STRING THEORY AND M-THEORY

A MODERN INTRODUCTION

Katrin Becker, Melanie Becker, and John Schwarz

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A MODERN INTRODUCTION

String theory is one of the most exciting and challenging areas of modern theoretical physics. This book guides the reader from the basics of string theory to very recent developments at the frontier of string theory research.

The book begins with the basics of perturbative string theory, world-sheet supersymmetry, space-time supersymmetry, conformal field theory and the heterotic string, and moves on to describe modern developments, including D-branes, string dualities and M-theory. It then covers string geometry (including Calabi–Yau compactifications) and flux compactifications, and applications to cosmology and particle physics. One chapter is dedicated to black holes in string theory and M-theory, and the microscopic origin of black-hole entropy. The book concludes by presenting matrix theory, AdS/CFT duality and its generalizations.

This book is ideal for graduate students studying modern string theory, and it will make an excellent textbook for a 1-year course on string theory. It will also be useful for researchers interested in learning about developments in modern string theory. The book contains about 120 solved exercises, as well as about 200 homework problems, solutions of which are available for lecturers on a password protected website at www.cambridge.org/9780521860697.

Katrin Becker is a Professor of physics at Texas A & M University. She was awarded the Radcliffe Fellowship from Harvard University in 2006 and received the Alfred Sloan Fellowship in 2003.

Melanie Becker is a Professor of physics at Texas A & M University. In 2006 she was awarded an Edward, Frances and Shirley B. Daniels Fellowship from the Radcliffe Institute for Advanced Studies at Harvard University. In 2001 she received the Alfred Sloan Fellowship.

John H. Schwarz is the Harold Brown Professor of Theoretical Physics at the California Institute of Technology. He is a MacArthur Fellow and a member of the National Academy of Sciences.

This is the first comprehensive textbook on string theory to also offer an up-to-date picture of the most important theoretical developments of the last decade, including the AdS/CFT correspondence and flux compactifications, which have played a crucial role in modern efforts to make contact with experiment. An excellent resource for graduate students as well as researchers in high-energy physics and cosmology.

Nima Arkani-Hamed, Harvard University

An exceptional introduction to string theory that contains a comprehensive treatment of all aspects of the theory, including recent developments. The clear pedagogical style and the many excellent exercises should provide the interested student or researcher a straightforward path to the frontiers of current research.

David Gross, Director of the Kavli Institute for Theoretical Physics, University of California, Santa Barbara and winner of the Nobel Prize for Physics in 2004

Masterfully written by pioneers of the subject, comprehensive, up-to-date and replete with illuminating problem sets and their solutions, *String Theory and M-theory: A Modern Introduction* provides an ideal preparation for research on the current forefront of the fundamental laws of nature. It is destined to become the standard textbook in the subject.

Andrew Strominger, Harvard University

This book is a magnificient resource for students and researchers alike in the rapidly evolving field of string theory. It is unique in that it is targeted for students without any knowledge of string theory and at the same time it includes the very latest developments of the field, all presented in a very fluid and simple form. The lucid description is nicely complemented by very instructive problems. I highly recommend this book to all researchers interested in the beautiful field of string theory.

Cumrun Vafa, Harvard University

This elegantly written book will be a valuable resource for students looking for an entry-way to the vast and exciting topic of string theory. The authors have skillfully made a selection of topics aimed at helping the beginner get up to speed. I am sure it will be widely read.

Edward Witten, Institute for Advanced Study, Princeton, winner of the Fields Medal in 1990

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To our parents

An Ode to the Unity of Time and Space

Time, ah, time,
how you go off like this!

Physical things, ah, things,
so abundant you are!

The Ruo's waters are three thousand,
how can they not have the same source?

Time and space are one body,
mind and things sustain each other.

Time, o time,
does not time come again?

Heaven, o heaven,
how many are the appearances of heaven!

From ancient days constantly shifting on,
black holes flaring up.

Time and space are one body,
is it without end?

Great indeed
is the riddle of the universe.

Beautiful indeed
is the source of truth.

To quantize space and time
the smartest are nothing.

To measure the Great Universe with a long thin tube
the learning is vast.

Shing-Tung Yau

Contents

Preface

String theory is one of the most exciting and challenging areas of modern theoretical physics. It was developed in the late 1960s for the purpose of describing the strong nuclear force. Problems were encountered that prevented this program from attaining complete success. In particular, it was realized that the spectrum of a fundamental string contains an undesired massless spin-two particle. Quantum chromodynamics eventually proved to be the correct theory for describing the strong force and the properties of hadrons. New doors opened for string theory when in 1974 it was proposed to identify the massless spin-two particle in the string's spectrum with the graviton, the quantum of gravitation. String theory became then the most promising candidate for a quantum theory of gravity unified with the other forces and has developed into one of the most fascinating theories of high-energy physics.

The understanding of string theory has evolved enormously over the years thanks to the efforts of many very clever people. In some periods progress was much more rapid than in others. In particular, the theory has experienced two major revolutions. The one in the mid-1980s led to the subject achieving widespread acceptance. In the mid-1990s a second superstring revolution took place that featured the discovery of nonperturbative dualities that provided convincing evidence of the uniqueness of the underlying theory. It also led to the recognition of an eleven-dimensional manifestation, called M-theory. Subsequent developments have made the connection between string theory, particle physics phenomenology, cosmology, and pure mathematics closer than ever before. As a result, string theory is becoming a mainstream research field at many universities in the US and elsewhere.

Due to the mathematically challenging nature of the subject and the above-mentioned rapid development of the field, it is often difficult for someone new to the subject to cope with the large amount of material that needs to be learned before doing actual string-theory research. One could spend several years studying the requisite background mathematics and physics, but by the end of that time, much more would have already been developed, and one still wouldn't be up to date. An alternative approach is to shorten the learning process so that the student can jump into research more quickly. In this spirit, the aim of this book is to guide the student through the fascinating subject of string theory in one academic year. This book starts with the basics of string theory in the first few chapters and then introduces the reader to some of the main topics of modern research. Since the subject is enormous, it is only possible to introduce selected topics. Nevertheless, we hope that it will provide a stimulating introduction to this beautiful subject and that the dedicated student will want to explore further.

The reader is assumed to have some familiarity with quantum field theory and general relativity. It is also very useful to have a broad mathematical background. Group theory is essential, and some knowledge of differential geometry and basics concepts of topology is very desirable. Some topics in geometry and topology that are required in the later chapters are summarized in an appendix.

The three main string-theory textbooks that precede this one are by Green, Schwarz and Witten (1987), by Polchinski (1998) and by Zwiebach (2004). Each of these was also published by Cambridge University Press. This book is somewhat shorter and more up-to-date than the first two, and it is more advanced than the third one. By the same token, those books contain much material that is not repeated here, so the serious student will want to refer to them, as well. Another distinguishing feature of this book is that it contains many exercises with worked out solutions. These are intended to be helpful to students who want problems that can be used to practice and assimilate the material.

This book would not have been possible without the assistance of many people. We have received many valuable suggestions and comments about the entire manuscript from Rob Myers, and we have greatly benefited from the assistance of Yu-Chieh Chung and Guangyu Guo, who have worked diligently on many of the exercises and homework problems and have carefully read the whole manuscript. Moreover, we have received extremely useful feedback from many colleagues including Keshav Dasgupta, Andrew Frey, Davide Gaiotto, Sergei Gukov, Michael Haack, Axel Krause, Hong Lu, Juan Maldacena, Lubos Motl, Hirosi Ooguri, Patricia Schwarz, Eric Sharpe, James Sparks, Andy Strominger, Ian Swanson, Xi Yin and especially Cumrun Vafa. We have further received great comments and suggestions from many graduate students at Caltech and Harvard University. We thank Ram Sriharsha for his assistance with some of the homework problems and Ketan Vyas for writing up solutions to the homework problems, which will be made available to instructors. We thank Sharlene Cartier and Carol Silberstein of Caltech for their help in preparing parts of the manuscript, Simon Capelin of Cambridge U. Press, whose help in coordinating the different aspects of the publishing process has been indispensable, Elisabeth Krause for help preparing some of the figures and Kovid Goyal for his assistance with computer-related issues. We thank Steven Owen for translating from Chinese the poem that precedes the preface.

