha_{-m}=\sqrt{m} a^{\dagger}(m), m>0 \tag{1.35}
\end{equation*}
$$

### 1.7 Radial quantization

For the case of a cylinder-like two-dimensional manifold, namely, where the space direction is compactified so that $x^{1} \equiv x^{1}+2 \pi$, it is natural to use the $z=\mathrm{e}^{x^{0}+i x^{1}}$ coordinates. Space translations $x^{1} \rightarrow x^{1}+a$ take the form of multiplying by a phase factor $z \rightarrow \mathrm{e}^{i a} z$, and time translations $x^{0} \rightarrow x^{0}+a$ turn into dilatations $z \rightarrow \mathrm{e}^{a} z$. Rotations $\left(x^{0}+i x^{1}\right) \rightarrow(c+i s)\left(x^{0}+i x^{1}\right)$, go into $z \rightarrow z^{(c+i s)}$, with $(c+i s)=\mathrm{e}^{i \theta}, \theta$ the rotation angle. Correspondingly the generators of these transformations change their geometrical operation. For instance the Hamiltonian obviously goes into the dilatation generator. Moreover, generators which are Noether charges transform into contour integrals. Recall that the Noether charge is $Q=\int \mathrm{d} x^{1} J_{0}\left(x^{1}\right)$ which in the new coordinates reads $Q=\int \mathrm{d} \theta J_{r}(\theta)$ so that we can write,

$$
\begin{equation*}
Q=\frac{1}{2 \pi i} \oint[\mathrm{~d} z J(z)+\mathrm{d} \bar{z} \bar{J}(\bar{z})] \tag{1.36}
\end{equation*}
$$

where the contour integral is performed at some radius and the sign convention we adopt is that both the $\mathrm{d} z$ and $\mathrm{d} \bar{z}$ integral are taken to be positive for the counter-clockwise sense.

The infinitesimal transformation of an operator generated by the Noether charge $Q$ is given by:

$$
\begin{equation*}
\delta_{\epsilon, \bar{\epsilon}} \mathcal{O}=\frac{1}{2 \pi i} \oint[\mathrm{~d} z J(z) \epsilon(z), \mathcal{O}(w, \bar{w})]+\mathrm{d} \bar{z}[\bar{J}(\bar{z}) \bar{\epsilon}(\bar{z}), \mathcal{O}(w, \bar{w})] . \tag{1.37}
\end{equation*}
$$

Define a product $R$ of two operators $A(z) B(w)$ as taken radially, namely ${ }^{4}$

$$
\begin{equation*}
R(A(z) B(w))=A(z) B(w),|z|>|w| ; \quad B(w) A(z),|w|>|z| \tag{1.38}
\end{equation*}
$$

In Fig. 1.2 we show the two contour integrals that lead to a contour integral around $w$, the location of the operator $\mathcal{O}$, so that the infinitesimal transformation is given by,

$$
\begin{equation*}
\delta_{\epsilon, \bar{\epsilon}} \mathcal{O}=\frac{1}{2 \pi i} \oint[\mathrm{~d} z \epsilon(z) R(J(z) \mathcal{O}(w, \bar{w}))+\mathrm{d} \bar{z} \bar{\epsilon}(\bar{z}) R(J(\bar{z}) \mathcal{O}(w, \bar{w}))] . \tag{1.39}
\end{equation*}
$$

[^2]

Fig. 1.2. A contour around $w$ from the commutator.

We now apply this formulation to the symmetry generators (discussed in Section 2.1):
(i) The infinitesimal affine current algebra transformation $\hat{X}(z, \bar{z}) \rightarrow \hat{X}(z, \bar{z})-$ $\epsilon(z)$ is generated by the holomorphic current $J(z)=\partial X$ via

$$
\begin{align*}
\delta_{\epsilon} \hat{X}(w, \bar{w}) & =\frac{1}{2 \pi i} \oint \mathrm{~d} z \epsilon(z) R(\partial X(z) \hat{X}(w, \bar{w})) \\
& =\frac{1}{2 \pi i} \oint \mathrm{~d} z \frac{-1}{z-w} \epsilon(z)=-\epsilon(w), \tag{1.40}
\end{align*}
$$

where we have used for the product of operators,

$$
\begin{equation*}
R(X(z) X(w))=-\log (z-w)+\text { finite terms. } \tag{1.41}
\end{equation*}
$$

This is an example of the concept of operator product expansion, which is addressed in the next section.
(ii) In a similar manner we can compute the transformation of $\partial X$ generated by the energy momentum tensor $T$

$$
\begin{array}{r}
\delta_{\epsilon} \partial X(w)=\frac{1}{2 \pi i} \oint \mathrm{~d} z \epsilon(z) R\left(-\frac{1}{2}: \partial X(z) \partial X(z): \partial X(w)\right) \\
=\frac{1}{2 \pi i} \oint \mathrm{~d} z \frac{1}{(z-w)^{2}} \partial X(z) \epsilon(z)=\partial \epsilon(w) \partial X(w)+\epsilon(w) \partial^{2} X(w), \tag{1.42}
\end{array}
$$

which is indeed the infinitesimal transformation of the holomorphic current $J=\partial X(z)$. The generator $T$ is normal ordered using the following expression:

$$
\begin{equation*}
T(w)=-\frac{1}{2}: \partial X(z) \partial X(w): \equiv-\frac{1}{2} \lim _{z \rightarrow w}\left[\partial X(z) \partial X(w)+\frac{1}{(z-w)^{2}}\right] \tag{1.43}
\end{equation*}
$$

### 1.8 Operator product expansion

In computing the contour integrals associated with infinitesimal transformations we have made use of the operator product expansions of pairs of operators. ${ }^{5}$ The singularities that occur when the points are taken to approach one another are captured in the notion of operator product expansion (OPE),

$$
\begin{equation*}
\mathcal{O}_{i}(x) \mathcal{O}_{j}(y)=\sum_{k} c_{i j}^{k}(x-y) \mathcal{O}_{k}(y) \tag{1.44}
\end{equation*}
$$

where $c_{i j}^{k}(x-y)$ are the coefficient functions which are singular in the limit of $x \rightarrow y$. Such expansions were proven to hold in renormalizable field theories. The OPEs are an essential tool in exploring quantum field theories. Recall that all of the information on the QFT is encoded into the values of all possible correlation functions of the complete set of local
operators $\mathcal{O}_{i}(x)$, namely, $<\mathcal{O}_{1}\left(x_{1}\right) \ldots \mathcal{O}_{n}\left(x_{n}\right)>$. In particular, one is interested in the behavior of these correlation functions when two or more points approach each other, which is encapsulated in the OPEs. For all applications discussed here the OPEs are treated as asymptotic expansions and only their singular terms will be specified. For the present case of two-dimensional free massless scalar field theory the OPE converges and in fact, as will be discussed in Section 3.7.2, a similar situation occurs in all 2d CFTs.

