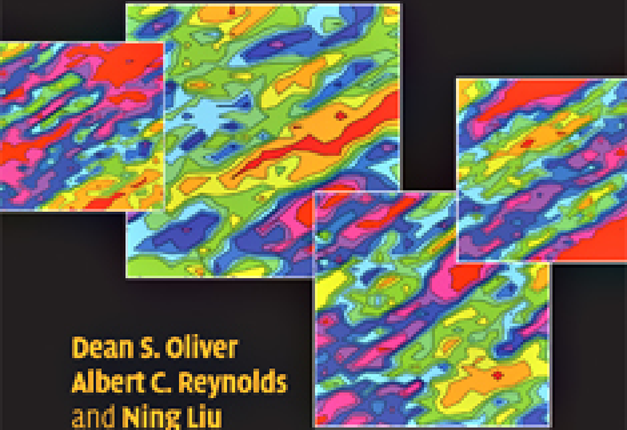
INVERSE THEORY FOR

Petroleum Reservoir Characterization and History Matching



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Inverse Theory for Petroleum Reservoir Characterization and History Matching

This book is a guide to the use of inverse theory for estimation and conditional simulation of flow and transport parameters in porous media. It describes the theory and practice of estimating properties of underground petroleum reservoirs from measurements of flow in wells, and it explains how to characterize the uncertainty in such estimates.

Early chapters present the reader with the necessary background in inverse theory, probability, and spatial statistics. The book then goes on to develop physical explanations for the sensitivity of well data to rock or flow properties, and demonstrates how to calculate sensitivity coefficients and the linearized relationship between models and production data. It also shows how to develop iterative methods for generating estimates and conditional realizations. Characterization of uncertainty for highly nonlinear inverse problems, and the methods of sampling from high-dimensional probability density functions, are discussed. The book then ends with a chapter on the development and application of methods for sequentially assimilating data into reservoir models.

This volume is aimed at graduate students and researchers in petroleum engineering and groundwater hydrology and can be used as a textbook for advanced courses on inverse theory in petroleum engineering. It includes many worked examples to demonstrate the methodologies, an extensive bibliography, and a selection of exercises.

Color figures that further illustrate the data in this book are available at
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Al Reynolds dedicates the book to Anne, his wife and partner in life.
Ning Liu dedicates the book to her parents and teachers.
Dean Oliver dedicates the book to his wife Mary
and daughters Sarah and Beth.

Contents

Preface

The intent of this book is to provide a rather broad overview of inverse theory as it might be applied to petroleum reservoir engineering and specifically to what has, in the past, been called history matching. It has been strongly influenced by the geophysicists' approach to inverse problems as opposed to that of mathematicians. In particular, we emphasize that measurements have errors, that the quantity of data are always limited, and that the dimension of the model space is usually infinite, so inverse problems are always underdetermined. The approach that we take to inverse theory is governed by the following philosophy.

1. All inverse problems are characterized by large numbers of parameters (conceptually infinite). We only limit the number of parameters in order to solve the forward problem.
2. The number of data is always finite, and the data always contain measurement errors.
3. It is impossible to correctly estimate all the parameters of a model from inaccurate, insufficient, and inconsistent data,[1] but reducing the number of parameters in order to get low levels of uncertainty is misleading.
4. On the other hand, we almost always have some prior information about the plausibility of models. This information might include positivity constraints (for density, permeability, and temperature), bounds (porosity between 0 and 1), or smoothness.
5. Most petroleum inverse problems related to fluid flow are nonlinear. The calculation of gradients is an important and expensive part of the problem; it must be done efficiently.
6. Because of the large cost of computing the output of a reservoir simulation model, trial and error approaches to inverting data are impractical.
7. Probabilisitic estimates or bounds are often the most meaningful. For nonlinear problems, this is usually best accomplished using Monte Carlo methods.
8. The ultimate goal of inverse theory (and history matching) is to make informed decisions on investments, data aquisition, and reservoir management. Good decisions can only be made if the uncertainty in future performance, and the consequences of actions can be accurately characterized.

[1] This is part of the title of a famous paper by Jackson [1]: "Interpretation of inaccurate, insufficient, and inconsistent data."

Other general references

Several good books on geophysical inverse theory are available. Menke [2] provides good introductory information on the probabilistic interpretation of an answer to an inverse problem, and much good material on the discrete inverse problem. Parker [3] contains good material on Hilbert space, norms, inner products, functionals, existence and uniqueness (for linear problems), resolution and inference, and functional differentiation. He does not, however, get very deeply into nonlinear problems or stochastic approaches. Tarantola [4] comes closest to covering the material on linear inverse problems, but has very little material on calculation of sensitivities. Sun [5] focusses on problems related to flow in porous media, and contains useful material on the calculation of sensitivities for flow and transport problems. A highly relevant free source of information on inverse theory is the book by John Scales [6].

No single book contains a thorough description of the nonlinear developments in inverse theory or the applications to petroleum engineering. Most of the material that is specifically related to petroleum engineering is based on our publications.

The choice of material for these notes is based on the observation that while many scientists and engineers have good intuition for the outcome of an experiment, they often have poor intuition regarding inverse problems. This is not to say that they can not estimate some parameter values that might result in a specified response, but that they have little feel for the degree of nonuniqueness of the answer, or of the relationship of their answer to other answers or to the true parameters. We feel that this intuition is best developed through a study of linear theory and that the method of Backus and Gilbert is good for promoting understanding of many important concepts at a fundamental level. On the other hand, the Backus and Gilbert method can produce solutions that are not *plausible* because they are too erratic or too smooth. We, therefore, introduce methods for incorporating prior information on smoothness and variability. One of the principal uses of these methods is to investigate risk and to make informed decisions regarding investment. For many petroleum engineering problems, evaluation of uncertainty requires the ability to generate a meaningful distribution multiple of models. Characterization of uncertainty for highly nonlinear inverse problems, and the methods of sampling from high-dimensional probability density functions are discussed in Chapter 10.

Most history-matching problems in petroleum engineering are strongly nonlinear. Efficient incorporation of production-type data (e.g. pressure, concentration, water-oil ratio, etc.) requires the calculation of sensitivity coefficients or the linearized relationship between model and data. This is the topic of Chapter 9.

Although history matching has typically been a “batch process” in which all data are assimilated simultaneously, the installation of permanent sensors in wells has increased the need for methods of updating reservoir models by sequentially assimilating data as it becomes available. A method for doing this is described in Chapter 11.

1 Introduction

If it were possible for geoloscientists and engineers to know the locations of oil and gas, the locations and transmissivity of faults, the porosity, the permeability, and the multi-phase flow properties such as relative permeability and capillary pressure at all locations in a reservoir, it would be conceptually possible to develop a mathematical model that could be used to predict the outcome of any action. The relationship of the *model* variables, m, describing the system to observable variables or *data*, d, is denoted

$$g(m) = d.$$

If the model variables are known, outcomes can be predicted, usually by running a numerical reservoir simulator that solves a discretized approximation to a set of partial differential equations. This is termed the *forward problem*.

Most oil and gas reservoirs are inconveniently buried beneath thousands of feet of overburden. Direct observations of the reservoir are available only at well locations that are often hundreds of meters apart. Indirect observations are typically made at the surface, either at the well-head (production rates and pressures) or at distributed locations (e.g. seismic). In the *inverse problem*, the observations are used to determine the variables that describe the system. Real observations are contaminated with errors, ϵ, so the inverse problem is to "solve" the set of equations

$$d_{\text{obs}} = g(m) + \epsilon$$

for the model variables, with the goal of making accurate predictions of future performance.