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Katrin Becker
Melanie Becker
John H. Schwarz

NOTATION AND CONVENTIONS

A	area of event horizon
AdS_D	D-dimensional anti-de Sitter space-time
A_3	three-form potential of $D = 11$ supergravity
b, c	fermionic world-sheet ghosts
b_n	Betti numbers
b^μ_r, $r \in \mathbb{Z} + 1/2$	fermionic oscillator modes in NS sector
B_2 or B	NS–NS two-form potential
c	central charge of CFT
$c_1 = [\mathcal{R}/2\pi]$	first Chern class
C_n	R–R n-form potential
d^μ_m, $m \in \mathbb{Z}$	fermionic oscillator modes in R sector
D	number of space-time dimensions
$F = dA + A \wedge A$	Yang–Mills curvature two-form (antihermitian)
$F = dA + iA \wedge A$	Yang–Mills curvature two-form (hermitian)
$F_4 = dA_3$	four-form field strength of $D = 11$ supergravity
F_m, $m \in \mathbb{Z}$	odd super-Virasoro generators in R sector
$F_{n+1} = dC_n$	$(n + 1)$-form R–R field strength
$g_{\mathrm{s}} = \langle \exp \Phi \rangle$	closed-string coupling constant
G_r, $r \in \mathbb{Z} + 1/2$	odd super-Virasoro generators in NS sector
G_D	Newton's constant in D dimensions
$H_3 = dB_2$	NS–NS three-form field strength
$h^{p,q}$	Hodge numbers
$j(\tau)$	elliptic modular function
$J = ig_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}$	Kähler form
$\mathcal{J} = J + iB$	complexified Kähler form
k	level of Kac–Moody algebra
K	Kaluza–Klein excitation number
$\mathcal{K}$	Kähler potential
$l_{\mathrm{p}} = 1.6 \times 10^{-33}\,\mathrm{cm}$	Planck length for $D = 4$
ℓ_{p}	Planck length for $D = 11$
$l_{\mathrm{s}} = \sqrt{2\alpha'}$, $\ell_{\mathrm{s}} = \sqrt{\alpha'}$	string length scale
L_n, $n \in \mathbb{Z}$	generators of Virasoro algebra
$m_{\mathrm{p}} = 1.2 \times 10^{19} \mathrm{GeV}/c^2$	Planck mass for $D = 4$
$M_{\mathrm{p}} = 2.4 \times 10^{18} \mathrm{GeV}/c^2$	reduced Planck mass $m_{\mathrm{p}}/\sqrt{8\pi}$
$M, N, \ldots$	space-time indices for $D = 11$
$\mathcal{M}$	moduli space

N_L, N_R	left- and right-moving excitation numbers
Q_{B}	BRST charge
$R = d\omega + \omega \wedge \omega$	Riemann curvature two-form
$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$	Ricci tensor
$\mathcal{R} = R_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}$	Ricci form
S	entropy
S^a	world-sheet fermions in light-cone gauge GS formalism
$T_{\alpha\beta}$	world-sheet energy–momentum tensor
T_p	tension of p-brane
W	winding number
x^{μ}, $\mu = 0, 1, \ldots D-1$	space-time coordinates
X^{μ}, $\mu = 0, 1, \ldots D-1$	space-time embedding functions of a string
$x^{\pm} = (x^0 \pm x^{D-1})/\sqrt{2}$	light-cone coordinates in space-time
x^I, $I = 1, 2, \ldots, D-2$	transverse coordinates in space-time
Z	central charge
α^{μ}_m, $m \in \mathbb{Z}$	bosonic oscillator modes
α'	Regge-slope parameter
β, γ	bosonic world-sheet ghosts
γ_{μ}	Dirac matrices in four dimensions
Γ_M	Dirac matrices in 11 dimensions
$\Gamma_{\mu\nu}{}^{\rho}$	affine connection
$\eta(\tau)$	Dedekind eta function
Θ^{Aa}	world-volume fermions in covariant GS formalism
λ^A	left-moving world-sheet fermions of heterotic string
$\Lambda \sim 10^{-120} M_{\mathrm{p}}^4$	observed vacuum energy density
σ^{α}, $\alpha = 0, 1, \ldots, p$	world-volume coordinates of a p-brane
$\sigma^0 = \tau$, $\sigma^1 = \sigma$	world-sheet coordinates of a string
$\sigma^{\pm} = \tau \pm \sigma$	light-cone coordinates on the world sheet
$\sigma^{\mu}_{\alpha\dot{\beta}}$	Dirac matrices in two-component spinor notation
Φ	dilaton field
$\chi(M)$	Euler characteristic of M
ψ^{μ}	world-sheet fermion in RNS formalism
Ψ_M	gravitino field of $D = 11$ supergravity
$\omega_{\mu}{}^{\alpha}{}_{\beta}$	spin connection
Ω	world-sheet parity transformation
Ω_n	holomorphic n-form

- $\hbar = c = 1$.
- The signature of any metric is 'mostly +', that is, $(-, +, \ldots, +)$.
- The space-time metric is $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.
- In Minkowski space-time $g_{\mu\nu} = \eta_{\mu\nu}$.
- The world-sheet metric tensor is $h_{\alpha\beta}$.
- A hermitian metric has the form $ds^2 = 2g_{a\bar{b}} dz^a d\bar{z}^{\bar{b}}$.
- The space-time Dirac algebra in $D = d + 1$ dimensions is $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}$.
- $\Gamma^{\mu_1\mu_2\cdots\mu_n} = \Gamma^{[\mu_1}\Gamma^{\mu_2} \cdots \Gamma^{\mu_n]}$.
- The world-sheet Dirac algebra is $\{\rho_\alpha, \rho_\beta\} = 2h_{\alpha\beta}$.
- $|F_n|^2 = \frac{1}{n!} g^{\mu_1\nu_1} \cdots g^{\mu_n\nu_n} F_{\mu_1\ldots\mu_n} F_{\nu_1\ldots\nu_n}$.
- The Levi–Civita tensor $\varepsilon^{\mu_1\cdots\mu_D}$ is totally antisymmetric with $\varepsilon^{01\cdots d} = 1$.

1
Introduction

There were two major breakthroughs that revolutionized theoretical physics in the twentieth century: general relativity and quantum mechanics. General relativity is central to our current understanding of the large-scale expansion of the Universe. It gives small corrections to the predictions of Newtonian gravity for the motion of planets and the deflection of light rays, and it predicts the existence of gravitational radiation and black holes. Its description of the gravitational force in terms of the curvature of space-time has fundamentally changed our view of space and time: they are now viewed as dynamical. Quantum mechanics, on the other hand, is the essential tool for understanding microscopic physics. The evidence continues to build that it is an exact property of Nature. Certainly, its exact validity is a basic assumption in all string theory research.

The understanding of the fundamental laws of Nature is surely incomplete until general relativity and quantum mechanics are successfully reconciled and unified. That this is very challenging can be seen from many different viewpoints. The concepts, observables and types of calculations that characterize the two subjects are strikingly different. Moreover, until about 1980 the two fields developed almost independently of one another. Very few physicists were experts in both. With the goal of unifying both subjects, string theory has dramatically altered the sociology as well as the science.

In relativistic quantum mechanics, called quantum field theory, one requires that two fields that are defined at space-time points with a space-like separation should commute (or anticommute if they are fermionic). In the gravitational context one doesn't know whether or not two space-time points have a space-like separation until the metric has been computed, which is part of the dynamical problem. Worse yet, the metric is subject to quantum fluctuations just like other quantum fields. Clearly, these are rather challenging issues. Another set of challenges is associated with the quantum description of black holes and the description of the Universe in the very early stages of its history.

The most straightforward attempts to combine quantum mechanics and general relativity, in the framework of perturbative quantum field theory, run into problems due to uncontrollable infinities. Ultraviolet divergences are a characteristic feature of radiative corrections to gravitational processes, and they become worse at each order in perturbation theory. Because Newton's constant is proportional to $(\text{length})^2$ in four dimensions, simple power-counting arguments show that it is not possible to remove these infinities by the conventional renormalization methods of quantum field theory. Detailed calculations demonstrate that there is no miracle that invalidates this simple dimensional analysis.[1]

String theory purports to overcome these difficulties and to provide a consistent quantum theory of gravity. How the theory does this is not yet understood in full detail. As we have learned time and time again, string theory contains many deep truths that are there to be discovered. Gradually a consistent picture is emerging of how this remarkable and fascinating theory deals with the many challenges that need to be addressed for a successful unification of quantum mechanics and general relativity.

1.1 Historical origins

String theory arose in the late 1960s in an attempt to understand the strong nuclear force. This is the force that is responsible for holding protons and neutrons together inside the nucleus of an atom as well as quarks together inside the protons and neutrons. A theory based on fundamental one-dimensional extended objects, called strings, rather than point-like particles, can account qualitatively for various features of the strong nuclear force and the strongly interacting particles (or hadrons).

The basic idea in the string description of the strong interactions is that specific particles correspond to specific oscillation modes (or quantum states) of the string. This proposal gives a very satisfying unified picture in that it postulates a single fundamental object (namely, the string) to explain the myriad of different observed hadrons, as indicated in Fig. 1.1.