The OPEs of the free massless scalar can be deduced from its propagator, which can be evaluated from the solution. It takes the form:

$$
\begin{equation*}
<\hat{X}(z \bar{z}) \hat{X}(w \bar{w})>=-\log |z-w|^{2} \tag{1.45}
\end{equation*}
$$

In terms of the separation of the solution into holomorphic and anti-holomorphic parts the two propagators read:

$$
\begin{equation*}
<X(z) X(w)>=-\log (z-w) ; \quad<\bar{X}(\bar{z}) \bar{X}(\bar{w})>=-\log (\bar{z}-\bar{w}) \tag{1.46}
\end{equation*}
$$

By differentiating the last relation with respect to $z$ and to $w$ one finds the short distance expansion of other operators like $J(z), T(z)$ etc. In particular the OPE of the currents is

$$
\begin{equation*}
J(z) J(w)=\partial X(z) \partial X(w)=-\frac{1}{(z-w)^{2}}+\text { finite terms } \tag{1.47}
\end{equation*}
$$

with a similar result for the anti-holomorphic currents.
A different, though equivalent, approach is to write the OPE as a Taylor expansion in $(z-w)$ and $(\bar{z}-\bar{w})$ in the following form:

$$
\begin{align*}
\hat{X}(z, \bar{z}) \hat{X}(w \bar{w})= & -\log |z-w|^{2}+\sum_{k=1}^{\infty} \frac{1}{k!}\left[(z-w)^{k}:\left(\partial^{k} \hat{X}(w, \bar{w})\right) \hat{X}(w, \bar{w}):\right. \\
& \left.+(\bar{z}-\bar{w})^{k}:\left(\bar{\partial}^{k} \hat{X}(w, \bar{w})\right) \hat{X}(w, \bar{w}):\right] \tag{1.48}
\end{align*}
$$

[^3]This form of expansion is based on the property that the normal ordered product of the scalar fields,

$$
\begin{equation*}
: \hat{X}(z, \bar{z}) \hat{X}(w \bar{w}):=\hat{X}(z, \bar{z}) \hat{X}(w, \bar{w})+\log |z-w|^{2}, \tag{1.49}
\end{equation*}
$$

obeys the equation of motion, namely,

$$
\begin{equation*}
\partial \bar{\partial}: \hat{X}(z, \bar{z}) \hat{X}(w, \bar{w}):=0 \tag{1.50}
\end{equation*}
$$

and hence can be decomposed to holomorphic and anti-holomorphic functions and thus is non singular.

In the previous subsection we used two OPEs to determine the symmetry transformation of $X$ and $\partial X$. We will work out now two additional examples of OPEs, involving the operator which will later be found to be very useful $: \mathrm{e}^{i \alpha X(w)}:$.
(i) The conformal properties of : $\mathrm{e}^{i \alpha X(w)}$ are being determined by its OPE with $T(z)$ which takes the form

$$
\begin{align*}
T(z): \mathrm{e}^{i \alpha X(w)}: & =-\frac{1}{2}(: \partial X(z) \partial X(z):)\left(: \mathrm{e}^{i \alpha X(w)}:\right) \\
& =\frac{\left(\frac{\alpha^{2}}{2}\right)}{(z-w)^{2}} \mathrm{e}^{i \alpha X(w)}+\frac{1}{(z-w)} \partial \mathrm{e}^{i \alpha X(w)} . \tag{1.51}
\end{align*}
$$

In language that will be developed in Section 2.2 this result will mean that : $\mathrm{e}^{i \alpha X(w)}$ : has a conformal dimension of $\frac{\alpha^{2}}{2}$.
(ii) The OPE of two operators of the form : $\mathrm{e}^{i \alpha X(w)}$ is

$$
\begin{equation*}
\left(: \mathrm{e}^{i \alpha X(z)}:\right)\left(: \mathrm{e}^{-i \beta X(w)}:\right)=\frac{: \mathrm{e}^{i \alpha X(w)} \mathrm{e}^{-i \beta X(w)}}{(z-w)^{\alpha \beta}} \tag{1.52}
\end{equation*}
$$

### 1.9 Path integral quantization

So far we have been using canonical quantization. Before proceeding to the general structure of affine current algebra and Virasoro algebra we introduce the quantization of a free massless scalar field using the Euclidean path integral approach. As usual the functional integration $D \hat{X}(z, \bar{z})$ can be approximated by discretizing the two-dimensional space and representing the functional integral by products of ordinary integrals. Expectation values of operators $\mathcal{O}(X)$ constructed from $X$ are given by,

$$
\begin{equation*}
<\mathcal{O}(\hat{X})>=\int D \hat{X}(z, \bar{z}) \mathcal{O}(\hat{X}) \mathrm{e}^{-S}=\int D \hat{X}(z, \bar{z}) \mathcal{O}(\hat{X}) \mathrm{e}^{-\frac{1}{2 \pi} \int \mathrm{~d}^{2} z \partial \hat{X} \bar{\partial} \hat{X}} \tag{1.53}
\end{equation*}
$$

Correlation functions have to obey the equation

$$
\begin{equation*}
\partial \bar{\partial}<\hat{X}(z, \bar{z}) \hat{X}(w, \bar{w})>=-2 \pi \delta^{2}(z-w, \bar{z}-\bar{w}) \tag{1.54}
\end{equation*}
$$

as can be deduced by using the fact that the path integral of a total derivative vanishes:

$$
\begin{align*}
0 & =\int D \hat{X}(z, \bar{z}) \frac{\delta}{\delta \hat{X}(z, \bar{z})}\left[\mathrm{e}^{-S} \hat{X}(w, \bar{w})\right] \\
& =\int D \hat{X}(z, \bar{z}) \mathrm{e}^{-S}\left[-\frac{\delta S}{\delta \hat{X}(z, \bar{z})} \hat{X}(w, \bar{w})+\delta^{2}(z-w, \bar{z}-\bar{w})\right] \\
& =\frac{1}{2 \pi}<\partial \bar{\partial} \hat{X}(z, \bar{z}) \hat{X}(w, \bar{w})>+<\delta^{2}(z-w, \bar{z}-\bar{w})> \tag{1.55}
\end{align*}
$$

Alternatively one can use (1.45) and (1.46) directly. Note that in that case care must be exercised, as naively we would get zero rather than the delta function, since the expression is a sum of two terms, one depending on $z$ only and the other on $\bar{z}$ only. The point is that the expressions (1.46) cannot be taken over at the origin. A working rule is:

$$
\bar{\partial}\left(\frac{1}{z}\right)=\partial\left(\frac{1}{\bar{z}}\right)=(2 \pi) \delta^{2}(z, \bar{z})
$$

This can be derived by going over from $\frac{1}{z}$ to $\frac{\bar{z}}{z \bar{z}+\epsilon^{2}}$, to regulate the singularity at the origin.

### 1.10 Affine current algebra

As was shown in Section 1.3 the classical action is invariant under both affine current algebra transformations and conformal transformations. We would like to study the algebraic structure of the generators of these symmetries. We start with the invariance under affine current algebra transformations. Recall that the corresponding generators are the holomorphic and anti-holomorphic currents $J$ and $\bar{J}$, given in (1.24). Expanding the currents in Laurant series,

$$
\begin{equation*}
J=\sum_{n=-\infty}^{\infty} J_{n} z^{-(n+1)} ; \quad \bar{J}=\sum_{n=-\infty}^{\infty} \bar{J}_{n} \bar{z}^{-(n+1)} \tag{1.56}
\end{equation*}
$$

it is obvious from the mode expansion (1.18) that the algebra of the currents is related to that of the $\alpha_{n}$ operators, namely

$$
\begin{equation*}
\left[J_{m}, J_{n}\right]=-m \delta_{m+n} ; \quad\left[\bar{J}_{m}, \bar{J}_{n}\right]=-m \delta_{m+n} \tag{1.57}
\end{equation*}
$$

This form of algebra will be shown in the next chapter to be associated with level $k=1$ abelian affine current algebra (or Kac-Moody algebra as it is sometimes referred to).