1.1 The forward problem

In a forward problem, the physical properties of some system (system or model parameters) are known, and a deterministic method is available for calculating the response or outcome of the system to a known stimulus. The physical properties are referred to as system or model parameters. A typical forward problem is represented by a differential equation with specified initial and/or boundary conditions. A simple example of a forward problem of interest to petroleum engineers is the following steady-state problem for a one-dimensional flow in a porous medium:

$$\frac{d}{dx}\left(\frac{k(x)A}{\mu}\frac{dp(x)}{dx}\right) = 0, \tag{1.1}$$

for $0 < x < L$, and

$$\left.\frac{dp}{dx}\right|_{x=L} = -\frac{q\mu}{k(L)A}, \tag{1.2}$$

$$p(0) = p_e \tag{1.3}$$

where A (cross sectional area to flow in cm^2), μ (viscosity in cp), q (flow rate in cm^3/s), and pressure p_e (atm) are assumed to be constant. The length of the system in cm is represented by L. The function $k(x)$ represents the permeability field in Darcies. This steady-state problem could describe linear flow in either a core or a reservoir. For this forward problem, the model parameters, which are assumed to be known, are A, L, μ, and $k(x)$. The stimulus for the system (reservoir or core) is provided by prescribing q (the flow rate out the right-hand end) and $p(0)$ (the pressure at the left-hand end), for example, by the boundary conditions, which are assumed to be known exactly. The system output or response is the pressure field, which can be determined by solving the boundary-value problem. The solution of this steady-state boundary-value problem is given by

$$p(x) = p_e - \frac{q\mu}{A}\int_0^x \frac{1}{k(\xi)}\,d\xi. \tag{1.4}$$

If the emphasis is on the relationship between the permeability field and the pressure, we might formally write the relationship between pressure, p_i, at a location, x_i, and the permeability field as $p_i = g_i(k)$. This expression indicates that the function g_i specifies the relation between the permeability field and pressure at the point x_i.

Forward problems of interest to us can usually be represented by a differential equation or system of differential equations together with initial and/or boundary conditions. Most such forward problems are well posed, or can be made to be well posed by imposing natural physical constraints on the coefficients of the differential equation(s) and the auxiliary conditions. Here, auxiliary conditions refer to the initial and boundary conditions. A boundary-value problem, or initial boundary-value problem, is said to be *well posed* in the sense of Hadamard [7], if the following three criteria are satisfied:

(a) the problem has a solution,
(b) the solution is unique, and
(c) the solution is a continuous function of the *problem data*.

It is important to note that the *problem data* include the functions defining the initial and boundary conditions and the coefficients in the differential equation. Thus, for the boundary-value problem of Eqs. (1.1)–(1.3), the problem data refers to p_e, $q\mu/k(L)A$ and $k(x)$.

If $k(x)$ were zero in some part of the core, then we can not obtain steady-state flow through the core and the pressure solution of Eq. (1.4) is not defined, i.e. the boundary-value problem of Eqs. (1.1)–(1.3) does not have a solution for $q > 0$. However, if we impose the restriction that $k(x) \geq \delta > 0$ for any arbitrarily small δ then the boundary-value problem is well posed.

If a problem is not well posed, it is said to be *ill posed*. At one time, most mathematicians believed that ill-posed problems were incorrectly formulated and nonphysical. We know now that this is incorrect and that a great deal of useful information can be obtained from ill-posed problems. If this were not so, there would be little reason to study inverse problems, as almost all inverse problems are ill posed.

1.2 The inverse problem

In its most general form, an inverse problem refers to the determination of the plausible physical properties of the system, or information about these properties, given the observed response of the system to some stimulus. The observed response will be referred to as observed data. For example, for the steady-state problem considered above, an inverse problem could represent the problem of determining the permeability field from pressure data measured at points in the interval $[0, L]$. Note that measured or observed data is different from the problem data introduced in the definition of a well-posed problem.

In both forward and inverse problems, the physical system is characterized by a set of model parameters, where here, a model parameter is allowed to be either a function or a scalar. For the steady single-phase flow problem, the model parameters can be chosen as the inverse permeability ($m(x) = 1/k(x)$), fluid viscosity, cross sectional area A and length L. Note, however, the model parameters could also be chosen as $(k(x)A)/\mu$ and L. If we were to attempt to solve Eq. (1.1) numerically, we might discretize the permeability function, and choose $k_i = k(x_i)$ for a limited number of integers i as our parameters. The choice of model parameters is referred to as a parameterization of the physical system. Observable parameters refer to those that can be observed or measured, and will simply be referred to as observed data. For the above steady-state problem, forcing fluid to flow through the porous medium at the specified rate q provides the stimulus and measured values of pressure at certain locations that represent observed data. Pressure can be measured only at a well location, or in the case where the system represents a core, at locations where pressure transducers are situated. Although the relation between observed data and model parameters is often referred to as the model, we will refer to this relationship as the (assumed) theoretical model, because we wish to refer to any feasible set of specific model parameters as a model. In the continuous inverse problem, the model or model parameters may represent a function or set of functions rather than simply a discrete set of parameters. For the steady-state problem of Eqs. (1.1)–(1.3), the boundary-value problem implicitly defines the theoretical model with the explicit relation between observable parameters and the model or model parameters given by Eq. (1.4).

The inverse problem is almost never well posed. In the cases of most interest to petroleum reservoir engineers and hydrogeologists, an infinite number of equally good solutions exist. For the steady-state problem, the general inverse problem represents the determination of information about model parameters (e.g. $1/k(x)$, μ, A, and L) from pressure measurements. As pressure measurements are subject to noise, measured pressure data will not, in general, be exact. The assumed theoretical model may also not be exact. For the example problem considered earlier, the theoretical model assumes constant viscosity and steady-state flow. If these assumptions are invalid, then we are using an approximate theoretical model and these modeling errors should be accounted for when generating inverse solutions.

For now, we state the general inverse problem as follows: determine plausible values of model parameters given inexact (uncertain) data and an assumed theoretical model relating the observed data to the model. For problems of interest to petroleum engineers, the theoretical model always represents an approximation to the true physical relation between physical and/or geometric properties and data. Left unsaid at this point is what is meant by plausible values (solutions) of the inverse problems. A plausible solution must of course be consistent with the observed data and physical constraints (permeability and porosity can not be negative), but for problems of interest in petroleum reservoir characterization, there will normally be an infinite number of models satisfying this criterion. Do we want to choose just one estimate? If so, which one? Do we want to determine several solutions? If so, how, why, and which ones? As readers will see, we have a very definite philosophical approach to inverse problems, one that is grounded in a Bayesian viewpoint of probability and assumes that prior information on model parameters is available. This prior information could be as simple as a geologist's statement that he or she believes that permeability is 200 md plus or minus 50. To obtain a mathematically tractable inverse problem, the prior information will always be encapsulated in a prior probability density function. Our general philosophy of the inverse problem can then be stated as follows: given prior information on some model parameters, inexact measurements of some observable parameters, and an uncertain relation between the data and the model parameters, how should one modify the prior probability density function (PDF) to include the information provided by the inexact measurements? The modified PDF is referred to as the a posteriori probability density function. In a sense, the construction of the a posteriori PDF represents the solution to the inverse problem. However, in a practical sense, one wishes to construct an estimate of the model (often, the maximum a posteriori estimate) or realizations of the model by sampling the a posteriori PDF. The process of constructing a particular estimate of the model will be referred to as estimation; the process of constructing a suite of realizations will be referred to as simulation.

Here, our emphasis is on estimating and simulating permeability and porosity fields. Our approach to the application of inverse problem theory to petroleum reservoir characterization problems may be summarized as follows.

1. Postulate a prior PDF for the model parameters from analog fields, core, logs, and seismic data. We will often assume that the prior PDF is multi-variate Gaussian, in which case the means and the covariance fully define the stochastic model.
2. Formulate the a posteriori PDF conditioned to all observed data. Data could include both production data and "hard" data (direct measurements of the variables to be estimated) for the rock property fields.
3. Construct a suite of realizations of the rock property fields by sampling the a posteriori PDF.
4. Generate a reservoir performance prediction under proposed operating conditions for each realization. This step is done using a reservoir simulator.
5. Construct statistics (e.g. histogram, mean, variance) from the set of predicted outcomes for each performance variable (e.g. cumulative oil production, water–oil ratio, breakthrough time). Determine the uncertainty in predicted performance from the statistics.