In the early 1970s another theory of the strong nuclear force – called quantum chromodynamics (or QCD) – was developed. As a result of this, as well as various technical problems in the string theory approach, string

1 Some physicists believe that perturbative renormalizability is not a fundamental requirement and try to "quantize" pure general relativity despite its nonrenormalizability. Loop quantum gravity is an example of this approach. Whatever one thinks of the logic, it is fair to say that despite a considerable amount of effort such attempts have not yet been very fruitful.

theory fell out of favor. The current viewpoint is that this program made good sense, and so it has again become an active area of research. The concrete string theory that describes the strong interaction is still not known, though one now has a much better understanding of how to approach the problem.

String theory turned out to be well suited for an even more ambitious purpose: the construction of a quantum theory that unifies the description of gravity and the other fundamental forces of nature. In principle, it has the potential to provide a complete understanding of particle physics and of cosmology. Even though this is still a distant dream, it is clear that in this fascinating theory surprises arise over and over.

1.2 General features

Even though string theory is not yet fully formulated, and we cannot yet give a detailed description of how the standard model of elementary particles should emerge at low energies, or how the Universe originated, there are some general features of the theory that have been well understood. These are features that seem to be quite generic irrespective of what the final formulation of string theory might be.

Gravity

The first general feature of string theory, and perhaps the most important, is that general relativity is naturally incorporated in the theory. The theory gets modified at very short distances/high energies but at ordinary distances and energies it is present in exactly the form as proposed by Einstein. This is significant, because general relativity is arising within the framework of a

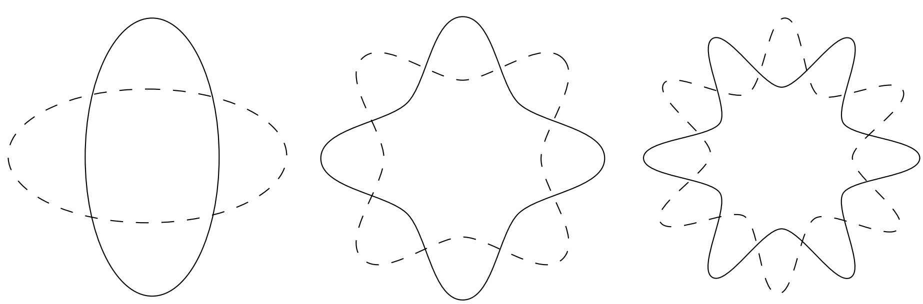

Fig. 1.1. Different particles are different vibrational modes of a string.

consistent quantum theory. Ordinary quantum field theory does not allow gravity to exist; string theory requires it.

Yang–Mills gauge theory

In order to fulfill the goal of describing all of elementary particle physics, the presence of a graviton in the string spectrum is not enough. One also needs to account for the standard model, which is a Yang–Mills theory based on the gauge group $SU(3)\times SU(2)\times U(1)$. The appearance of Yang–Mills gauge theories of the sort that comprise the standard model is a general feature of string theory. Moreover, matter can appear in complex chiral representations, which is an essential feature of the standard model. However, it is not yet understood why the specific $SU(3) \times SU(2) \times U(1)$ gauge theory with three generations of quarks and leptons is singled out in nature.

Supersymmetry

The third general feature of string theory is that its consistency requires supersymmetry, which is a symmetry that relates bosons to fermions is required. There exist nonsupersymmetric bosonic string theories (discussed in Chapters 2 and 3), but lacking fermions, they are completely unrealistic. The mathematical consistency of string theories with fermions depends crucially on local supersymmetry. Supersymmetry is a generic feature of all potentially realistic string theories. The fact that this symmetry has not yet been discovered is an indication that the characteristic energy scale of supersymmetry breaking and the masses of supersymmetry partners of known particles are above experimentally determined lower bounds.

Space-time supersymmetry is one of the major predictions of superstring theory that could be confirmed experimentally at accessible energies. A variety of arguments, not specific to string theory, suggest that the characteristic energy scale associated with supersymmetry breaking should be related to the electroweak scale, in other words in the range 100 GeV to a few TeV. If this is correct, superpartners should be observable at the CERN Large Hadron Collider (LHC), which is scheduled to begin operating in 2007.

Extra dimensions of space

In contrast to many theories in physics, superstring theories are able to predict the dimension of the space-time in which they live. The theory is only consistent in a ten-dimensional space-time and in some cases an eleventh dimension is also possible.

To make contact between string theory and the four-dimensional world of everyday experience, the most straightforward possibility is that six or seven of the dimensions are compactified on an internal manifold, whose size is sufficiently small to have escaped detection. For purposes of particle physics, the other four dimensions should give our four-dimensional space-time. Of course, for purposes of cosmology, other (time-dependent) geometries may also arise.

Fig. 1.2. From far away a two-dimensional cylinder looks one-dimensional.

The idea of an extra compact dimension was first discussed by Kaluza and Klein in the 1920s. Their goal was to construct a unified description of electromagnetism and gravity in four dimensions by compactifying five-dimensional general relativity on a circle. Even though we now know that this is not how electromagnetism arises, the essence of this beautiful approach reappears in string theory. The Kaluza–Klein idea, nowadays referred to as *compactification*, can be illustrated in terms of the two cylinders of Fig. 1.2. The surface of the first cylinder is two-dimensional. However, if the radius of the circle becomes extremely small, or equivalently if the cylinder is viewed from a large distance, the cylinder looks effectively one-dimensional. One now imagines that the long dimension of the cylinder is replaced by our four-dimensional space-time and the short dimension by an appropriate six, or seven-dimensional compact manifold. At large distances or low energies the compact internal space cannot be seen and the world looks effectively four-dimensional. As discussed in Chapters 9 and 10, even if the internal manifolds are invisible, their topological properties determine the particle content and structure of the four-dimensional theory. In the mid-1980s Calabi–Yau manifolds were first considered for compactifying six extra dimensions, and they were shown to be phenomenologically rather promising, even though some serious drawbacks (such as the moduli space problem discussed in Chapter 10) posed a problem for the predictive power of string theory. In contrast to the circle, Calabi–Yau manifolds do not have isometries, and part of their role is to break symmetries rather than to make them.

The size of strings

In conventional quantum field theory the elementary particles are mathematical points, whereas in perturbative string theory the fundamental objects are one-dimensional loops (of zero thickness). Strings have a characteristic length scale, denoted l_{s}, which can be estimated by dimensional analysis. Since string theory is a relativistic quantum theory that includes gravity it must involve the fundamental constants c (the speed of light), $\hbar$ (Planck's constant divided by 2π), and G (Newton's gravitational constant). From these one can form a length, known as the Planck length

$$l_{\mathrm{p}} = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.6 \times 10^{-33}\,\mathrm{cm}.$$

Similarly, the Planck mass is

$$m_{\mathrm{p}} = \left(\frac{\hbar c}{G}\right)^{1/2} = 1.2 \times 10^{19}\,\mathrm{GeV}/c^2.$$

The Planck scale is the natural first guess for a rough estimate of the fundamental string length scale as well as the characteristic size of compact extra dimensions. Experiments at energies far below the Planck energy cannot resolve distances as short as the Planck length. Thus, at such energies, strings can be accurately approximated by point particles. This explains why quantum field theory has been so successful in describing our world.

1.3 Basic string theory

As a string evolves in time it sweeps out a two-dimensional surface in spacetime, which is called the string *world sheet* of the string. This is the string counterpart of the world line for a point particle. In quantum field theory, analyzed in perturbation theory, contributions to amplitudes are associated with Feynman diagrams, which depict possible configurations of world lines. In particular, interactions correspond to junctions of world lines. Similarly, perturbation expansions in string theory involve string world sheets of various topologies.

The existence of interactions in string theory can be understood as a consequence of world-sheet topology rather than of a local singularity on the world sheet. This difference from point-particle theories has two important implications. First, in string theory the structure of interactions is uniquely determined by the free theory. There are no arbitrary interactions to be chosen. Second, since string interactions are not associated with short-distance singularities, string theory amplitudes have no ultraviolet divergences. The string scale $1/l_s$ acts as a UV cutoff.

World-volume actions and the critical dimension

A string can be regarded as a special case of a p-brane, which is an object with p spatial dimensions and tension (or energy density) T_p. In fact, various p-branes do appear in superstring theory as nonperturbative excitations. The classical motion of a p-brane extremizes the $(p+1)$-dimensional volume V that it sweeps out in space-time. Thus there is a p-brane action that is given by $S_p = -T_p V$. In the case of the fundamental string, which has $p = 1$, V is the area of the string world sheet and the action is called the Nambu–Goto action.

Classically, the Nambu–Goto action is equivalent to the string sigma-model action

$$S_\sigma = -\frac{T}{2} \int \sqrt{-h} h^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu d\sigma d\tau,$$

where $h_{\alpha\beta}(\sigma, \tau)$ is an auxiliary world-sheet metric, $h = \det h_{\alpha\beta}$, and $h^{\alpha\beta}$ is the inverse of $h_{\alpha\beta}$. The functions $X^\mu(\sigma, \tau)$ describe the space-time embedding of the string world sheet. The Euler–Lagrange equation for $h^{\alpha\beta}$ can be used to eliminate it from the action and recover the Nambu–Goto action.