This algebra translates into the following algebra of the currents:

$$
\begin{equation*}
[J(z), J(w)]=\delta^{\prime}(z-w) \tag{1.58}
\end{equation*}
$$

Using the technique developed in Section 1.7 we can derive this result also from the operator product expansion of two currents,

$$
\begin{equation*}
J(z) J(w)=\frac{1}{(z-w)^{2}}+\text { finite terms } \tag{1.59}
\end{equation*}
$$

### 1.11 Virasoro algebra

Next we address the algebraic structure of the generators of conformal transformations (1.27). Upon inserting (1.18) into the Laurent expansion of the energy momentum tensor,

$$
\begin{equation*}
T=\sum_{n=-\infty}^{\infty} L_{n} z^{-(n+2)} ; \quad \bar{T}=\sum_{n=-\infty}^{\infty} \bar{L}_{n} \bar{z}^{-(n+2)} \tag{1.60}
\end{equation*}
$$

one finds that for $L_{n}$ with $n \neq 0$,

$$
\begin{equation*}
L_{n}=1 / 2 \sum_{m=-\infty}^{\infty}: \alpha_{n-m} \alpha_{m}: \tag{1.61}
\end{equation*}
$$

For $n \neq 0$ the operators $\alpha_{n-m}$ and $\alpha_{m}$ commute, and so the product equals the normal ordered one. The situation is different for $L_{0}$. Here one encounters an infinity in the product of chiral fields, which normal ordering removes, resulting in,

$$
\begin{equation*}
L_{0}=1 / 2 \mathcal{P}^{2}+\sum_{1}^{\infty} \alpha_{-m} \alpha_{m} \tag{1.62}
\end{equation*}
$$

We shall later see that it is sometimes necessary to shift $L_{0}$ by a constant. Using the commutation relation of $\alpha_{n}$ one finds the following "naive" expression for the commutator of $L_{n}$ operators:

$$
\begin{align*}
{\left[L_{m}, L_{n}\right] } & =\frac{1}{4} \sum_{k, l}\left[\alpha_{m-k} \alpha_{k}, \alpha_{n-l} \alpha_{l}\right] \\
& =\frac{1}{2} \sum_{k} k \alpha_{m-k} \alpha_{k+n}+\frac{1}{2} \sum_{k}(m-k) \alpha_{m-k+n} \alpha_{k} \\
& =(m-n) L_{m+n} \tag{1.63}
\end{align*}
$$

where to get to the third line we have changed a variable in the first sum from $k \rightarrow k-n$. This is the classical Virasoro algebra, ${ }^{6}$ and in fact in the quantum theory it is further corrected. The correction appears only for the case $m+n=0$, so for $m \neq-n$ the classical form (1.63) is exact. For generators with $m+n=0$ the two sums in the second line of (1.63) have to be brought to normal order. As re-ordering means using the commutator, one gets divergent series for which, in the case at hand, one cannot shift the variable of summation without changing

[^4]the result. Taking this into account, one gets a c-number shift in the commutation rule,
\[

$$
\begin{equation*}
\left[L_{m}, L_{n}\right]=(m-n) L_{m+n}+\mathcal{A}(m) \delta_{m+n} \tag{1.64}
\end{equation*}
$$

\]

To compute the anomaly term $\mathcal{A}(m)$ we introduce a cutoff function $f_{\Lambda}(k)$, which tends to 1 in the limit of infinite regulator $\Lambda$ for any $k$, but for every finite $\Lambda$ goes to zero sufficiently rapidly at infinite $k$. Thus we view the operators $L_{n}$ as regularized sums,

$$
\begin{equation*}
L_{n}=1 / 2 \sum_{m=-\infty}^{\infty}: \alpha_{n-m} \alpha_{m}: f_{\Lambda}(m) \tag{1.65}
\end{equation*}
$$

to replace (1.61). With this regularized expression, a direct computation gives for the anomaly,

$$
\begin{align*}
\mathcal{A}(m)=1 / 4 \sum_{k=1}^{\infty} & \left\{k(m-k) f_{\Lambda}(m-k)\left[f_{\Lambda}(k-m)+f_{\Lambda}(-k)\right]\right. \\
& \left.+k(m+k) f_{\Lambda}(-k)\left[f_{\Lambda}(k)+f_{\Lambda}(-m-k)\right]\right\} \tag{1.66}
\end{align*}
$$

If we now take $f_{\Lambda}(k)$ to 1 , without being careful, we get the divergent sum,

$$
\begin{equation*}
\mathcal{A}(m) \rightarrow m \sum_{k=1}^{\infty} k \tag{1.67}
\end{equation*}
$$

Using $\zeta$-function regularization, namely replacing $k$ by $k^{-s}$, we get a convergent sum for any $s>1$, and then we continue analytically to $s=-1$, to get $-m / 12$ for the right-hand side of the last equation.

To compute $\mathcal{A}(m)$ with the regulators $f_{\Lambda}$, we now look at,

$$
\begin{align*}
\mathcal{A}(m)+\frac{m}{12}= & 1 / 4 \sum_{k=1}^{\infty}\left\{k(m-k) f_{\Lambda}(m-k)\left[f_{\Lambda}(k-m)+f_{\Lambda}(-k)\right]\right. \\
& \left.+k(m+k) f_{\Lambda}(-k)\left[f_{\Lambda}(k)+f_{\Lambda}(-m-k)\right]-4 m k\right\} \tag{1.68}
\end{align*}
$$

Only large $k$ is relevant now, as for any finite $k$ we can take $\Lambda$ to infinity first, obtaining zero on the right-hand side. We now take,

$$
\begin{equation*}
f_{\Lambda}(q) \approx|q|^{-p} \quad|q| \rightarrow \infty \tag{1.69}
\end{equation*}
$$

with $p \rightarrow 0$ as $\Lambda \rightarrow \infty$. Expanding in powers of $\frac{m}{k}$, and recalling that $\zeta(s)$ has a pole only at $s=1$, we get by summing first and then letting $p \rightarrow 0$, the result,

$$
\begin{equation*}
\mathcal{A}(m)+\frac{m}{12}=\frac{m^{3}}{12} . \tag{1.70}
\end{equation*}
$$

The anomaly term $\mathcal{A}(m)$ can also be determined using the Jacobi identity $\left[L_{k},\left[L_{m}, L_{n}\right]\right]+\left[L_{m},\left[L_{n}, L_{k}\right]+\left[L_{n},\left[L_{k}, L_{m}\right]=0\right.\right.$. One finds that for $k+m+$ $n=0$ the anomaly term obeys $(m-n) \mathcal{A}(k)+(n-k) \mathcal{A}(m)+(k-n) \mathcal{A}(m)=0$. Recall also that $\mathcal{A}(0)=0$ and $\mathcal{A}(m)=-\mathcal{A}(-m)$ so it is enough to determine $\mathcal{A}(m)$ for positive $m$. The relation derived from the Jacobi identity can be used
to get a recursion relation which is determined by values of $\mathcal{A}(1)$ and $\mathcal{A}(2)$.
In fact the general solution is of the form $\mathcal{A}(n)=b_{3} n^{3}+b_{1} n$. The coefficient $b_{1}$ is correlated with the normal ordering ambiguity constant of $L_{0}$. One can determine the coefficients $b_{1}$ and $b_{3}$ by computing the vacuum expectation values of $\langle 0|\left[L_{1}, L_{-1}\right]|0\rangle=0$ and $\langle 0|\left[L_{2}, L_{-2}\right]|0\rangle=\frac{1}{2}$, so that altogether one finds $\mathcal{A}(n)=\frac{1}{12} n\left(n^{2}-1\right)$ and the full Virasoro algebra associated with the massless free scalar field is,

$$
\begin{equation*}
\left[L_{m}, L_{n}\right]=(m-n) L_{m+n}+\frac{1}{12} m\left(m^{2}-1\right) \delta_{m+n} \tag{1.71}
\end{equation*}
$$

In the next chapter the Virasoro algebra will be discussed in a broader perspective. In that context it will become clear that the algebra of (1.71) associated with the massless free scalar is characterized by a $c=1$ Virasoro anomaly.