In our view, steps 2 and 3 are both vital, albeit difficult, and most of our research effort has focussed either on step 3 or on issues related to computational efficiency including the development of methods to efficiently generate sensitivity coefficients. Note that if one simply generates a set of rock property fields consistent with all observed data, but the set does not characterize the true uncertainty in the rock property fields (in our language, does not represent a correct sampling of the a posteriori PDF), steps 4 and 5 can not be expected to yield a meaningful characterization of the uncertainty in predicted reservoir performance.

2 Examples of inverse problems

The inverse problems examples presented in this chapter illustrate the concepts of data, model, uniqueness, and sensitivity. Each of these concepts will be developed in much greater depth in subsequent chapters. The examples are all quite simple to describe and understand, but several are difficult to solve.

2.1 Density of the Earth

The mass, M, and moment of inertia, I, of the Earth are related to the density distribution, $\rho(r)$, (assuming mass density is only a function of radius) by the following formulas:

$$M = 4\pi \int_0^a r^2 \rho(r)\, dr, \tag{2.1}$$

$$I = \frac{8\pi}{3} \int_0^a r^4 \rho(r)\, dr, \tag{2.2}$$

where a is the radius of the Earth. If the true density is known for all r, then it is easy to compute the mass and the moment of inertia. In reality, the mass and moment of inertia can be estimated from measurements of the precession of the axis of rotation and the gravitational constant; the density distribution must be estimated. The data vector consists of the "observed" mass and moment of inertia of the Earth:

$$d = [M \quad I]^{\mathrm{T}} \tag{2.3}$$

and the model variable, $m = \rho(r)$, is the density. (Throughout this book, the superscript T on a matrix or vector denotes its transpose.) The relationship between the model variable and the theoretical data is

$$d = \int_0^a \begin{bmatrix} 4\pi r^2 \\ \frac{8\pi}{3} r^4 \end{bmatrix} m\, dr. \tag{2.4}$$

Note that, in this example, the dimension of the model to be estimated is infinite, while the dimension of the data space is just 2. Prior information might be a lower

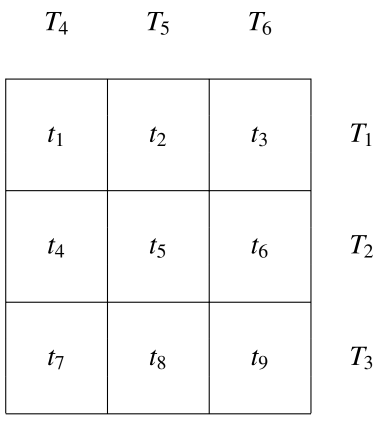

Figure 2.1. The array of nine blocks with traveltime parameters, t_i, and the six measurement locations for total traveltime, T_i, across the array.

bound on the density. A loose lower bound would be that density is positive. A reasonable lower bound with more information is that density is greater than or equal to 2250 kg/m^3. Although it is easy to generate a model that fits the data exactly, unless other information is available, the uncertainty in the estimated density at a point or a radius is unbounded.

Note also that the theoretical relationship between the density and the data in this example is only approximate as the Earth is not exactly spherical, and there is no a priori reason to believe that the density is only a function of radius.

2.2 Acoustic tomography

One of the simplest examples that demonstrates the concepts of sensitivity, nonuniqueness, and inconsistency is the problem of estimation of the spatial distribution of acoustic slowness (1/velocity) from measurements of traveltime along several ray paths through a solid body. For simplicity, we assume that the material properties are uniform within each of the nine blocks (Fig. 2.1) and we only consider paths that are orthogonal to the block boundaries so that refraction can be ignored and the paths remain straight. If t denotes the acoustic slowness of a homogeneous block, and T denotes the time required to travel a distance D within or across a block, then $T = tD$. Consider a 3×3 array of blocks of various materials shown in Fig. 2.1. Each homogeneous block is 1 unit in width by 1 unit in height. Measurements of traveltime have been made for each column and each row of blocks. If the slowness of the (1, 1) block is t_1, the slowness of the (1, 2) block is t_2, and the slowness of the (1, 3) block is t_3, then T_1, the total traveltime for a sound wave to travel across the first row of blocks, is given by $T_1 = t_1 + t_2 + t_3$. Similar relations hold for the other rows and columns. If the measurements of traveltime are exact, the entire set of relations between measurements and slowness in each block is

$$\begin{aligned}
T_1 &= t_1 + t_2 + t_3 \\
T_2 &= t_4 + t_5 + t_6 \\
T_3 &= t_7 + t_8 + t_9 \\
T_4 &= t_1 + t_4 + t_7 \\
T_5 &= t_2 + t_5 + t_8 \\
T_6 &= t_3 + t_6 + t_9.
\end{aligned} \tag{2.5}$$

Given measured values of T_i, $i = 1, 2, \ldots, 6$, the inverse problem is to determine information about the acoustic slownesses, t_j, $j = 1, 2, \ldots, 9$. More specifically, we may wish to determine the set of all solutions of Eq. (2.6)

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \end{bmatrix}. \tag{2.6}$$

With the notation commonly used in this book, Eq. (2.6) is written as

$$d = Gm, \tag{2.7}$$

where the data, d, is the vector of traveltime measurements, i.e.

$$d = [T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5 \quad T_6]^{\mathrm{T}}, \tag{2.8}$$

the model, m, is the vector of slowness values given by

$$m = [t_1 \quad t_2 \quad \ldots \quad t_9]^{\mathrm{T}} \tag{2.9}$$

and the sensitivity matrix, G, is the matrix that relates the data to the model variables and is given by

$$G = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}. \tag{2.10}$$

The reason for calling G the sensitivity matrix is easily understood by examining the particular row of G associated with a particular measurement. Note that there are as many rows as there are measurements. Each row has nine elements in this example, one for each model variable. The element in the ith row and jth column of G gives the "sensitivity" $(\partial T_i/\partial t_j)$ of the ith measurement to a change in the jth model variable. So, for example, the fourth measurement is only sensitive to t_1, t_4, and t_7. As can be seen easily from Eq. (2.5) or (2.6), $\partial T_4/\partial t_j = 1$ for $j = 1, 4, 7$ and $\partial T_4/\partial t_j = 0$ otherwise. Note when $\partial T_i/\partial t_j = 0$, a change in the acoustic slowness t_j will not affect the value of the traveltime T_i, thus we can find no information on the value of t_j from the measured value of T_i.

When we want to visualize the sensitivity for a particular measurement, we often display the row in a natural ordering, one that corresponds to the spatial distribution of model parameters; see Fig. 2.1. Here, we let G_i denote the ith row of G and display G_2 as:

0	0	0
1	1	1
0	0	0

. This display is convenient as it indicates that the second traveltime measurement only depends on the slowness values in the second row. Similarly, G_4 can be displayed as:

1	0	0
1	0	0
1	0	0

, which, when compared to Fig. 2.1 shows clearly that the fourth traveltime measurement is only sensitive to the slowness values of the first column of blocks. Of course, when the models become very large, we will not display all of the numbers. Instead we will use a shading scheme that shows the strength of the sensitivity by the darkness of the grayscale.