Quantum mechanically, the story is more subtle. Instead of eliminating h via its classical field equations, one should perform a Feynman path integral, using standard machinery to deal with the local symmetries and gauge fixing. When this is done correctly, one finds that there is a conformal anomaly unless the space-time dimension is $D = 26$. These matters are explored in Chapters 2 and 3. An analogous analysis for superstrings gives the critical dimension $D = 10$.

Closed strings and open strings

The parameter τ in the embedding functions $X^\mu(\sigma, \tau)$ is the world-sheet time coordinate and σ parametrizes the string at a given world-sheet time. For a closed string, which is topologically a circle, one should impose periodicity in the spatial parameter σ. Choosing its range to be π one identifies both ends of the string $X^\mu(\sigma,\tau) = X^\mu(\sigma+\pi,\tau)$. All string theories contain closed strings, and the graviton always appears as a massless mode in the closed-string spectrum of critical string theories.

For an open string, which is topologically a line interval, each end can be required to satisfy either Neumann or Dirichlet boundary conditions (for each value of μ). The Dirichlet condition specifies a space-time hypersurface on which the string ends. The only way this makes sense is if the open string ends on a physical object, which is called a D-brane. (D stands for Dirichlet.) If all the open-string boundary conditions are Neumann, then the ends of the string can be anywhere in the space-time. The modern interpretation is that this means that space-time-filling D-branes are present.

Perturbation theory

Perturbation theory is useful in a quantum theory that has a small dimensionless coupling constant, such as quantum electrodynamics (QED), since it allows one to compute physical quantities as expansions in the small parameter. In QED the small parameter is the fine-structure constant $\alpha \sim 1/137$. For a physical quantity $T(\alpha)$, one computes (using Feynman diagrams)

$$T(\alpha) = T_0 + \alpha T_1 + \alpha^2 T_2 + \dots$$

Perturbation series are usually asymptotic expansions with zero radius of convergence. Still, they can be useful, if the expansion parameter is small, because the first terms in the expansion provide an accurate approximation.

The heterotic and type II superstring theories contain oriented closed strings only. As a result, the only world sheets in their perturbation expansions are closed oriented Riemann surfaces. There is a unique world-sheet topology at each order of the perturbation expansion, and its contribution is UV finite. The fact that there is just one string theory Feynman diagram at each order in the perturbation expansion is in striking contrast to the large number of Feynman diagrams that appear in quantum field theory. In the case of string theory there is no particular reason to expect the coupling constant $g_{\rm s}$ to be small. So it is unlikely that a realistic vacuum could be analyzed accurately using only perturbation theory. For this reason, it is important to understand nonperturbative effects in string theory.

Superstrings

The *first superstring revolution* began in 1984 with the discovery that quantum mechanical consistency of a ten-dimensional theory with $\mathcal{N} = 1$ supersymmetry requires a local Yang–Mills gauge symmetry based on one of two possible Lie algebras: $SO(32)$ or $E_8 \times E_8$. As is explained in Chapter 5, only for these two choices do certain quantum mechanical anomalies cancel. The fact that only these two groups are possible suggested that string theory has a very constrained structure, and therefore it might be very predictive. [2]

When one uses the superstring formalism for both left-moving modes and right-moving modes, the supersymmetries associated with the left-movers and the right-movers can have either opposite handedness or the same handedness. These two possibilities give different theories called the type IIA and type IIB superstring theories, respectively. A third possibility, called type I superstring theory, can be derived from the type IIB theory by modding out by its left–right symmetry, a procedure called orientifold projection. The strings that survive this projection are unoriented. The type I and type II superstring theories are described in Chapters 4 and 5 using formalisms with world-sheet and space-time supersymmetry, respectively.

A more surprising possibility is to use the formalism of the 26-dimensional bosonic string for the left-movers and the formalism of the 10-dimensional superstring for the right-movers. The string theories constructed in this way are called "heterotic." Heterotic string theory is discussed in Chapter 7. The mismatch in space-time dimensions may sound strange, but it is actually exactly what is needed. The extra 16 left-moving dimensions must describe a torus with very special properties to give a consistent theory. There are precisely two distinct tori that have the required properties, and they correspond to the Lie algebras $SO(32)$ and $E_8 \times E_8$.

Altogether, there are five distinct superstring theories, each in ten dimensions. Three of them, the type I theory and the two heterotic theories, have $\mathcal{N} = 1$ supersymmetry in the ten-dimensional sense. The minimal spinor in ten dimensions has 16 real components, so these theories have 16 conserved supercharges. The type I superstring theory has the gauge group $SO(32)$, whereas the heterotic theories realize both $SO(32)$ and $E_8 \times E_8$. The other two theories, type IIA and type IIB, have $\mathcal{N} = 2$ supersymmetry or equivalently 32 supercharges.

1.4 Modern developments in superstring theory

The realization that there are five different superstring theories was somewhat puzzling. Certainly, there is only one Universe, so it would be most satisfying if there were only one possible theory. In the late 1980s it was

2 Anomaly analysis alone also allows $U(1)^{496}$ and $E_8 \times U(1)^{248}$. However, there are no string theories with these gauge groups.

realized that there is a property known as T-duality that relates the two type II theories and the two heterotic theories, so that they shouldn't really be regarded as distinct theories.

Progress in understanding nonperturbative phenomena was achieved in the 1990s. Nonperturbative S-dualities and the opening up of an eleventh dimension at strong coupling in certain cases led to new identifications. Once all of these correspondences are taken into account, one ends up with the best possible conclusion: there is a unique underlying theory. Some of these developments are summarized below and are discussed in detail in the later chapters.

T-duality

String theory exhibits many surprising properties. One of them, called T-duality, is discussed in Chapter 6. T-duality implies that in many cases two different geometries for the extra dimensions are physically equivalent! In the simplest example, a circle of radius R is equivalent to a circle of radius ℓ_{s}^2/R, where (as before) ℓ_{s} is the fundamental string length scale.

T-duality typically relates two different theories. For example, it relates the two type II and the two heterotic theories. Therefore, the type IIA and type IIB theories (also the two heterotic theories) should be regarded as a single theory. More precisely, they represent opposite ends of a continuum of geometries as one varies the radius of a circular dimension. This radius is not a parameter of the underlying theory. Rather, it arises as the vacuum expectation value of a scalar field, and it is determined dynamically.

There are also fancier examples of duality equivalences. For example, there is an equivalence of type IIA superstring theory compactified on a Calabi–Yau manifold and type IIB compactified on the "mirror" Calabi–Yau manifold. This mirror pairing of topologically distinct Calabi–Yau manifolds is discussed in Chapter 9. A surprising connection to T-duality will emerge.

S-duality

Another kind of duality – called S-duality – was discovered as part of the *second superstring revolution* in the mid-1990s. It is discussed in Chapter 8. S-duality relates the string coupling constant g_{s} to $1/g_{\mathrm{s}}$ in the same way that T-duality relates R to ℓ_{s}^2/R. The two basic examples relate the type I superstring theory to the $SO(32)$ heterotic string theory and the type IIB superstring theory to itself. Thus, given our knowledge of the small g_{s} behavior of these theories, given by perturbation theory, we learn how these three theories behave when $g_s \gg 1$. For example, strongly coupled type I theory is equivalent to weakly coupled $SO(32)$ heterotic theory. In the type IIB case the theory is related to itself, so one is actually dealing with a symmetry. The string coupling constant g_s is given by the vacuum expectation value of $\exp\phi$, where ϕ is the dilaton field. S-duality, like T-duality, is actually a field transformation, $\phi \to -\phi$, and not just a statement about vacuum expectation values.

D-branes

When studied nonperturbatively, one discovers that superstring theory contains various p-branes, objects with p spatial dimensions, in addition to the fundamental strings. All of the p-branes, with the single exception of the fundamental string (which is a 1-brane), become infinitely heavy as $g_s \to 0$, and therefore they do not appear in perturbation theory. On the other hand, when the coupling g_s is not small, this distinction is no longer significant. When that is the case, all of the p-branes are just as important as the fundamental strings, so there is p-brane democracy.

The type I and II superstring theories contain a class of p-branes called D-branes, whose tension is proportional $1/g_s$. As was mentioned earlier, their defining property is that they are objects on which fundamental strings can end. The fact that fundamental strings can end on D-branes implies that quantum field theories of the Yang–Mills type, like the standard model, reside on the world volumes of D-branes. The Yang–Mills fields arise as the massless modes of open strings attached to the D-branes. The fact that theories resembling the standard model reside on D-branes has many interesting implications. For example, it has led to the speculation that the reason we experience four space-time dimensions is because we are confined to live on three-dimensional D-branes (D3-branes), which are embedded in a higher-dimensional space-time. Model-building along these lines, sometimes called the *brane-world* approach or scenario, is discussed in Chapter 10.