## 2

## Conformal field theory

Conformal invariance of two-dimensional massless scalar field theory was shown in the previous chapter to associate with the infinite algebra of conserved charges, the Virasoro algebra. In this chapter we describe the basic building blocks of any two-dimensional conformal field theory (CFT). The notions of primary and descendant operators will be introduced and the structure of the Hilbert space of states will be described. We will discuss and classify certain classes of unitary CFTs. Crossing symmetry, duality and bootstrap equations will be defined and applied to computing correlators of CFTs. We then discuss the Verlinde formula which relates the fusion rules and the $S$ transformation. We will end up with two examples of CFTs that demonstrate all of the concepts that have been introduced before. The first one is the theory of a Majorana fermion and the second is the $m=3$ unitary minimal model, which is shown to be the continuum limit of the two-dimensional Ising model.

Conformal field theory in two dimensions is covered by many review articles and books. The former include [109] which we use intensively in this chapter, also [25], [13], [59], [233] and many others.

Among the books that discuss 2d CFT is [140] and books on string theories [113], [154], [174], [138], [237], [142], [30].

The most complete book on the topic is [77].
The basics of conformal field theory were stated in the seminal paper by Belavin, Polyakov and Zamolodchikov [33]. This includes the introduction of primary fields, the behavior of the energy-momentum tensor and the central charge. Conformal Ward identity and the use of OPEs appears in [93], [95] and [94].

### 2.1 Conformal symmetry in two dimensions

The theory of the free massless scalar field in two dimensions was shown to be invariant under the holomorphic and anti-holomorphic coordinate transformations

$$
\begin{equation*}
z \rightarrow z^{\prime}=f(z) ; \quad \bar{z} \rightarrow \bar{z}^{\prime}=\bar{f}(\bar{z}) \tag{2.1}
\end{equation*}
$$

Under such a transformation the metric transforms as

$$
\begin{equation*}
\mathrm{d} s^{2}=\mathrm{d} z \mathrm{~d} \bar{z} \rightarrow \mathrm{~d} z^{\prime} \mathrm{d} \bar{z}^{\prime}=\frac{\partial z^{\prime}}{\partial z} \frac{\partial \bar{z}^{\prime}}{\partial \bar{z}} \mathrm{~d} z \mathrm{~d} \bar{z} \tag{2.2}
\end{equation*}
$$

At this point we can understand why we referred to these transformations as conformal transformations. In general in $d$ space-time dimensions the conformal group is the subgroup of coordinate transformations that leaves the metric invariant up to a scale, namely,

$$
\begin{equation*}
g_{\mu \nu}(x) \rightarrow g_{\mu \nu}^{\prime}\left(x^{\prime}\right)=\Omega(x) g_{\mu \nu}(x) \tag{2.3}
\end{equation*}
$$

It is obvious from (2.2) that the 2 d conformal transformations (2.1) indeed produce such a variation of the metric. An important property of conformal transformations in any dimension is that they preserve the angle $\frac{\vec{A} \cdot \vec{B}}{\sqrt{A^{2} B^{2}}}$ between two vectors $\vec{A}$ and $\vec{B}$.

Starting from flat space, the general infinitesimal coordinate transformations $x^{\mu} \rightarrow x^{\mu}+\epsilon^{\mu}(x)$ induces a change of the metric $\mathrm{d} s^{2} \rightarrow \mathrm{~d} s^{2}+\left(\partial_{\mu} \epsilon_{\nu}+\right.$ $\left.\partial_{\nu} \epsilon_{\mu}\right) \mathrm{d} x^{\mu} \mathrm{d} x^{\nu}$, so that the condition for conformal transformations reads,

$$
\begin{equation*}
\partial_{\mu} \epsilon_{\nu}+\partial_{\nu} \epsilon_{\mu}=\frac{2}{d}(\partial \cdot \epsilon) g_{\mu \nu} \tag{2.4}
\end{equation*}
$$

where $g_{\mu \nu}$ is $\eta_{\mu \nu}$ or $\delta_{\mu \nu}$ for a Minkowskian signature, or Euclidean signature, respectively.

It is thus obvious that for two-dimensional Euclidean space-time $\epsilon=\epsilon(z)$ and $\bar{\epsilon}=\bar{\epsilon}(\bar{z})$ are the unique solutions of (2.4), which reduces to the Cauchy-Riemann equation $\partial_{1} \epsilon_{1}=\partial_{2} \epsilon_{2}$ and $\partial_{1} \epsilon_{2}=-\partial_{2} \epsilon_{1}$.

We would like now to put aside scalar field theory and explore the general properties of conformal field theories in two dimensions. Any theory with a vanishing trace of the energy-momentum tensor $T_{\mu}^{\mu}=0$, or in complex coordinates $T_{z \bar{z}}=0$, has necessarily an independent holomorphically (and anti-holomorphically) conserved energy-momentum tensor components, namely,

$$
\begin{equation*}
\bar{\partial} T \equiv \bar{\partial} T_{z z}=0 \quad \partial \bar{T} \equiv \partial T_{\bar{z} \bar{z}}=0 \tag{2.5}
\end{equation*}
$$

This follows trivially from the usual conservation law $\bar{\partial} T_{z z}+\partial T_{z \bar{z}}=0$, and its complex conjugation. It is also clear that in fact there are infinitely many conserved currents, since $g(z) T(z)$ for any analytic function $g(z)$ is also a holomorphically conserved current (we sometimes call any conserved tensor "current").

We show in the following section that indeed the energy-momentum tensor $T(z)$ and $\bar{T}(\bar{z})$ generate the conformal transformations given in (2.1).