Solutions

Suppose that the values of acoustic slowness are such that the exact measurement of one-way traveltime in each of the columns and rows is equal to 6 units (i.e. $T_i = 6$ for all i). Clearly, a homogeneous model for which the slowness of each block is 2 will satisfy this data exactly, i.e. with all $t_j = 2$ and all $T_i = 6$, Eq. (2.6) is satisfied. Similarly, it is easy to see that

$$\hat{m} = [2 \quad 2 \quad 2 \quad 2+b \quad 2-b \quad 2 \quad 2-b \quad 2+b \quad 2]^{\mathrm{T}}. \tag{2.11}$$

is a solution of Eq. (2.6), for any real constant b, when all entries of the data vector are equal to 6. A little examination shows that the following models also satisfy the data exactly:

1	2	3
2	2	2
3	2	1

−2	0	8
−2	6	2
10	0	−4

2+a	2	2−a
2	2	2
2−a	2	2+a

2+b	2−b	2
2−b	2+b	2
2	2	2

,

Box 1. Nonuniqueness

The null space of G is the set of all real, nine-dimensional column vectors m such that $Gm = 0$. It is easy to verify that each of the following models represent vectors in the null space of G,

0	1	−1		1	−1	0		0	0	0		0	0	0
0	−1	1		−1	1	0		0	1	−1		1	−1	0
0	0	0		0	0	0		0	−1	1		−1	1	0

.

In fact, the four vectors represented by these four models represent a basis for the null space of G, so any vector in the null space of G can be written as a unique linear combination of these four vectors. If v is any vector in the null space of G and m is a vector of acoustic slownesses that satisfies $Gm = d$ where d is the vector of measured traveltimes, then the model $m + v$ also satisfies the data because

$$G(m + v) = Gm + Gv = d. \tag{2.12}$$

Thus, we can add any linear combination of models (vectors) in the null space of G to a model that satisfies the traveltime data and obtain another model which also satisfies the data.

This acoustic tomography problem has an infinite number of models that satisfy the data exactly for certain data. As there are fewer traveltime data than model variables, this is not surprising. We show next, however, that for other values of the traveltime data, there are no values of acoustic slowness that satisfy Eq. (2.6).

No solution

As measurements are always noisy, let us assume that because of the inaccuracy of the timing, the following measurements were made:

$$T_{obs} = [6.07 \quad 6.07 \quad 5.77 \quad 5.93 \quad 5.93 \quad 6.03]^T. \tag{2.13}$$

Interestingly, despite the fact that there are fewer data than model parameters, there are *no* models that satisfy this data. Eq. (2.5) indicates that T_1 should be the sum of the slowness values in the first row, T_2 should be the sum of the slowness values in the second row, and T_3 should be the sum of the slowness values in the third row. Thus

$$T_1 + T_2 + T_3 = t_1 + t_2 + \cdots + t_9. \tag{2.14}$$

But T_4 is the sum of slowness values in column one, and similarly for T_5 and T_6 so if there are values of the model parameters that satisfy these data, we must also have

$$T_4 + T_5 + T_6 = t_1 + t_2 + \cdots + t_9. \tag{2.15}$$

From these results, it is clear that in order for a solution to exist, we must have $T_1 + T_2 + T_3 = T_4 + T_5 + T_6$, but when the data contain noise this is extremely unlikely. For the data of Eq. (2.13), $T_1 + T_2 + T_3 = 17.91$ and $T_4 + T_5 + T_6 = 17.89$, so that with these data, Eq. (2.6) has no solution. Generally, in this case one seeks a solution that comes as close as possible to satisfying the data. A reasonable measure of the goodness of fit is the sum of the squared errors,

$$O(m) = \sum_{j=1}^{6}(d_{\mathrm{obs},j} - d_j(m))^2 = (d_{\mathrm{obs}} - Gm)^T(d_{\mathrm{obs}} - Gm). \tag{2.16}$$

Here, we have introduced notation that will be used throughout this book. Specifically, $d_{\mathrm{obs},j}$ denotes the j component of the vector of measured or observed data (traveltimes in this example), and d_j denotes the corresponding data that would be calculated (predicted) from the assumed theoretical model relationship (Eq. (2.7) in this example) for a given model variable, m. $O(m)$ denotes an objective function to be minimized and is defined by the first equality of Eq. (2.16). The second equality of Eq. (2.16) follows from standard matrix vector algebra. One solution that has the minimum data mismatch is

2.011	2.011	2.044
2.011	2.011	2.044
1.911	1.911	1.944

, or equivalently,

$$\hat{m} = [2.011\;\; 2.011\;\; 2.044\;\; 2.011\;\; 2.011\;\; 2.044\;\; 1.911\;\; 1.911\;\; 1.944]^{\mathrm{T}}. \tag{2.17}$$

From the last equality of Eq. (2.16), it is clear that if m is a least-squares solution then so is $m + v$ where v is a solution in the null space of G. Thus, similar to the case where data are exact, an infinite number of solutions satisfy the data equally well in the least-squares sense.

2.3 Steady-state 1D flow in porous media

Here, the steady-state flow problem introduced in Section 1.1 is formulated as a linear inverse problem. It is assumed that the cross sectional area A, the viscosity μ, the flow rate q, and the end pressure p_e in Eq. (1.4) are known exactly. Although many other characteristics of the porous medium are also unknown (e.g. color, mineralogy, grain size, porosity), we will treat the permeability field as the only unknown. Let

$$d(x) = p_e - p(x) \tag{2.18}$$

and

$$d_i = d(x_i), \tag{2.19}$$

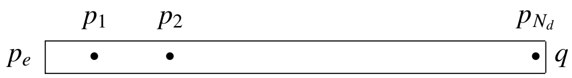

Figure 2.2. A porous medium with constant pressure p_e at the left-hand end, constant production rate q at the right-hand end, and N_d measurements of pressure at various locations along the medium.

for $i = 1, 2, \ldots, N_d$ where the x_is define N_d distinct locations in the interval $[0, L]$ at which pressure measurements are recorded. If the inverse problem under consideration involves linear flow in a reservoir, the x_is would correspond to points at which wells are located. However, the steady-state problem could also represent flow through a core with the x_is representing locations of pressure transducers. The d_is now represent pressure drops, or more generally, pressure changes. However, for simplicity, the data d_i for this problem will be referred to simply as pressure data.

For linear flow problems, it is convenient to define the model variable, $m(x)$, as inverse permeability

$$m(x) = \frac{1}{k(x)}. \tag{2.20}$$

With this notation, Eq. (1.4) can be written as

$$d(x) = \int_0^L G(x, \xi) m(\xi)\, d\xi, \tag{2.21}$$

where

$$G(x, \xi) = \begin{cases} q\mu/A & \text{for } \xi \le x, \\ 0 & \text{for } \xi > x. \end{cases} \tag{2.22}$$

Note that G is only nonzero in the region between the constant pressure boundary location and the measurement location, so the data (pressure drop) are only sensitive to the permeability in that region; changing the permeability beyond the measurement location would have no effect on the measurement.

Assuming pressure data, $d_i = d(x_i)$, are recorded at $x_1 < x_2 < \cdots < x_{N_d}$, Eq. (2.21) is replaced by the inverse problem

$$d_i = \frac{q\mu}{A} \int_0^{x_i} m(\xi)\, d\xi = \int_0^L G_i(\xi) m(\xi)\, d\xi, \tag{2.23}$$

for $i = 1, 2, \ldots, N_d$ where

$$G_i(\xi) = G(x_i, \xi). \tag{2.24}$$

In a general sense, solving this inverse problem means determining the set of functions that satisfy Eq. (2.23) given the values of the d_is.

If only a single pressure drop measurement, $d_1 = d(L)$ is recorded at the right-hand end ($x = L$) of the system, the problem is to solve

$$d_1 = p_e - p(L) = \frac{q\mu}{A} \int_0^L m(x)\, dx, \tag{2.25}$$

for $m(x)$. Clearly there is not a unique function that satisfies Eq. (2.25) since if $m(x)$ satisfies this equation, and $u(x)$ is any function such that

$$\int_0^L u(\xi)\, d\xi = 0, \tag{2.26}$$

then the function $m(x) + u(x)$ also satisfies Eq. (2.25).