What is M-theory?

S-duality explains how three of the five original superstring theories behave at strong coupling. This raises the question: What happens to the other two superstring theories – type IIA and $E_8 \times E_8$ heterotic – when g_s is large? The answer, which came as quite a surprise, is that they grow an eleventh dimension of size $g_s \ell_s$. This new dimension is a circle in the type IIA case and a line interval in the heterotic case. When the eleventh dimension is large, one is outside the regime of perturbative string theory, and new techniques are required. Most importantly, a new type of quantum theory in 11 dimensions, called M-theory, emerges. At low energies it is approximated by a classical field theory called 11-dimensional supergravity, but M-theory is much more than that. The relation between M-theory and the two superstring theories previously mentioned, together with the T and S dualities discussed above, imply that the five superstring theories are connected by a web of dualities, as depicted in Fig. 1.3. They can be viewed as different corners of a single theory.

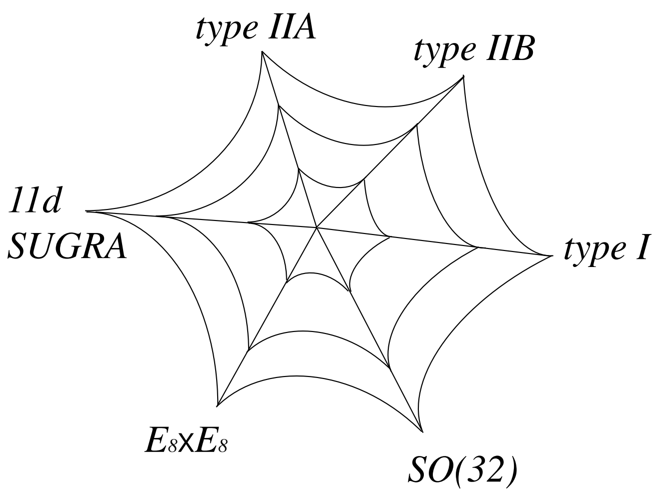

Fig. 1.3. Different string theories are connected through a web of dualities.

There are techniques for identifying large classes of superstring and M-theory vacua, and describing them exactly, but there is not yet a succinct and compelling formulation of the underlying theory that gives rise to these vacua. Such a formulation should be completely unique, with no adjustable dimensionless parameters or other arbitrariness. Many things that we usually take for granted, such as the existence of a space-time manifold, are likely to be understood as emergent properties of specific vacua rather than identifiable features of the underlying theory. If this is correct, then the missing formulation of the theory must be quite unlike any previous theory. Usual approaches based on quantum fields depend on the existence of an ambient space-time manifold. It is not clear what the basic degrees of freedom should be in a theory that does not assume a space-time manifold at the outset.

There is an interesting proposal for an exact quantum mechanical description of M-theory, applicable to certain space-time backgrounds, that goes by the name of *Matrix theory.* Matrix theory gives a dual description of M-theory in flat 11-dimensional space-time in terms of the quantum mechanics of $N \times N$ matrices in the large N limit. When n of the spatial dimensions are compactified on a torus, the dual Matrix theory becomes a quantum field theory in n spatial dimensions (plus time). There is evidence that this conjecture is correct when n is not too large. However, it is unclear how to generalize it to other compactification geometries, so Matrix theory provides only pieces of a more complete description of M-theory.

F-theory

As previously discussed, the type IIA and heterotic $E_8 \times E_8$ theories can be viewed as arising from a more fundamental eleven-dimensional theory, M-theory. One may wonder if the other superstring theories can be derived in a similar fashion. An approach, called F-theory, is described in Chapter 9. It utilizes the fact that ten-dimensional type IIB superstring theory has a nonperturbative $SL(2, \mathbb{Z})$ symmetry. Moreover, this is the modular group of a torus and the type IIB theory contains a complex scalar field τ that transforms under $SL(2, \mathbb{Z})$ as the complex structure of a torus. Therefore, this symmetry can be given a geometric interpretation if the type IIB theory is viewed as having an auxiliary two-torus T^2 with complex structure τ. The $SL(2, \mathbb{Z})$ symmetry then has a natural interpretation as the symmetry of the torus.

Flux compactifications

One question that already bothered Kaluza and Klein is why should the fifth dimension curl up? Another puzzle in those early days was the size of the circle, and what stabilizes it at a particular value. These questions have analogs in string theory, where they are part of what is called *the moduli-space problem.* In string theory the shape and size of the internal manifold is dynamically determined by the vacuum expectation values of scalar fields. String theorists have recently been able to provide answers to these questions in the context of flux compactifications , which is a rapidly developing area of modern string theory research. This is discussed in Chapter 10.

Even though the underlying theory (M-theory) is unique, it admits an enormous number of different solutions (or quantum vacua). One of these solutions should consist of four-dimensional Minkowski space-time times a compact manifold and accurately describes the world of particle physics.

One of the major challenges of modern string theory research is to find this solution.

It would be marvelous to identify the correct vacuum, and at the same time to understand *why* it is the right one. Is it picked out by some special mathematical property, or is it just an environmental accident of our particular corner of the Universe? The way this question plays out will be important in determining the extent to which the observed world of particle physics can be deduced from first principles.

Black-hole entropy

It follows from general relativity that macroscopic *black holes* behave like thermodynamic objects with a well-defined *temperature* and *entropy*. The entropy is given (in gravitational units) by 1/4 the area of the event horizon, which is the *Bekenstein–Hawking* entropy formula. In quantum theory, an entropy S ordinarily implies that there are a large number of quantum states (namely, $\exp S$ of them) that contribute to the corresponding microscopic description. So a natural question is whether this rule also applies to black holes and their higher-dimensional generalizations, which are called *black p-branes*. D-branes provide a set-up in which this question can be investigated.

In the early work on this subject, reliable techniques for counting microstates only existed for very special types of black holes having a large amount of supersymmetry. In those cases one found agreement with the entropy formula. More recently, one has learned how to analyze a much larger class of black holes and black p-branes, and even how to compute corrections to the area formula. This subject is described in Chapter 11. Many examples have been studied and no discrepancies have been found, aside from corrections that are expected. It is fair to say that these studies have led to a much deeper understanding of the thermodynamic properties of black holes in terms of string-theory microphysics, a fact that is one of the most striking successes of string theory so far.

AdS/CFT duality

A remarkable discovery made in the late 1990s is the exact equivalence (or duality) of conformally invariant quantum field theories and superstring theory or M-theory in special space-time geometries. A collection of coincident p-branes produces a space-time geometry with a horizon, like that of a black hole. In the vicinity of the horizon, this geometry can be approximated by a product of an *anti-de Sitter space* and a sphere. In the example that arises from considering N coincident D3-branes in the type IIB superstring theory, one obtains a duality between $SU(N)$ Yang–Mills theory with $\mathcal{N} = 4$ supersymmetry in four dimensions and type IIB superstring theory in a ten-dimensional geometry given by a product of a five-dimensional anti-de Sitter space (AdS_5) and a five-dimensional sphere (S^5). There are N units of five-form flux threading the five sphere. There are also analogous M-theory dualities.

These dualities are sometimes referred to as AdS/CFT dualities. AdS stands for *anti-de Sitter space*, a maximally symmetric space-time geometry with negative scalar curvature. CFT stands for *conformal field theory*, a quantum field theory that is invariant under the group of conformal transformations. This type of equivalence is an example of a *holographic duality*, since it is analogous to representing three-dimensional space on a two-dimensional emulsion. The study of these dualities is teaching us a great deal about string theory and M-theory as well as the dual quantum field theories. Chapter 12 gives an introduction to this vast subject.

String and M-theory cosmology

The field of *superstring cosmology* is emerging as a new and exciting discipline. String theorists and string-theory considerations are injecting new ideas into the study of cosmology. This might be the arena in which predictions that are specific to string theory first confront data.

In a quantum theory that contains gravity, such as string theory, the *cosmological constant*, Λ, which characterizes the energy density of the vacuum, is (at least in principle) a computable quantity. This energy (sometimes called *dark energy*) has recently been measured to fairly good accuracy, and found to account for about 70% of the total mass/energy in the present-day Universe. This fraction is an increasing function of time. The observed value of the cosmological constant/dark energy is important for cosmology, but it is extremely tiny when expressed in Planck units (about 10^{-120}). The first attempts to account for $\Lambda > 0$ within string theory and M-theory, based on compactifying 11-dimensional supergravity on time-independent compact manifolds, were ruled out by "no-go" theorems. However, certain nonperturbative effects allow these no-go theorems to be circumvented.