### 2.2 Primary fields

Conformal invariance constrains the OPEs of the theory. In particular, since $T$ is holomorphic, the OPE of $T(z)$ with a general operator can be expanded in terms of a Laurent expansion in integer powers of $z$. The singular part of the OPE takes the form,

$$
\begin{equation*}
T(z) \tilde{\mathcal{O}}(w, \bar{w})=\sum_{n=0}^{\infty} \frac{1}{(z-\omega)^{n+1}} \tilde{\mathcal{O}}^{(n)}(w, \bar{w}) \tag{2.6}
\end{equation*}
$$

where the sum is usually finite, and the operators $\tilde{\mathcal{O}}^{(n)}(w, \bar{w})$ have to be determined. Using radial quantization as in Section 1.7 and the OPE above, we get for the transformation generated by $T(z)$,

$$
\begin{equation*}
\delta_{\epsilon} \tilde{\mathcal{O}}(w, \bar{w})=\sum_{n} \frac{1}{n!}\left[\left(\partial^{n} \epsilon\right) \tilde{\mathcal{O}}^{(n)}(w, \bar{w})\right] \tag{2.7}
\end{equation*}
$$

We now consider operators that transform under conformal transformation in a way that generalizes the transformation of the metric, (2.2),

$$
\begin{equation*}
\mathcal{O}(z, \bar{z}) \rightarrow \mathcal{O}^{\prime}\left(z^{\prime} \bar{z}^{\prime}\right)=\left(\frac{\partial z^{\prime}}{\partial z}\right)^{h}\left(\frac{\partial \bar{z}^{\prime}}{\partial \bar{z}}\right)^{\bar{h}} \mathcal{O}\left(z^{\prime} \bar{z}^{\prime}\right) \tag{2.8}
\end{equation*}
$$

An operator with such conformal transformations is a primary field or a tensor operator with conformal weights $(h, \bar{h})$, which are sometimes referred to as the holomorphic and anti-holomorphic conformal dimensions. ${ }^{1}$ The sum of the weights $h+\bar{h}$ is the total dimension that determines the behavior under scaling, whereas $h-\bar{h}$ is the spin that controls the behavior under rotations. The infinitesimal transformations that correspond to (2.8) are,

$$
\begin{equation*}
\delta_{\epsilon, \bar{\epsilon}} \mathcal{O}(z, \bar{z})=[(h \partial \epsilon+\epsilon \partial)+(\bar{h} \bar{\partial} \bar{\epsilon}+\bar{\epsilon} \bar{\partial})] \mathcal{O}(z \bar{z}) \tag{2.9}
\end{equation*}
$$

This form of transformation implies that the singular part of the OPE of $T$ and $\mathcal{O}(w, \bar{w})$ reduces to,

$$
\begin{equation*}
T(z) \mathcal{O}(w, \bar{w})=\frac{h}{(z-\omega)^{2}} \mathcal{O}(w, \bar{w})+\frac{1}{(z-\omega)} \partial \mathcal{O}(w, \bar{w}) \tag{2.10}
\end{equation*}
$$

Applying these notions to the free scalar field we find that $\partial X(z)$ has $(1,0)$ weights, $\bar{\partial} \bar{X}(\bar{z})$ has $(0,1)$ and the weights of : $\mathrm{e}^{i \alpha X(z, \bar{z})}$ : are $\left(\frac{\alpha^{2}}{2}, \frac{\alpha^{2}}{2}\right)$.

In Chapter 1 the notion of OPE was discussed in the context of scalar field theory. The generalization to any CFT is straightforward. Normalize the operators with fixed conformal weights as,

$$
\begin{equation*}
\left\langle\mathcal{O}_{i}(z, \bar{z}) \mathcal{O}_{j}(w, \bar{w})\right\rangle=\delta_{i j} \frac{1}{(z-w)^{2 h_{i}}} \frac{1}{(\bar{z}-\bar{w})^{2 \bar{h}_{i}}} \tag{2.11}
\end{equation*}
$$

then, for a complete set, the OPE of any pair of such operators is, to leading singularity,

$$
\begin{equation*}
\mathcal{O}_{i}(z, \bar{z}) \mathcal{O}_{j}(w, \bar{w}) \sim \sum_{k} C_{i j k}(z-w)^{h_{k}-h_{i}-h_{j}}(\bar{z}-\bar{w})^{\bar{h}_{k}-\bar{h}_{i}-\bar{h}_{j}} \mathcal{O}_{k}(w, \bar{w}) \tag{2.12}
\end{equation*}
$$

where $C_{i j k}$ are the product coefficients of the theory.

[^5]
### 2.3 Conformal properties of the energy-momentum tensor

For the free massless scalar field we found that the OPE of $T(z) T(w)$ is not of the form shown as (2.6), due to the anomaly term as in (1.71). The form of $T(z) T(w)$ OPE for any CFT is rather,

$$
\begin{equation*}
T(z) T(w)=\frac{c / 2}{(z-w)^{4}}+\frac{2 T(w)}{(z-w)^{2}}+\frac{\partial_{w} T(w)}{(z-w)} \tag{2.13}
\end{equation*}
$$

where $c$ is the central charge (or the Virasoro anomaly), a constant that characterizes the theory. The second term represents the dimensions and the third the property of translations under $T$. For theories with positive semi-definite Hilbert space $c \geq 0$, as follows from,

$$
<T(z) T(w)>=\frac{c / 2}{(z-w)^{4}}
$$

This type of OPE implies the following infinitesimal transformation of $T$ :

$$
\begin{equation*}
\delta_{\epsilon(z)} T(z)=\frac{c}{12} \partial^{3} \epsilon(z)+2(\partial \epsilon(z)) T(z)+\epsilon(z) \partial T(z) . \tag{2.14}
\end{equation*}
$$

The corresponding finite transformation $T(z) \rightarrow T^{\prime}\left(z^{\prime}\right)$ takes the form,

$$
\begin{equation*}
T^{\prime}\left(z^{\prime}\right)=\left(\partial z^{\prime}\right)^{2} T(z)+\frac{c}{12}\left\{z^{\prime}, z\right\} \tag{2.15}
\end{equation*}
$$

where $\left\{z^{\prime}, z\right\}$ is the Schwarzian derivative,

$$
\begin{equation*}
\{f, z\}=\frac{2 \partial^{3} f \partial f-3 \partial^{2} f \partial^{2} f}{2 \partial f \partial f} \tag{2.16}
\end{equation*}
$$

To derive (2.16), we first note that by applying a second transformation $f \rightarrow \omega$ we get,

$$
\begin{equation*}
\{w, z\}=\left(\partial_{z} f\right)^{2}\{w, f\}+\{f, z\} \tag{2.17}
\end{equation*}
$$

Then, we take $\omega=f+\delta f$, thus obtaining a functional equation,

$$
\begin{equation*}
\delta f \frac{\delta}{\delta f}\{f, z\}=\left(\partial_{z} f\right)^{2} \frac{\partial^{3} \delta f}{\partial^{3} f} \tag{2.18}
\end{equation*}
$$

Expressing the right-hand side as derivatives with respect to z,

$$
\frac{1}{f^{\prime}}(\delta f)^{\prime \prime \prime}-\frac{3 f^{\prime \prime}}{\left(f^{\prime}\right)^{2}}(\delta f)^{\prime \prime}+\left[\frac{3\left(f^{\prime \prime}\right)^{2}}{\left(f^{\prime}\right)^{3}}-\frac{f^{\prime \prime \prime}}{\left(f^{\prime}\right)^{2}}\right](\delta f)^{\prime}
$$

we can integrate the equation to get (2.16). The first term suggests integrating to $f^{\prime \prime \prime} / f^{\prime}$, the variation of which gives $1 / f^{\prime}(\delta f)^{\prime \prime \prime}-f^{\prime \prime \prime} /\left(f^{\prime}\right)^{2}(\delta f)^{\prime}$, while the second term suggests $-3\left(f^{\prime \prime}\right)^{2} / 2\left(f^{\prime}\right)^{2}$, the variation of which gives $-3 f^{\prime \prime} /\left(f^{\prime}\right)^{2}(\delta f)^{\prime \prime}+$ $3\left(f^{\prime \prime}\right)^{2} /\left(f^{\prime}\right)^{3}(\delta f)^{\prime}$.