Discretization

A discrete inverse problem for the estimation of permeability in steady-state flow can be formulated in more than one way. By approximating the integral in Eq. (2.23) or (2.25) using numerical quadrature, a discrete inverse problem can be obtained. A second procedure for obtaining a discrete inverse problem would be to discretize the differential equation, i.e. write down a finite-difference scheme for the steady-state flow problem of Eqs. (1.1)–(1.3). There is no guarantee that these two approaches are equivalent. Most work on petroleum reservoir characterization is focussed on the second approach, i.e. when observed and predicted data correspond to production data, the forward problem is represented by a reservoir simulator. Here, however, we consider the general continuous inverse problem, Eq. (2.23), and use a numerical quadrature formula to obtain a discrete inverse problem.

In many cases, the best choice of a numerical integration procedure would be a Gauss–Legendre formula (see, for example, chapter 18 of Press *et al.* [8]). But, since our purpose is only illustrative, a midpoint rectangular rule is applied here to perform numerical integration. Let M be a positive integer,

$$x_{1/2} = 0 \tag{2.27}$$

and

$$\Delta x = \frac{L}{M}. \tag{2.28}$$

Then let

$$x_{j+1/2} = x_{j-1/2} + \Delta x \tag{2.29}$$

and

$$x_j = \frac{x_{j-1/2} + x_{j+1/2}}{2}, \tag{2.30}$$

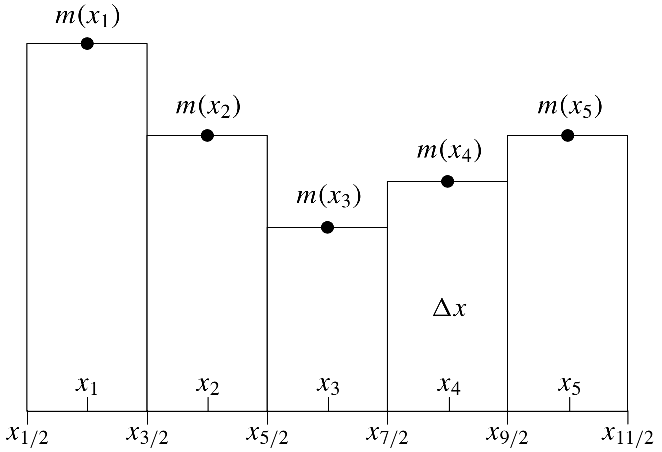

Figure 2.3. Discretization of the porous medium for integration using the midpoint rectangular method. In this figure, $m(x_i)$ is the value of $m(x)$ in the middle of the interval that extends from $x_{i-1/2}$ to $x_{i+1/2}$.

for $j = 1, 2, \ldots, M$. Using the preceding partitioning of the interval [0, L], defining the constant β by

$$\beta = \frac{q\mu}{A}\Delta x, \tag{2.31}$$

and applying the midpoint rectangular rule [9] for integration, Eq. (2.25) can be approximated by

$$d_1 = \beta \sum_{j=1}^{M} m(x_j). \tag{2.32}$$

For simplicity in notation, it is again assumed that pressure data are measured at $x_{r_i+1/2}$, $i = 1, 2, \ldots, N_d$, where the r_is are a subset of $\{i\}_{i=1}^{M}$ and $r_1 < r_2 < \cdots < r_{N_d}$. The pressure change data at $x_{r_i+1/2}$ is denoted by $d_{\text{obs},i}$ with corresponding calculated data represented by d_i for $i = 1, 2, \ldots, N_d$. With this notation, applying the midpoint rectangular integration rule to Eq. (2.23) (with i replaced by $r_i + 1/2$) gives the approximation

$$d_i = \beta \sum_{j=1}^{r_i} m(x_j), \tag{2.33}$$

for $i = 1, 2, \ldots, N_d$.

Now let d denote the vector of calculated data given by

$$d = [d_1, d_2, \ldots, d_{N_d}]^{\mathrm{T}}, \tag{2.34}$$

and let d_{obs} denote the corresponding vector of observed (measured) pressure drop data. Also let $G = [g_{i,j}]$ be the $N_d \times M$ matrix with the entry in the ith row and jth column defined by

$$g_{i,j} = \beta, \tag{2.35}$$

for $j \leq r_i$ and

$$g_{i,j} = 0, \tag{2.36}$$

for $j > r_i$. Then defining $m_i = m(x_i)$ for all i, and using the notation of Eqs. (3.4) and (3.5), Eq. (2.33) can be written as

$$d = Gm, \tag{2.37}$$

where G is an $N_d \times M$ matrix. Solutions of Eq. (2.23) are functions and as such represent elements of an infinite-dimensional linear space $L^2[0, L]$, whereas, "solutions" of Eq. (2.37) are vectors and are elements of a finite-dimensional linear space. In replacing $m(x)$ by its values at discrete points, the model has been reparameterized. To approximate $m(x)$ from its values at discrete points would require interpolation. Alternately, one could set $m(x) = m_i$, for $x_{i-1/2} < x_{i+1/2}$, i.e. $k_i = 1/m_i$, which corresponds to defining one permeability for each "gridblock" in the interval $[0, L]$.

For the problem under consideration, the discrete inverse problem is specified as

$$d_{\text{obs}} = Gm + \varepsilon, \tag{2.38}$$

where d_{obs} is the vector of observed "pressure drop data" and ε represents measurement errors. The objective is to characterize the set of vectors m that in some sense satisfy or are consistent with Eq. (2.38).

In the case where pressure drop data is available at $x_{i+1/2}$ for $i = 1, 2, \ldots, N_d = M$, G is a square $N_d \times N_d$ matrix which can be written as

$$G = \beta \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}. \tag{2.39}$$

Note that G is a lower triangular matrix with all diagonal elements nonzero. Thus, G is nonsingular and the unique solution of Eq. (2.38) is $m = G^{-1}d_{\text{obs}}$.

If the number of data is fewer than the number of model parameters (components of m), $N_d < M$, then Eq. (2.38) represents N_d equations in M unknowns. As the number of equations is fewer than the number of unknowns, the system of equations is said to be underdetermined. Similarly, if the number of equations is greater than the number of unknowns, $N_d > M$, the problem is said to be overdetermined. A detailed classification of underdetermined, overdetermined and mixed determined problems is presented later.

Underdetermined problem

Suppose the interval $[0, L]$ is partitioned into five gridblocks of equal size and pressure drop data $d_{\text{obs},1}$ is observed at $x_{7/2}$ and $d_{\text{obs},2}$ is observed at $x_{11/2} = L$. Then Eq. (2.38) becomes

$$\begin{bmatrix} d_{\text{obs},1} \\ d_{\text{obs},2} \end{bmatrix} = \beta \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{bmatrix}, \tag{2.40}$$

or, equivalently,

$$\frac{m_1 + m_2 + m_3}{3} = \frac{d_{\text{obs},1}}{3\beta} \tag{2.41}$$

and

$$\frac{m_1 + m_2 + m_3 + m_4 + m_5}{5} = \frac{d_{\text{obs},2}}{5\beta}. \tag{2.42}$$

Clearly the preceding two equations uniquely determine the average value of the first three model parameters and the average value of all five model parameters, but do not uniquely determine the values of the individual m_is. There are, in fact, an infinite number of vectors m that satisfy Eq. (2.40).