A viewpoint that has gained in popularity recently is that string theory can accommodate almost any value of Λ, but only solutions for which Λ is sufficiently small describe a Universe that can support life. So, if it were much larger, we wouldn't be here to ask the question. This type of reasoning is called *anthropic*. While this may be correct, it would be satisfying to have another explanation of why Λ is so small that does not require this type of reasoning.

Another important issue in cosmology concerns the accelerated expansion of the very early Universe, which is referred to as *inflation*. The observational case for inflation is quite strong, and it is an important question to understand how it arises from a fundamental theory. Before the period of inflation was the *Big Bang*, the origin of the observable Universe, and much effort is going into understanding that. Two radically different proposals are quantum tunneling from nothing and a collision of branes.

2

The bosonic string

This chapter introduces the simplest string theory, called the bosonic string. Even though this theory is unrealistic and not suitable for phenomenology, it is the natural place to start. The reason is that the same structures and techniques, together with a number of additional ones, are required for the analysis of more realistic superstring theories. This chapter describes the free (noninteracting) theory both at the classical and quantum levels. The next chapter discusses various techniques for introducing and analyzing interactions.

A string can be regarded as a special case of a p-brane, a p-dimensional extended object moving through space-time. In this notation a point particle corresponds to the $p = 0$ case, in other words to a zero-brane. Strings (whether fundamental or solitonic) correspond to the $p = 1$ case, so that they can also be called one-branes. Two-dimensional extended objects or two-branes are often called membranes. In fact, the name p-brane was chosen to suggest a generalization of a membrane. Even though strings share some properties with higher-dimensional extended objects at the classical level, they are very special in the sense that their two-dimensional world-volume quantum theories are renormalizable, something that is not the case for branes of higher dimension. This is a crucial property that makes it possible to base quantum theories on them. In this chapter we describe the string as a special case of p-branes and describe the properties that hold only for the special case $p = 1$.

2.1 p-brane actions

This section describes the free motion of p-branes in space-time using the principle of minimal action. Let us begin with a point particle or zero-brane.

Relativistic point particle

The motion of a relativistic particle of mass m in a curved D-dimensional space-time can be formulated as a variational problem, that is, an action principle. Since the classical motion of a point particle is along geodesics, the action should be proportional to the invariant length of the particle's trajectory

$$S_0 = -\alpha \int ds, \tag{2.1}$$

where α is a constant and $\hbar = c = 1$. This length is extremized in the classical theory, as is illustrated in Fig. 2.1.

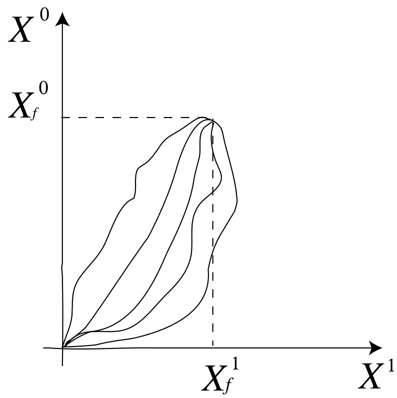

Fig. 2.1. The classical trajectory of a point particle minimizes the length of the world line.

Requiring the action to be dimensionless, one learns that α has the dimensions of inverse length, which is equivalent to mass in our units, and hence it must be proportional to m. As is demonstrated in Exercise 2.1, the action has the correct nonrelativistic limit if $\alpha = m$, so the action becomes

$$S_0 = -m \int ds. \tag{2.2}$$

In this formula the line element is given by

$$ds^2 = -g_{\mu\nu}(X)dX^\mu dX^\nu. \tag{2.3}$$

Here $g_{\mu\nu}(X)$, with $\mu, \nu = 0, \ldots, D-1$, describes the background geometry, which is chosen to have Minkowski signature $(- + \cdots +)$. The minus sign has been introduced here so that ds is real for a time-like trajectory. The particle's trajectory $X^\mu(\tau)$, also called the world line of the particle, is parametrized by a real parameter τ, but the action is independent of the choice of parametrization (see Exercise 2.2). The action (2.2) therefore takes the form

$$S_0 = -m \int \sqrt{-g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu} d\tau, \tag{2.4}$$

where the dot represents the derivative with respect to τ.

The action S_0 has the disadvantage that it contains a square root, so that it is difficult to quantize. Furthermore, this action obviously cannot be used to describe a massless particle. These problems can be circumvented by introducing an action equivalent to the previous one at the classical level, which is formulated in terms of an auxiliary field $e(\tau)$

$$\widetilde{S}_0 = \frac{1}{2}\int d\tau \left(e^{-1}\dot{X}^2 - m^2 e\right), \tag{2.5}$$

where $\dot{X}^2 = g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu$. Reparametrization invariance of $\widetilde{S}_0$ requires that $e(\tau)$ transforms in an appropriate fashion (see Exercise 2.3). The equation of motion of $e(\tau)$, given by setting the variational derivative of this action with respect to $e(\tau)$ equal to zero, is $m^2 e^2 + \dot{X}^2 = 0$. Solving for $e(\tau)$ and substituting back into $\widetilde{S}_0$ gives S_0.

Generalization to the p-brane action

The action (2.4) can be generalized to the case of a string sweeping out a two-dimensional world sheet in space-time and, in general, to a p-brane sweeping out a $(p+1)$-dimensional world volume in D-dimensional space-time. It is necessary, of course, that $p < D$. For example, a membrane or two-brane sweeps out a three-dimensional world volume as it moves through a higher-dimensional space-time. This is illustrated for a string in Fig. 2.2.

The generalization of the action (2.4) to a p-brane naturally takes the form

$$S_p = -T_p \int d\mu_p. \tag{2.6}$$

Here T_p is called the p-brane tension and $d\mu_p$ is the $(p+1)$-dimensional volume element given by

$$d\mu_p = \sqrt{-\det G_{\alpha\beta}}\, d^{p+1}\sigma, \tag{2.7}$$

where the induced metric is given by

$$G_{\alpha\beta} = g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu \quad \alpha, \beta = 0, \ldots, p. \tag{2.8}$$

To write down this form of the action, one has taken into account that p-brane world volumes can be parametrized by the coordinates $\sigma^0 = \tau$, which is time-like, and σ^i, which are p space-like coordinates. Since $d\mu_p$ has units of $(\text{length})^{p+1}$ the dimension of the p-brane tension is

$$[T_p] = (\text{length})^{-p-1} = \frac{\text{mass}}{(\text{length})^p}, \tag{2.9}$$

or energy per unit p-volume.

Exercises

Exercise 2.1

Show that the nonrelativistic limit of the action (2.1) in flat Minkowski space-time determines the value of the constant α to be the mass of the point particle.

Solution

In the nonrelativistic limit the action (2.1) becomes

$$S_0 = -\alpha \int \sqrt{dt^2 - d\vec{x}^2} = -\alpha \int dt \sqrt{1 - \vec{v}^2} \approx -\alpha \int dt \left(1 - \frac{1}{2}\vec{v}^2 + \ldots\right).$$

Comparing the above expansion with the action of a nonrelativistic point

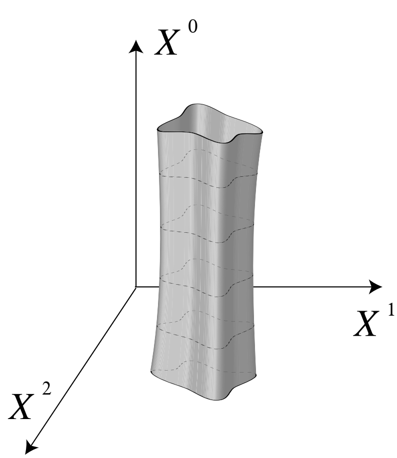

Fig. 2.2. The classical trajectory of a string minimizes the area of the world sheet.

particle, namely

$$S_{\rm nr} = \int dt \frac{1}{2} m\vec{v}^2,$$

gives $\alpha = m$. In the nonrelativistic limit an additional constant (the famous $E = mc^2$ term) appears in the above expansion of S_0. This constant does not contribute to the classical equations of motion. □

EXERCISE **2.2**

One important requirement for the point-particle world-line action is that it should be invariant under reparametrizations of the parameter τ. Show that the action S_0 is invariant under reparametrizations of the world line by substituting $\tau' = f(\tau)$.