For the massless scalar case $T$ can be written as $T(z)=-\frac{1}{2}: J(z) J(z):$, as we saw in (1.5). In fact, as will be discussed in Chapter 3, there is a large class of theories that share this so-called Sugawara form. For this type of theory the proof that the finite transformation is of the form of (2.15) is as follows. Recall
that as a primary field of weights $(1,0), J(z) \rightarrow \frac{\partial z^{\prime}}{\partial z} J\left(z^{\prime}\right)$. If we write $T(z)=$ $-\frac{1}{2} \lim _{z \rightarrow w}\left(J(z) J(w)+\frac{1}{(z-w)^{2}}\right)$ and substitute the transformation of the currents we end up after some lengthy but straightforward calculation with (2.15).

### 2.4 Virasoro algebra for CFT

Let us use the Laurent expansion of $T$ for CFT, following (1.60),

$$
\begin{equation*}
T=\sum_{n=-\infty}^{\infty} L_{n} z^{-(n+2)} \quad \bar{T}=\sum_{n=-\infty}^{\infty} \frac{\bar{L}_{n}}{\bar{z}^{-(n+2)}}, \tag{2.19}
\end{equation*}
$$

so that,

$$
\begin{equation*}
L_{n}=\frac{1}{2 \pi i} \oint \mathrm{~d} z z^{n+1} T(z) \tag{2.20}
\end{equation*}
$$

The expansion is chosen such that $L_{n}$ has scale dimension $n$ under $z \rightarrow z / a$, namely, $L_{n} \rightarrow a^{n} L_{n}$.

The Virasoro algebra ${ }^{2}$ can now be derived using the OPE of $T(z) T(w)$ given in (2.13),

$$
\begin{equation*}
\left[L_{n}, L_{m}\right]=\left(\frac{1}{2 \pi i}\right)^{2} \oint \mathrm{~d} z \oint \mathrm{~d} w\left[z^{n+1} w^{m+1}-z^{m+1} w^{n+1}\right] T(z) T(w) \tag{2.21}
\end{equation*}
$$

The double integral is performed by fixing $w$ and transforming the difference of the two $\oint \mathrm{d} z$ integrals into one integral around $w$,

$$
\begin{align*}
{\left[L_{n}, L_{m}\right]=} & \left(\frac{1}{2 \pi i}\right)^{2} \oint \mathrm{~d} z \oint \mathrm{~d} w\left[z^{n+1} w^{m+1}-z^{m+1} w^{n+1}\right] \\
& {\left[\frac{c / 2}{(z-w)^{4}}+\frac{2 T(w)}{(z-w)^{2}}+\frac{\partial_{w} T(w)}{(z-w)}\right] } \\
= & \left(\frac{1}{2 \pi i}\right) \oint \mathrm{d} w\left[c / 12\left(n^{3}-n\right) w^{n+m-1}\right. \\
& \left.+[2(n+1)-(n+m+2)] w^{n+m+1} T(w)\right] \\
= & \frac{c}{12}\left(n^{3}-n\right) \delta(n+m)+(n-m) L_{n+m} \tag{2.22}
\end{align*}
$$

Performing identical steps for $\bar{L}_{n}$ we get that $\bar{L}_{n}$ obeys the same infinite algebra, with some central charge $\bar{c}$, and that $\left[L_{n}, \bar{L}_{m}\right]=0$.

Any CFT is a representation of the Virasoro algebra characterized by $c$ and $\bar{c}$. It is straightforward to identify the following properties of the algebra:

- The generators $\left(L_{ \pm 1}, L_{0}\right)$ span an $S L(2, \mathcal{R})$ algebra,

$$
\begin{equation*}
\left[L_{+1}, L_{-1}\right]=2 L_{0} \quad\left[L_{0}, L_{ \pm 1}\right]=\mp L_{ \pm} \tag{2.23}
\end{equation*}
$$

[^6]Table 2.1. The conformal family

| Level | Weight | Fields |
| :---: | :---: | :---: |
| 0 | h | $\phi$ |
| 1 | $h+1$ | $L_{-1} \phi$ |
| 2 | $h+2$ | $L_{-2} \phi, L_{-1}^{2} \phi$ |
| 3 | $h+3$ | $L_{-3} \phi, L_{-2} L_{-1} \phi, L_{-1}^{3} \phi$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $N$ | $h+N$ | $P(N)$ fields |

- For $n>0, L_{-n}$ is a raising operator and $L_{n}$ is a lowering one, since $\left[L_{0}, L_{n}\right]=$ $-n L_{n}$ so that if $|\psi\rangle$ is an eigenstate of $L_{0}, L_{0}|\psi\rangle=h|\psi\rangle$ then $L_{0}\left|L_{n} \psi\right\rangle=$ $(h-n) \mid L_{n} \psi>$.


### 2.5 Descendant operators

From every primary operator $\phi(z, \bar{z})$ one can construct an infinite tower of Virasoro descendant operators,

$$
\begin{equation*}
\left(L_{-n} \phi(w, \bar{w})\right)=\frac{1}{2 \pi i} \oint \mathrm{~d} z \frac{1}{z^{n-1}} T(z) \phi(w, \bar{w}) \tag{2.24}
\end{equation*}
$$

A distinguished descendant operator is the energy momentum tensor $T(z)$ since,

$$
\begin{equation*}
L_{-2} \mathbf{1}=\frac{1}{2 \pi i} \oint \frac{\mathrm{~d} z}{z} T(z) \mathbf{1}=T(0) \tag{2.25}
\end{equation*}
$$

The set containing the primary field $\phi(z, \bar{z})$ and all its descendant operators is called a conformal family and it is denoted by $[\phi]$. A conformal family is a tower of operators where each layer is characterized by its level as shown in Table 2.1, where $P(N)$ is the number of partitions of $N$ into positive integer parts, which can be written in terms of the generating function $\prod_{n=1} \frac{1}{\left(1-q^{n}\right)}=$ $\sum_{N=0}^{\infty} P(N) q^{N}$.

We can now use the conformal family to rewrite the expression of the OPE (2.12) of two primary fields,

$$
\begin{align*}
& \phi_{i}(z, \bar{z}) \phi_{j}(w, \bar{w}) \\
& \quad=\sum_{k\{l \bar{l}\}} C_{i j k}^{\{\bar{l}\}}(z-w)^{h_{k}-h_{i}-h_{j}+\sum_{n} l_{n}}(\bar{z}-\bar{w})^{\bar{h}_{k}-\bar{h}_{i}-\bar{h}_{j}+\sum_{n} \bar{l}_{n}} \phi_{k}^{l \bar{l}}(w, \bar{w}), \tag{2.26}
\end{align*}
$$

where we denote by $\phi_{k}^{l \bar{l}}(w, \bar{w})$ the descendants $L_{-l_{1}} \ldots L_{-l_{n}} \bar{L}_{-\bar{l}_{1}} \ldots \bar{L}_{-\bar{l}_{n}} \phi_{k}(w, \bar{w})$ with the normalization given in (2.11). The product coefficients $C_{i j k}^{\{l \bar{l}\}}$ are given in terms of those of (2.12) $C_{i j k}$ as,

$$
\begin{equation*}
C_{i j k}^{\{l \bar{l}\}}=C_{i j k} \beta_{i j}^{k\{l\}} \bar{\beta}_{i j}^{k\{\bar{l}\}} \tag{2.27}
\end{equation*}
$$

where $\beta_{i j}^{k\{l\}}$ are determined by conformal invariance and are functions of $c$ and $h_{i}, h_{j}, h_{k}$, and similarly for $\bar{\beta}_{i j}^{k\{\bar{\jmath}\}}$. This follows from a detailed analysis that we do not show here.