Integral equation

Many inverse problems are naturally formulated as integral equations, instead of matrix equations. In Chapter 1, we considered a boundary-value problem for one-dimensional, single-phase, steady-state flow; see Eqs. (1.1)–(1.3). Here we assume that the constant flow rate q, viscosity μ and cross sectional area A are known exactly, and rewrite Eq. (1.4) as

$$p_e - p(x) = C \int_0^x \frac{1}{k(\xi)}\, d\xi, \tag{2.43}$$

where the constant C is defined by $C = (q\mu)/A$, q is the volumetric flow rate, μ is the viscosity, and A is the cross sectional area to flow. If the function $p_e - p(x)$ is also known at a measurement location x_0, then Eq. (1.4) represents a Fredholm integral equation of the first kind [10]. The inverse problem is then to find a solution, or characterize the solutions, of the integral equation, i.e. to find a model $m(x) = k(x)$ which satisfies Eq. (2.43). Stated this way the integral equation, and hence the inverse problem, is nonlinear. This particular problem is somewhat atypical as it is possible to reformulate the problem as a linear inverse problem by defining the model as

$$m(x) = 1/k(x) \tag{2.44}$$

and rewrite the integral equation as

$$p_e - p(x) = C \int_0^x m(\xi)\, d\xi. \tag{2.45}$$

Although for the physical problem under consideration, $m(x)$ must be positive for $k(x) = 1/m(x)$ to represent a plausible permeability field, here it is convenient to define the inverse problem as the problem of finding piecewise continuous real functions, $m(x)$, defined on $[0, L]$ which satisfy Eq. (2.43) and to define the model space M as the set of all positive piecewise continuous functions defined on $[0, L]$. (M is a real vector space, whereas the subset of M consisting of all positive real-valued functions defined on $[0, L]$ is not a vector space.) The operator G defined on the model space by

$$[Gm](x) = C \int_0^x m(\xi)\, d\xi, \tag{2.46}$$

is a linear operator, i.e. for any constants α and β and any two models $m_1(x)$ and $m_2(x)$

$$G\big(\alpha m_1 + \beta m_2\big) = \alpha G m_1 + \beta G m_2. \tag{2.47}$$

Thus, by replacing the parameter $k^{-1}(x)$ by $m(x)$, we have converted the original nonlinear inverse problem (nonlinear integral equation) to a linear inverse problem. Also note Gm is a continuous function of x. Defining

$$d(x) = p_e - p(x), \tag{2.48}$$

Eq. (2.45) can be written as

$$d(x) = [Gm](x). \tag{2.49}$$

Note the similarity to Eq. (2.7).

If the pressure change across the core, $d(L) = p_e - p(L) = p(0) - p(L)$, is measured, the inverse problem becomes to find models $m(x)$ such that

$$d(L) = [Gm](L), \tag{2.50}$$

where the linear operator G is now defined by

$$[Gm](L) = C \int_0^L m(\xi)\, d\xi = \frac{q\mu}{A} \int_0^L m(\xi)\, d\xi. \tag{2.51}$$

Note that G defined by Eq. (2.51) maps functions $m(x)$ in the model space into the set of real numbers.

2.4 History matching in reservoir simulation

A major inverse problem of interest to reservoir engineers is the estimation of rock property fields by history-matching production data. Here, we introduce the complexities, using a single-phase, flow problem.

The finite-difference equations for one-dimensional single-phase flow can be obtained from the differential equation,

$$C_1 \frac{\partial}{\partial x}\left(\frac{k(x)A}{\mu}\frac{\partial p(x,t)}{\partial x}\right) - q\delta(x-x_0) = C_2\phi(x)c_t A\frac{\partial p(x,t)}{\partial t}, \quad \text{for } 0 < x < L \text{ and } t > 0, \tag{2.52}$$

$$\frac{\partial p(0,t)}{\partial x} = \frac{\partial p(L,t)}{\partial x} = 0, \qquad \text{for all } t > 0 \tag{2.53}$$

and

$$p(x,0) = p_{\text{in}}, \qquad \text{for all } t > 0, \tag{2.54}$$

where p_{in} is the initial pressure which is assumed to be uniform. The constants C_1 and C_2 which appear in Eq. (2.52) depend on the system of units. In SI units, both constants are equal to unity. Here, we use oil field units in which case, $C_1 = 1.127 \times 10^{-3}$ and $C_2 = 5.615$. Eq. (2.53) represents no flow boundaries at the ends of the system. In Eq. (2.52), A has units of ft^2 and represents the cross sectional area to flow which we assume to be uniform; μ in centipoise represents the fluid viscosity which we assume to be constant; $k(x)$ in millidarcies represents a heterogeneous permeability field; $\phi(x)$ represents a heterogeneous porosity field; c_t is the total compressibility in psi^{-1} and is assumed to be constant. In Eq. (2.52), the Dirac delta function, $\delta(x-x_0)$, is used to model a production well at x_0 produced at a rate q. The units of the delta function are ft^{-1}. For simplicity, we partition the reservoir into N uniform gridblocks of width Δx in the x direction, let x_i denote the center of the ith gridblock, let $x_{i+1/2}$ and $x_{i-1/2}$, respectively, denote the right- and left-hand boundaries of gridblock i. The grid system is shown in Fig. 2.4, where the circles represent the gridblock centers.

We assume that a single producing well is located in gridblock k. Integrating Eq. (2.52) with respect to x over the ith gridblock, i.e. from $x_{i-1/2}$ to $x_{i+1/2}$, and using the fact that the resulting integral of the Dirac delta function is equal to 1 gives

$$\begin{aligned} C_1\left(\frac{k(x)A}{\mu}\frac{\partial p}{\partial x}\right)_{(x_{i+1/2},t)} &- C_1\left(\frac{k(x)A}{\mu}\frac{\partial p}{\partial x}\right)_{(x_{i-1/2},t)} - q\delta_{i,k} \\ &= C_2\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\phi(x)c_t A\frac{\partial p(x,t)}{\partial t}\right)dx \\ &= \phi_i c_t A\Delta x\left(\frac{\partial p}{\partial t}\right)_{(x_i,t)}, \end{aligned} \tag{2.55}$$

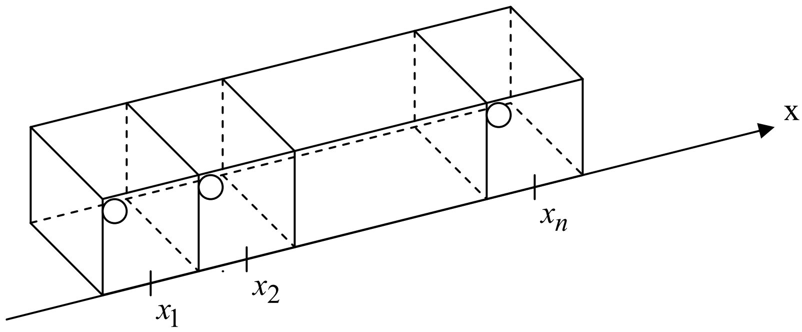

Figure 2.4. One-dimensional grid system.

for $i = 1, 2, \ldots, N$ and $t > 0$. In Eq. (2.55), the last equality assumes $\phi(x)$ and the time derivative of pressure are constant on the interval $(x_{i-1/2}, x_{i+1/2})$ and equal to their values at the gridblock center. If this assumption is invalid then Eq. (2.55) represents an approximation of Eq. (2.52). Throughout, $\delta_{i,k}$ denotes the Kronecker delta function defined by

$$\delta_{i,k} = \begin{cases} 0 & \text{for } k \neq i, \\ 1 & \text{for } k = i. \end{cases} \tag{2.56}$$