SOLUTION

The action

$$S_0 = -m \int \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}} d\tau$$

can be written in terms of primed quantities by taking into account

$$d\tau' = \frac{df(\tau)}{d\tau} d\tau = \dot{f}(\tau) d\tau \qquad \text{and} \qquad \frac{dX^\mu}{d\tau} = \frac{dX^\mu}{d\tau'}\frac{d\tau'}{d\tau} = \frac{dX^\mu}{d\tau'} \cdot \dot{f}(\tau).$$

This gives,

$$S_0' = -m \int \sqrt{-\frac{dX^\mu}{d\tau'}\frac{dX_\mu}{d\tau'}} \dot{f}(\tau) \cdot \frac{d\tau'}{\dot{f}(\tau)} = -m \int \sqrt{-\frac{dX^\mu}{d\tau'}\frac{dX_\mu}{d\tau'}} \cdot d\tau',$$

which shows that the action S_0 is invariant under reparametrizations. □

EXERCISE **2.3**

The action $\widetilde{S}_0$ in Eq. (2.5) is also invariant under reparametrizations of the particle world line. Even though it is not hard to consider finite transformations, let us consider an infinitesimal change of parametrization

$$\tau \to \tau' = f(\tau) = \tau - \xi(\tau).$$

Verify the invariance of $\widetilde{S}_0$ under an infinitesimal reparametrization.

SOLUTION

The field X^μ transforms as a world-line scalar, $X^{\mu\prime}(\tau') = X^\mu(\tau)$. Therefore, the first-order shift in X^μ is

$$\delta X^\mu = X^{\mu\prime}(\tau) - X^\mu(\tau) = \xi(\tau)\dot{X}^\mu.$$

Notice that the fact that X^μ has a space-time vector index is irrelevant to this argument. The auxiliary field $e(\tau)$ transforms at the same time according to

$$e'(\tau')d\tau' = e(\tau)d\tau.$$

Infinitesimally, this leads to

$$\delta e = e'(\tau) - e(\tau) = \frac{d}{d\tau}(\xi e).$$

Let us analyze the special case of a flat space-time metric $g_{\mu\nu}(X) = \eta_{\mu\nu}$, even though the result is true without this restriction. In this case the vector index on X^μ can be raised and lowered inside derivatives. The expression $\widetilde{S}_0$ has the variation

$$\delta \widetilde{S}_0 = \frac{1}{2}\int d\tau \left(\frac{2\dot{X}^\mu \delta \dot{X}_\mu}{e} - \frac{\dot{X}^\mu \dot{X}_\mu}{e^2}\delta e - m^2 \delta e \right).$$

Here $\delta \dot{X}_\mu$ is given by

$$\delta \dot{X}_\mu = \frac{d}{d\tau}\delta X_\mu = \dot{\xi}\dot{X}_\mu + \xi \ddot{X}_\mu.$$

Together with the expression for δe, this yields

$$\delta \widetilde{S}_0 = \frac{1}{2}\int d\tau \left[\frac{2\dot{X}^\mu}{e}\left(\dot{\xi}\dot{X}_\mu + \xi\ddot{X}_\mu\right) - \frac{\dot{X}^\mu \dot{X}_\mu}{e^2}\left(\dot{\xi}e + \xi\dot{e}\right) - m^2 \frac{d(\xi e)}{d\tau} \right].$$

The last term can be dropped because it is a total derivative. The remaining terms can be written as

$$\delta \widetilde{S}_0 = \frac{1}{2}\int d\tau \cdot \frac{d}{d\tau}\left(\frac{\xi}{e}\dot{X}^\mu \dot{X}_\mu\right).$$

This is a total derivative, so it too can be dropped (for suitable boundary conditions). Therefore, $\widetilde{S}_0$ is invariant under reparametrizations. □

EXERCISE **2.4**

The reparametrization invariance that was checked in the previous exercise allows one to choose a gauge in which $e = 1$. As usual, when doing this one should be careful to retain the e equation of motion (evaluated for $e = 1$). What is the form and interpretation of the equations of motion for e and X^μ resulting from $\widetilde{S}_0$?

Solution

The equation of motion for e derived from the action principle for $\widetilde{S}_0$ is given by the vanishing of the variational derivative

$$\frac{\delta \widetilde{S}_0}{\delta e} = -\frac{1}{2}\left(e^{-2}\dot{X}^\mu \dot{X}_\mu + m^2\right) = 0.$$

Choosing the gauge $e(\tau) = 1$, we obtain the equation

$$\dot{X}^\mu \dot{X}_\mu + m^2 = 0.$$

Since $p^\mu = \dot{X}^\mu$ is the momentum conjugate to X^μ, this equation is simply the mass-shell condition $p^2 + m^2 = 0$, so that m is the mass of the particle, as was shown in Exercise 2.1. The variation with respect to X^μ gives the second equation of motion

$$-\frac{d}{d\tau}(g_{\mu\nu}\dot{X}^\nu) + \frac{1}{2}\partial_\mu g_{\rho\lambda}\dot{X}^\rho \dot{X}^\lambda$$

$$= -(\partial_\rho g_{\mu\nu})\dot{X}^\rho \dot{X}^\nu - g_{\mu\nu}\ddot{X}^\nu + \frac{1}{2}\partial_\mu g_{\rho\lambda}\dot{X}^\rho \dot{X}^\lambda = 0.$$

This can be brought to the form

$$\ddot{X}^\mu + \Gamma^\mu_{\rho\lambda}\dot{X}^\rho \dot{X}^\lambda = 0, \tag{2.10}$$

where

$$\Gamma^\mu_{\rho\lambda} = \frac{1}{2}g^{\mu\nu}(\partial_\rho g_{\lambda\nu} + \partial_\lambda g_{\rho\nu} - \partial_\nu g_{\rho\lambda})$$

is the Christoffel connection (or Levi–Civita connection). Equation (2.10) is the geodesic equation. Note that, for a flat space-time, $\Gamma^\mu_{\rho\lambda}$ vanishes in Cartesian coordinates, and one recovers the familiar equation of motion for a point particle in flat space. Note also that the more conventional normalization ($\dot{X}^\mu \dot{X}_\mu + 1 = 0$) would have been obtained by choosing the gauge $e = 1/m$. □

Exercise **2.5**

The action of a p-brane is invariant under reparametrizations of the $p + 1$ world-volume coordinates. Show this explicitly by checking that the action (2.6) is invariant under a change of variables $\sigma^\alpha \to \sigma^\alpha(\widetilde{\sigma})$.

Solution

Under this change of variables the induced metric in Eq. (2.8) transforms in the following way:

$$G_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta} g_{\mu\nu} = (f^{-1})^\gamma_\alpha \frac{\partial X^\mu}{\partial \widetilde{\sigma}^\gamma}(f^{-1})^\delta_\beta \frac{\partial X^\nu}{\partial \widetilde{\sigma}^\delta} g_{\mu\nu},$$

where

$$f^\alpha_\beta(\widetilde{\sigma}) = \frac{\partial \sigma^\alpha}{\partial \widetilde{\sigma}^\beta}.$$

Defining J to be the Jacobian of the world-volume coordinate transformation, that is, $J = \det f^\alpha_\beta$, the determinant appearing in the action becomes

$$\det\left(g_{\mu\nu}\frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\right) = J^{-2}\det\left(g_{\mu\nu}\frac{\partial X^\mu}{\partial \widetilde{\sigma}^\gamma}\frac{\partial X^\nu}{\partial \widetilde{\sigma}^\delta}\right).$$

The measure of the integral transforms according to

$$d^{p+1}\sigma = J d^{p+1}\widetilde{\sigma},$$

so that the Jacobian factors cancel, and the action becomes

$$\widetilde{S}_p = -T_p \int d^{p+1}\widetilde{\sigma}\sqrt{-\det\left(g_{\mu\nu}\frac{\partial X^\mu}{\partial \widetilde{\sigma}^\gamma}\frac{\partial X^\nu}{\partial \widetilde{\sigma}^\delta}\right)}.$$

Therefore, the action is invariant under reparametrizations of the world-volume coordinates. □

2.2 The string action

This section specializes the discussion to the case of a string (or one-brane) propagating in D-dimensional flat Minkowski space-time. The string sweeps out a two-dimensional surface as it moves through space-time, which is called the *world sheet*. The points on the world sheet are parametrized by the two coordinates $\sigma^0 = \tau$, which is time-like, and $\sigma^1 = \sigma$, which is space-like. If the variable σ is periodic, it describes a closed string. If it covers a finite interval, the string is open. This is illustrated in Fig. 2.3.

The Nambu-Goto action

The space-time embedding of the string world sheet is described by functions $X^\mu(\sigma, \tau)$, as shown in Fig. 2.4. The action describing a string propagating in a flat background geometry can be obtained as a special case of the more general p-brane action of the previous section. This action, called the *Nambu–Goto action*, takes the form

$$S_{\rm NG} = -T\int d\sigma d\tau \sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}, \qquad (2.11)$$

where

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} \qquad \text{and} \qquad X^{\mu\prime} = \frac{\partial X^\mu}{\partial \sigma}, \tag{2.12}$$

and the scalar products are defined in the case of a flat space-time by $A \cdot B = \eta_{\mu\nu} A^\mu B^\nu$. The integral appearing in this action describes the area of the world sheet. As a result, the classical string motion minimizes (or at least extremizes) the world-sheet area, just as classical particle motion makes the length of the world line extremal by moving along a geodesic.