The OPEs of any pair of descendant fields can also be deduced from (2.12) which implies in fact that all the information about the OPE is encoded in the product coefficients $C_{i j k}$. Moreover since the structure of (2.26) holds for all the primaries and their descendants, one can write the so-called fusion algebra for conformal, families, which takes the form,

$$
\begin{equation*}
\left[\phi_{i}\right]\left[\phi_{j}\right]=\sum_{k} N_{i j}^{k}\left[\phi_{k}\right] . \tag{2.28}
\end{equation*}
$$

### 2.6 Hilbert space of states

Our next task is to construct the Hilbert space of states. First we define the ground state $\mid 0>$ by,

$$
\begin{equation*}
L_{n} \mid 0>=0 \quad n \geq 0 \tag{2.29}
\end{equation*}
$$

The next step in this program is to build the highest weight states (hws). Consider the state generated from the vacuum by a primary field $\phi(z)$ of dimension $h$,

$$
\begin{equation*}
|h>=\phi(0)| 0> \tag{2.30}
\end{equation*}
$$

It is easy to check that for $n>0,\left[L_{n}, \phi(0)\right]=0$ since,

$$
\begin{equation*}
\left[L_{n}, \phi(w)\right]=\frac{1}{2 \pi i} \oint \mathrm{~d} z z^{n+1} T(z) \phi(w)=h(n+1) w^{n} \phi(w)+w^{n+1} \partial \phi(w) \tag{2.31}
\end{equation*}
$$

Hence the highest weight state $\mid h>$ obeys

$$
\begin{equation*}
L_{0}|h>=h| h>\quad L_{n} \mid h>=0 \quad n>0 . \tag{2.32}
\end{equation*}
$$

Expanding the primary field $\phi(z)$ in a Laurent series $\sum_{n} \phi_{n} z^{(n-h)}$, one can write the highest weight state symbolically as $\phi_{h} \mid 0>$.

Descendant states are generated by applying the descendant operators $L_{-n} \phi$ on the vacuum or alternatively by applying $L_{-n}$ on highest weight states, namely,

$$
\begin{equation*}
L_{-n}\left|h>=L_{-n} \phi(0)\right| 0>=\left(L_{-n} \phi\right) \mid 0>. \tag{2.33}
\end{equation*}
$$

It is thus clear that the highest weight states, or equivalently the primary operators, play a major role in constructing representations of the Virasoro algebra. In fact one can show that every representation is characterized by a primary operator. Consider an eigenstate of $L_{0}, L_{0}|\psi\rangle=h_{\psi}|\psi\rangle$. Now act on it with the lowering operator $L_{n}$ with $n>0$. The $L_{0}$ eigenvalue of the new state $L_{n} \mid \psi>$ is $h_{\psi}-n$. Since we require that the Hamiltonian is bounded from below, $L_{0}$ has to be also bounded. This implies that after repeating the lowering process one finally hits a state that is annihilated by $L_{n}$ for every $n>0$ and hence an hws.

It is thus clear that any state in a positive Hilbert space is a linear combination of hws, and their descendants. The representation given in Table 2.1 is referred to as the Verma module. Denoting it by $\mathcal{V}(c, h)$ and its analogous representation for the anti-holomorphic Virasoro algebra by $\overline{\mathcal{V}}(\bar{c}, \bar{h})$, the Hilbert space of the theory is a direct sum of the products $\mathcal{V}(c, h) \otimes \overline{\mathcal{V}}(\bar{c}, \bar{h})$, namely,

$$
\begin{equation*}
\mathcal{H}=\sum_{h, \bar{h}} \mathcal{V}(c, h) \otimes \overline{\mathcal{V}}(\bar{c}, \bar{h}) \tag{2.34}
\end{equation*}
$$

The Verma module may be reducible in the sense that there is a submodule that is by itself a Verma module. Such a submodule whose states transform amongst themselves under any conformal transformation, is built from a $\left|h_{\text {null }}\right\rangle$. The latter is both an hws., namely $L_{n} \mid h_{\text {null }}>=0$ for $n>0$, as well as a descendant. Such a state is called null state or null vector, motivated by what follows. It generates its own Verma module which is included in the parent module. It is orthogonal to the whole Verma module as well as to itself $\left\langle h_{\text {null }} \mid h_{\text {null }}\right\rangle=0$, since $<h_{\text {null }}\left|L_{-k_{1}} \ldots L_{-k_{n}}\right| h>=<h\left|L_{k_{n}} \ldots L_{k_{1}}\right| h_{\text {null }}>^{*}=0$, and in particular it has a zero norm $<h_{\text {null }} \mid h_{\text {null }}>=0$ and similarly also its descendants. The null state corresponds to a null operator which is simultaneously a primary and a secondary field.

Let us now demonstrate the construction of a null vector. Consider a general linear combination of the states of level 2 ,

$$
\begin{equation*}
L_{-2}\left|h>+a L_{-1}^{2}\right| h> \tag{2.35}
\end{equation*}
$$

we would like to check whether for certain values of the mixing coefficient $a$, this state is a null state. If indeed it is |null>, then so is the state [ $L_{n} \mid$ null $>$ ]. In fact it is easy to verify that at level 2 , one has to check these consistency conditions only for $L_{1}$ and $L_{2}$. Now using the Virasoro algebra we find that,

$$
\begin{align*}
{\left[L_{1}, L_{-2}\right]\left|h>+a\left[L_{1}, L_{-1}^{2}\right]\right| h>} & =(3+2 a(2 h+1)) L_{-1} \mid h> \\
{\left[L_{2}, L_{-2}\right]\left|h>+a\left[L_{2}, L_{-1}^{2}\right]\right| h>} & \left.=\left(4 h+\frac{c}{2}+6 a h\right) \right\rvert\, h> \tag{2.36}
\end{align*}
$$

It is thus clear that for the following values of $a$ and $c$,

$$
\begin{equation*}
a=-\frac{3}{2(2 h+1)} \quad c=\frac{2 h}{2 h+1}(5-8 h) \tag{2.37}
\end{equation*}
$$

the linear combination state (2.35) is a null state. In the unitary case we have $h$ and $c$ positive (see next section). Hence in this example $h<\frac{5}{8}$.

An irreducible representation of the Virasoro algebra can be constructed from a Verma module that contains a null vector by a quotient procedure, taking out of the Verma module the null module. In the next section we discuss this construction.