Note that Eq. (2.55) applies at any value of time. A sequence of discrete times is defined using a constant time step, Δt, by $t_n = t_{n-1} + \Delta t$ for $n = 0, 1, 2, \ldots$, where $t_0 = 0$. If we consider Eq. (2.55) at any $t = t_n > 0$ and use standard Taylor series approximations for the spatial and time derivatives, we obtain the following finite-difference equation:

$$\begin{aligned} C_1 \frac{k_{i+1/2} A}{\mu} \left(\frac{p_{i+1}^n - p_i^n}{\Delta x} \right) - C_1 \frac{k_{i-1/2} A}{\mu} \left(\frac{p_i^n - p_{i-1}^n}{\Delta x} \right) - q \delta_{i,k} \\ = \phi_i c_t A \Delta x \left(\frac{p_i^n - p_i^{n-1}}{\Delta t} \right) \end{aligned} \tag{2.57}$$

for $i = 2, 3, \ldots, N-1$ and $n = 1, 2, \ldots$. At $i = 1$ and $i = N$, respectively, we impose the no flow boundary conditions of Eq. (2.53) and obtain instead of Eq. (2.57), the following two equations:

$$C_1 \frac{k_{3/2} A}{\mu} \frac{p_2^n - p_1^n}{\Delta x} - q \delta_{1,k} = \phi_1 c_t A \Delta x \left(\frac{p_1^n - p_1^{n-1}}{\Delta t} \right), \tag{2.58}$$

and

$$-C_1 \frac{k_{N-1/2} A}{\mu} \frac{p_N^n - p_{N-1}^n}{\Delta x} - q \delta_{N,k} = \phi_N c_t A \Delta x \left(\frac{p_N^n - p_N^{n-1}}{\Delta t} \right), \tag{2.59}$$

for $n = 1, 2, \ldots$. The initial condition is imposed on the finite-difference problem by requiring that

$$p_i^0 = p_{\text{in}}. \tag{2.60}$$

In general, the solution $p(x, t)$ of the initial boundary-value problem specified by Eqs. (2.52)–(2.54) will not satisfy the finite-difference system, Eqs. (2.57)–(2.59), exactly because of the approximations we have used in deriving the finite-difference equations, for example in approximating partial derivatives by difference quotients. The expectation is that the solution, p_i^n $(i = 1, 2, \ldots, N, n = 1, 2, \ldots)$, of Eqs. (2.57)–(2.60) will be close to $p(x_i, t_n)$ if Δt and Δx are sufficiently small.

Given the cross sectional area to flow, rock and fluid properties, the initial pressure and the flow rate, the forward problem is to solve the system of finite-difference equations (Eqs. (2.57)–(2.59)) for p_i^n, $i = 1, 2, \ldots, N$, given p_i^{n-1}, $i = 1, 2, \ldots, N$. At the first time step, $n = 1$ and $p_i^{n-1} = p_i^0 = p_{\text{in}}$.

As is usually done in reservoir simulation, we now assume that permeability is constant on each gridblock, $x_{i-1/2} < x < x_{i+1/2}$, with $k(x) = k_i$ for $i = 1, 2, \ldots, N$. Using the standard harmonic average to relate the permeabilities at a gridblock boundary to the permeabilities of the two adjacent gridblocks gives

$$k_{i+1/2} = \frac{2k_i k_{i+1}}{k_i + k_{i+1}}, \tag{2.61}$$

for $i = 1, 2, \ldots, N - 1$. A typical history-matching problem would be to estimate the permeability and porosity fields given the value of flow rate, A, μ, c_t and observations of gridblock pressure at a few locations.

Multiple solutions

Using a numerical reservoir simulator, we have generated a solution of the system of finite-difference equations given by Eqs. (2.57)–(2.59) for parameter values given in Table 2.1.

Table 2.1. *Reservoir data.*

Number of gridblocks, N	9
Cross sectional area, A, ft^2	2500
Porosity, ϕ	0.25
Permeability, k, md	150
Δx, ft	500
Well location	$i = 9$
Well production rate, q, RB/D	250
System compressibility, c_t, psi^{-1}	10^{-5}
Fluid viscosity, μ, cp	0.5
Initial pressure, p_{in}, psi	3500

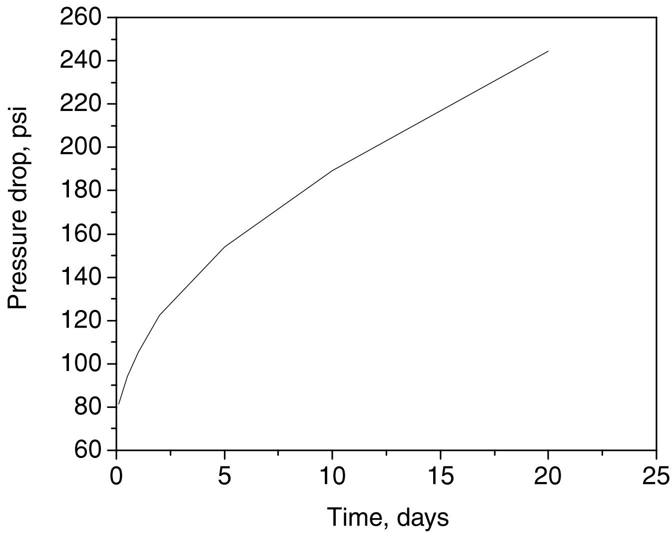

Figure 2.5. Pressure drop for one-dimensional single-phase flow example.

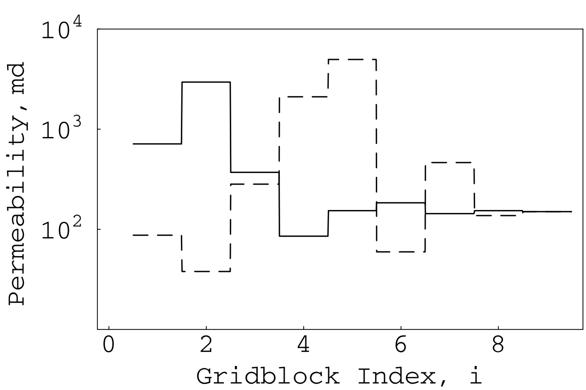

Figure 2.6. Two permeability fields which honor the wellbore pressure data.

Note that the "true" reservoir is homogeneous. Also note that the reservoir is produced by a single well located in gridblock 9. The wellbore pressure at the well in gridblock 9 was obtained by using a Peaceman [11] type equation to relate gridblock and flowing bottomhole wellbore pressure, $p_{wf}(t)$. A plot of the wellbore pressure drop, $\Delta p = p_{\text{in}} - p_{wf}$, versus time for twenty days of production is shown in Fig. 2.5.

Figure 2.6 shows two different permeability fields that were obtained as solutions to the history-matching problem, assuming that $\phi = 0.25$ in all gridblocks. Both solutions match the wellbore pressure data of Fig. 2.5 to within 0.01 psi. This example illustrates clearly that the inverse problem of determining the gridblock porosities and permeabilities from flowing wellbore pressure will not have a unique solution when the data are inaccurate and measurements are obtained at only a few locations. In Fig. 2.6,

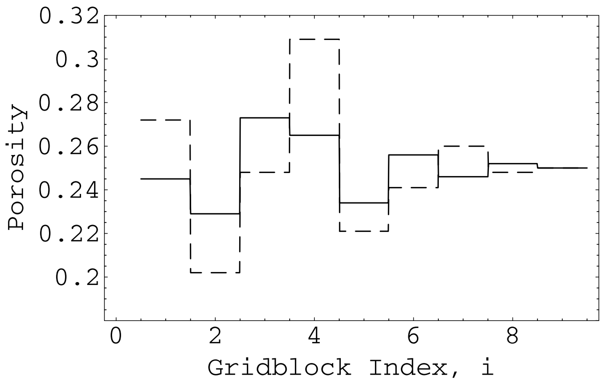

Figure 2.7. Two porosity fields which honor the wellbore pressure data.

we have plotted the estimated value of permeability on each of the nine gridblocks, versus i where i represents the gridblock index. The solid curve represents the first permeability field estimate and the dashed curve represents a second permeability field estimate. Each solution assumes that permeability k_i is constant on the interval $(x_{i-1/2}, x_{i+1/2})$. Note, the two permeability fields are quite different, even though both honor the pressure data equally well.

Interestingly, we can also reproduce the transient wellbore pressure drop shown in Fig. 2.5 to within 0.01 psi using $k = 150$ md in all gridblocks and either of the porosity fields shown in Fig. 2.7, which again illustrates the nonuniqueness of the inverse problem.

2.5 Summary

The examples in this chapter would all have been infinite dimensional in their parameterization, if a natural parameterization had been chosen. It was often necessary, however, to discretize the system in order to solve the forward problem. That is typical for systems that are described mathematically by differential equations. Even with a reduced parameterization, however, the inverse solutions were not unique. When the measurements contain noise (which is always the case), there may be no solutions to the problem that match the data exactly. In the acoustic tomography example, there were no solutions that honored the noisy data exactly, but infinitely many that approximately honored the data equally well.

The relationships of the data to the model variables varied from very simple weighted integrals for the relationship between mass of the Earth (data) and the mass density distribution (model), to a highly complex, nonlinear relationship between pressure (data) and permeability (model) for transient flow in a heterogeneous porous medium.

One of the difficult features of petroleum inverse problems is that the relationship between measurements (water-cut, pressure, seismic amplitude) and variables to be estimated (permeability, porosity, fault transmissibility) is difficult to compute.

For those cases where the solutions are nonunique or no exact solutions exist, it is useful to relax the definition of a solution. It will sometimes be useful to identify a "best estimate" after carefully specifying the meaning of best. In some cases it might be the estimate with the fewest features not required by the data, or the smoothest estimate. In any case, it is also useful to provide an estimate of uncertainty, either in the parameters or in some function of the parameters.

3 Estimation for linear inverse problems

In this chapter, the notions of underdetermined problems, overdetermined problems, mixed determined problems, the null space, the generalized inverse, methods of constructing estimates, sensitivities and resolution are explored for linear finite-dimensional inverse problems. In petroleum reservoir characterization, neither permeabilities nor pressure data are available at every point in the reservoir. Thus, it is assumed that a discrete set of measured data, d_i, $i = 1, 2, \ldots, N_d$, are available and the solution of the inverse problem means the construction of estimates or realizations of the model conditional to these data. The concepts are illustrated by considering the steady-state flow problem introduced in Section 1.1.

Linear inverse problems are those for which the theoretical relation between data and the model can be represented by

$$d_i = (G_i, m), \tag{3.1}$$

where m represents the model variables, d_i represents the data predicted by a particular model m, and $(\cdot, \cdot)$ represents an inner product on some suitably chosen inner product space which contains all feasible models. In many continuous inverse problems of interest, Eq. (3.1) can be represented by the equation,

$$d_i = \int_a^b G_i(x) m(x)\, dx, \tag{3.2}$$

for $i = 1, 2, \ldots, N_d$, where a and b are constants and the inner product is defined on the space of functions which are square integrable on the interval $[a, b]$, i.e. $L^2[a, b]$. Any $m(x)$ in $L^2[a, b]$ which satisfies Eq. (3.2) is a solution of the inverse problem. In general, it is not required that $m(x)$ exactly satisfy this equation to qualify as a solution, if for no other reason than the measured data are corrupted by noise. Classical least-squares fitting of data and nonlinear regression are commonly used in pressure transient analysis to generate models that "honor" data, but do not exactly reproduce the measured data. The continuous inverse problem of Eq. (3.2) is said to be linear because for any $u(x)$ and $v(x)$ in $L^2[a, b]$ and any constants α and β,

$$(G_i, \alpha u + \beta v) = \alpha(G_i, u) + \beta(G_i, v), \tag{3.3}$$

for all i.

Although most natural systems are best modeled using continuous functional representations, we primarily consider discrete inverse problems. Discrete problems refer to those where the physical system under consideration is characterized by a finite number of model variables, say $m_1, m_2, \ldots, m_{\mathrm{M}}$. For discrete inverse problems, it is often convenient to describe a model by the vector of model variables

$$m = [m_1 \quad m_2 \quad \cdots \quad m_{\mathrm{M}}]^{\mathrm{T}}, \tag{3.4}$$

where the superscript T on a matrix or vector denotes its transpose. It is convenient to also include all calculated data in a vector

$$d = [d_1 \quad d_2 \quad \cdots \quad d_{N_d}]^{\mathrm{T}}. \tag{3.5}$$

Then, a discrete linear inverse problem can be represented by

$$d = Gm, \tag{3.6}$$

where G is an $N_d \times M$ matrix representing the sensitivity of data to model variables.

Eq. (3.2) (or more generally, Eq. (3.1)) and Eq. (3.6) predict the data that will be calculated given a model m. Thus, d is referred to as the calculated or theoretical data. If measured data are exact (zero measurement error) and m is the true (actual) physical model, then d will be identical to measured data. In general, however, the observed data will be corrupted by measurement error. The vector of measured or observed data is denoted by d_{obs}.

3.1 Characterization of discrete linear inverse problems

Throughout this section, m denotes a real M-dimensional column vector of model variables; d_{obs} denotes an N_d-dimensional vector of observed data; d denotes the associated calculated data; G denotes an $N_d \times M$ matrix and the inverse problem is formally represented by

$$d_{\mathrm{obs}} = Gm. \tag{3.7}$$

The relation between calculated data and any model m is given by

$$d = Gm. \tag{3.8}$$

We assume that the vectors d and m and the matrix G have been normalized so that the entries of each of them are dimensionless. This will be important when eigenvalues and eigenvectors are discussed.

3.1.1 The null space and range

Assume that the unknown vector of model variables is an element of a linear vector space. We refer to this vector space as the model space. Since all solutions of Eq. (3.7) must be M-dimensional vectors, it is convenient to assume that the model space is R^M for this particular inverse problem. The set of all possible vectors of observed data is referred to as the data space. Since such data vectors are N_d dimensional, it is convenient to assume that the data space is R^{N_d}.

Two definitions are useful in characterizing the matrix G in the linear inverse problem. The **null space** of G is defined to be the set of all vectors in the model space that satisfy $Gm = 0$. The **range** of G is the set of all vectors d such that there is at least one m which satisfies $Gm = d$.

The dimension of the range of G is called the **rank** of G. Two important results from linear algebra, relating the rank of G and the dimension of the null space of G, are
(i) the rank of G is equal to the rank of G^{T} and
(ii) the sum of the dimension of the null space of G and the rank of G is equal to M, where G is an $N_d \times M$ matrix [see, for example, 12].

For $i = 1, 2, \ldots, M$, let g_i denote the ith column of the $N_d \times M$ matrix G so

$$G = [g_1 \quad g_2 \quad \ldots \quad g_{\mathrm{M}}]. \tag{3.9}$$

Note that for any M-dimensional vector x

$$Gx = [g_1 \quad g_2 \quad \ldots \quad g_{\mathrm{M}}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{\mathrm{M}} \end{bmatrix} = \sum_{j=1}^{M} x_j g_j. \tag{3.10}$$

Then, it is clear that every vector in the range of G is a linear combination of the columns of G. It can be shown that if the columns of G are linearly independent then there are no nontrivial solutions to $Gm = 0$ and vice versa.

If the number of linearly independent columns of G is equal to N_d, then there is at least one vector m which satisfies $Gm = d_{\mathrm{obs}}$. On the other hand, if $M < N_d$, then there exist data vectors d_{obs} such that the equation $Gm = d_{\mathrm{obs}}$ has **no solution**.

We can also show that $Gm = d_{\mathrm{obs}}$ has a **unique solution** for every d_{obs} in the data space if all of the rows of G are independent and the number of model variables M is equal to the number of data N_d. This result indicates that Eq. (3.7) has a unique solution for **every** $d_{\mathrm{obs}} \in R(N_d)$ if and only if G is a nonsingular square matrix. However, if G is not a square matrix, there may exist a particular vector d_{obs} such that $Gm = d_{\mathrm{obs}}$ has a unique solution. This could occur if some individual equations represented in $Gm = d_{\mathrm{obs}}$ are linear combinations of other equations. For example, if rows k and l are identical, then the kth and lth components of the calculated data would be identical, and assuming no measurement error, $d_{\mathrm{obs},k}$ and $d_{\mathrm{obs},l}$ will be equal. In general, however,