### 2.7 Unitary CFT and Kac determinant

Unitarity is obviously lost if there are negative norm states in the Verma module. Hence, our task is to derive the conditions for having a negative norm state. In the basis of the Verma module,

$$
\begin{equation*}
L_{-k_{1}} \ldots L_{-k_{i}}|h>\equiv| s>\quad\left(1 \leq k_{1} \leq \ldots \leq k_{i}\right) \tag{2.38}
\end{equation*}
$$

the matrix of inner products $\mathbf{I}_{s s^{\prime}}=\langle s| s^{\prime}>$ is block diagonal with blocks $\mathbf{I}^{(N)}$ for states at level $N\left(\sum_{i} k_{i}=N\right)$. For a given Verma module the elements of $\mathbf{I}$ are functions of $(h, c)$. It is easy to realize that unitarity dictates $c>0$ and $h>0$. This follows from $\langle h| L_{n} L_{-n}|h\rangle=\left[2 n h+1 / 12 c n\left(n^{2}-1\right)\right]\langle h \mid h\rangle$, which is positive for $n=1$ only if $h>0$ and for large enough $n$ only for $c>0$. To determine the full set of constraints for unitarity let us analyze further the properties of $\mathbf{I}$. A general state $|\hat{s}\rangle=\sum_{k} c_{k}|s\rangle$ has a norm $<\hat{s}|\hat{s}\rangle=\hat{c}^{\dagger} \mathbf{I} \hat{c}$, with $\hat{c}$ the vector of the $c_{k}$. Now since $\mathbf{I}$ is hermitian it can be diagonalized by a unitary matrix $U$ so that the norm can be written as $\langle\hat{s} \mid \hat{s}\rangle=\sum_{k} l_{k}\left|t_{k}\right|^{2}$ where $t=U \hat{c}$ and $l_{k}$ are the eigenvalues of $\mathbf{I}$, which are real. It is thus clear that there are negative norm states if and only if $\mathbf{I}$ has negative eigenvalues. A vanishing eigenvalue indicates a null vector which means a reducible Verma module.

For the low lying levels these matrices take the following form:

$$
\begin{align*}
& \mathbf{I}^{(0)}=1 \\
& \mathbf{I}^{(1)}=2 h \\
& \mathbf{I}^{(2)}=\left(\begin{array}{cc}
4 h(2 h+1) & 6 h \\
6 h & 4 h+c / 2
\end{array}\right) \tag{2.39}
\end{align*}
$$

The derivation of the various elements is straightforwad, for instance,

$$
\begin{align*}
\mathbf{I}_{11}^{(2)} & =<h\left|L_{1} L_{1} L_{-1} L_{-1}\right| h>=<h\left|L_{1} L_{-1} L_{1} L_{-1}\right| h>+2<h\left|L_{1} L_{0} L_{-1}\right| h> \\
& =4<h\left|L_{1} L_{-1} L_{0}\right| h>+2<h\left|L_{1} L_{-1}\right| h>=8 h^{2}+4 h \tag{2.40}
\end{align*}
$$

The determinant of $\mathbf{I}^{(2)}$ is given by

$$
\begin{equation*}
\operatorname{det}\left[\mathbf{I}^{(2)}\right]=32\left(h-h_{1,1}\right)\left(h-h_{1,2}\right)\left(h-h_{2,1}\right), \tag{2.41}
\end{equation*}
$$

where $h_{1,1}=0$ and $h_{1,2}, h_{2,1}$ are $(1 / 16)[(5-c) \pm \sqrt{(1-c)(25-c)}]$. The trace of $\mathbf{I}^{(2)}$ is $\operatorname{Tr}\left[\mathbf{I}^{(2)}\right]=8 h(h+1)+c / 2$. Since the trace and the determinant are the sum and product of the two eigenvalues, unitarity is lost if either the trace or the determinant is negative.

The determinant for $\mathbf{I}^{(N)}$ at general level $N$, which is referred to as the Kac determinant, ${ }^{3}$ has the form

$$
\begin{equation*}
\operatorname{det}\left[\mathbf{I}^{(N)}\right]=\alpha_{N} \prod_{p q \leq N}\left[h-h_{p, q}(c)\right]^{P(N-p q)}, \tag{2.42}
\end{equation*}
$$

[^7]where $\alpha_{N}$ are constants independent of $(c, h)$ and $h_{p, q}(c)$ can be expressed in terms of $m=-\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{25-c}{1-c}}$ as,
\[

$$
\begin{equation*}
h_{p, q}(c)=\frac{[(m+1) p-m q]^{2}-1}{4 m(m+1)} . \tag{2.43}
\end{equation*}
$$

\]

Note that we can choose either the plus or the minus sign in the expression for $m$, as their interchange is like interchanging $p$ with $q$, which does not change the determinant. Note also that $h_{p, q}$ is invariant under $p \rightarrow m-p, q \rightarrow m+1-q$. Let us also mention that for $N=2$ the result is identical to (2.41).

In the $(h, c)$ plane the determinant vanishes along the curves $h=h_{p, q}(c)$ which are therefore called the vanishing curves. If the determinant (2.42) is negative it means that there is an odd number of negative eigenvalues and hence the corresponding Virasoro representation is not unitary. If the determinant is vanishing or positive one needs to further analyze the determinant as follows:

- For $c>1$ and $h>0$ it is straightforward to show that the determinant does not vanish.
In the domain $1<c<25$ the value for $m$ has an imaginary part. Thus $h_{p, q}$ are complex for $p \neq q$, and as they come in complex conjugate pairs the product of the appropriate two factors in the determinant is positive. For $p=q$ the value of $h_{p, q}$ is negative. Thus the determinant is positive in that domain.
For $c>25$ the $h_{p, q}$ are negative.
For large $h$ the matrix is dominated by its diagonal elements.
Since these elements are positive, the eigenvalues for large $h$ are all positive. Now since the determinant never vanishes in the region considered $(h>0, c>1)$ all the eigenvalues have to be positive on the entire region.
Note that in $\mathbf{I}^{(2)}$ the off-diagonal element is larger at large h than the 22 element, but still the determinant is dominated at large h by the diagonal elements, and thus also the eigenvalues, as a $2 \times 2$ matrix.
- For $c=1$ we have $h_{p, q}=(p-q)^{2} / 4$, and so the determinant is never negative. However, it vanishes when $h=n^{2} / 4$ for some integer $n$.
- For $0<c<1, h>0$ a closer look at the determinant is required. We draw $h_{p, q}(c)$ in Fig. 2.1.
By expanding the curves around $c=1$ one can show that any point in the region can be connected to the right of $c=1$ by crossing a single vanishing curve. The vanishing of the determinant is due to one eigenvalue that reverses its sign which implies that there are negative norm states at any point in the region that are not on the vanishing curve. In fact it turns out that there are additional negative norm states at points along the vanishing curve except at


[^0]:    ${ }^{1}$ The content of this chapter comprises the basics of massless scalar fields in two dimensions. This is covered in many textbooks.
    2 The use of complex coordinates in the context of the bosonic string theory is described by Polyakov in [177].

[^1]:    ${ }^{3}$ Affine Lie algebras describing a physical system were first discussed in [27]. More references will be given in the next two chapters.

[^2]:    ${ }^{4}$ The notion of radial quantization was introduced in [104]. This construction was used in the context of complex geometry in [93].

[^3]:    ${ }^{5}$ Wilson introduced for the first time the concept of an operator product expansion [219]. It was used for two-dimensional conformal field theories in [33].

[^4]:    ${ }^{6}$ The Virasoro algebra was presented in [212]. More references will be given in the next two chapters.

[^5]:    ${ }^{1}$ The notion of conformal primary field and its descendants was introduced in [33] and further discussed in [236].

[^6]:    ${ }^{2}$ The first use of the Virasoro algebra was by M. Virasoro in the context of the dual resonance model [212]. Its application to two-dimensional CFT was presented in [33].

[^7]:    ${ }^{3}$ The proof of the Kac determinant is detailed in [89], [206] and [95].