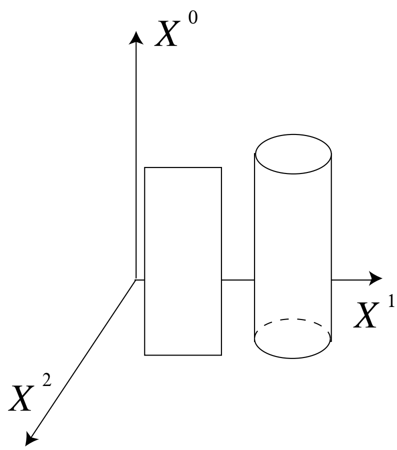

Fig. 2.3. The world sheet for the free propagation of an open string is a rectangular surface, while the free propagation of a closed string sweeps out a cylinder.

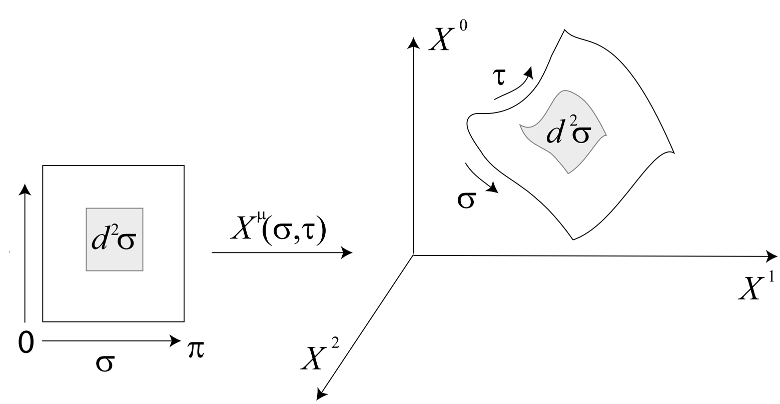

Fig. 2.4. The functions $X^\mu(\sigma, \tau)$ describe the embedding of the string world sheet in space-time.

The string sigma model action

Even though the Nambu–Goto action has a nice physical interpretation as the area of the string world sheet, its quantization is again awkward due to the presence of the square root. An action that is equivalent to the Nambu–Goto action at the classical level, because it gives rise to the same equations of motion, is the *string sigma model action.*[1]

The string sigma-model action is expressed in terms of an auxiliary world-sheet metric $h_{\alpha\beta}(\sigma,\tau)$, which plays a role analogous to the auxiliary field $e(\tau)$ introduced for the point particle. We shall use the notation $h_{\alpha\beta}$ for the world-sheet metric, whereas $g_{\mu\nu}$ denotes a space-time metric. Also,

$$h = \det h_{\alpha\beta} \qquad \text{and} \qquad h^{\alpha\beta} = (h^{-1})_{\alpha\beta}\,, \tag{2.13}$$

as is customary in relativity. In this notation the string sigma-model action is

$$S_\sigma = -\frac{1}{2}T\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X. \tag{2.14}$$

At the classical level the string sigma-model action is equivalent to the Nambu–Goto action. However, it is more convenient for quantization.

Exercises

Exercise 2.6

Derive the equations of motion for the auxiliary metric $h_{\alpha\beta}$ and the bosonic field X^μ in the string sigma-model action. Show that classically the string sigma-model action (2.14) is equivalent to the Nambu–Goto action (2.11).

Solution

As for the point-particle case discussed earlier, the auxiliary metric $h_{\alpha\beta}$ appearing in the string sigma-model action can be eliminated using its equations of motion. Indeed, since there is no kinetic term for $h_{\alpha\beta}$, its equation of motion implies the vanishing of the world-sheet energy–momentum tensor

1 This action, traditionally called the *Polyakov action*, was discovered by Brink, Di Vecchia and Howe and by Deser and Zumino several years before Polyakov skillfully used it for path-integral quantization of the string.

$T_{\alpha\beta}$, that is,

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S_\sigma}{\delta h^{\alpha\beta}} = 0.$$

To evaluate the variation of the action, the following formula is useful:

$$\delta h = -h h_{\alpha\beta}\delta h^{\alpha\beta},$$

which implies that

$$\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h}h_{\alpha\beta}\delta h^{\alpha\beta}. \tag{2.15}$$

After taking the variation of the action, the result for the energy–momentum tensor takes the form

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X = 0.$$

This is the equation of motion for $h_{\alpha\beta}$, which can be used to eliminate $h_{\alpha\beta}$ from the string sigma-model action. The result is the Nambu–Goto action. The easiest way to see this is to take the square root of minus the determinant of both sides of the equation

$$\partial_\alpha X \cdot \partial_\beta X = \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X.$$

This gives

$$\sqrt{-\det(\partial_\alpha X \cdot \partial_\beta X)} = \frac{1}{2}\sqrt{-h}\, h^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X.$$

Finally, the equation of motion for X^μ, obtained from the Euler–Lagrange condition, is

$$\Delta X^\mu = -\frac{1}{\sqrt{-h}}\partial_\alpha\left(\sqrt{-h}h^{\alpha\beta}\partial_\beta X^\mu\right) = 0.$$

□

EXERCISE **2.7**

Calculate the nonrelativistic limit of the Nambu–Goto action

$$S_{\rm NG} = -T\int d\tau d\sigma\sqrt{-\det G_{\alpha\beta}}, \qquad G_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu$$

for a string in Minkowski space-time. Use the static gauge, which fixes the longitudinal directions $X^0 = \tau$, $X^1 = \sigma$, while leaving the transverse directions X^i free. Show that the kinetic energy contains only the transverse velocity. Determine the mass per unit length of the string.

SOLUTION

In the static gauge

$$\det G_{\alpha\beta} = \det \begin{pmatrix} \partial_\tau X^\mu \partial_\tau X_\mu & \partial_\tau X^\mu \partial_\sigma X_\mu \\ \partial_\sigma X^\mu \partial_\tau X_\mu & \partial_\sigma X^\mu \partial_\sigma X_\mu \end{pmatrix}$$

$$= \det \begin{pmatrix} -1 + \partial_\tau X^i \partial_\tau X_i & \partial_\tau X^i \partial_\sigma X_i \\ \partial_\sigma X^i \partial_\tau X_i & 1 + \partial_\sigma X^i \partial_\sigma X_i \end{pmatrix}.$$

Then,

$$\det G_{\alpha\beta} \approx -1 + \partial_\tau X^i \partial_\tau X_i - \partial_\sigma X^i \partial_\sigma X_i + \ldots$$

Here the dots indicate higher-order terms that can be dropped in the non-relativistic limit for which the velocities are small. In this limit the action becomes (after a Taylor expansion)

$$S_{\text{NG}} = -T \int d\tau d\sigma \sqrt{|-1 + \partial_\tau X^i \partial_\tau X_i - \partial_\sigma X^i \partial_\sigma X_i|}$$

$$\approx T \int d\tau d\sigma \left(-1 + \frac{1}{2} \partial_\tau X^i \partial_\tau X_i - \frac{1}{2} \partial_\sigma X^i \partial_\sigma X_i \right).$$

The first term in the parentheses gives $-m \int d\tau$, if L is the length of the σ interval and $m = LT$. This is the rest-mass contribution to the potential energy. Note that L is a distance in space, because of the choice of static gauge. Thus the tension T can be interpreted as the mass per unit length, or mass density, of the string. The last two terms of the above formula are the kinetic energy and the negative of the potential energy of a nonrelativistic string of tension T. □

EXERCISE **2.8**

Show that if a *cosmological constant term* is added to the string sigma-model action, so that

$$S_\sigma = -\frac{T}{2} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + \Lambda \int d^2\sigma \sqrt{-h},$$

it leads to inconsistent classical equations of motion.

SOLUTION

The equation of motion for the world-sheet metric is

$$\frac{2}{\sqrt{-h}} \frac{\delta S_\sigma}{\delta h^{\gamma\delta}} = -T[\partial_\gamma X^\mu \partial_\delta X_\mu - \frac{1}{2} h_{\gamma\delta} (h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu)] - \Lambda h_{\gamma\delta} = 0,$$

where we have used Eq. (2.15). Contracting with $h^{\gamma\delta}$ gives

$$h_{\gamma\delta}h^{\gamma\delta}\Lambda = T(\frac{1}{2}h_{\gamma\delta}h^{\gamma\delta} - 1)h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu.$$

Since $h_{\gamma\delta}h^{\gamma\delta} = 2$, the right-hand side vanishes. Thus, assuming $h \neq 0$, consistency requires $\Lambda = 0$. In other words, adding a cosmological constant term gives inconsistent classical equations of motion. □

EXERCISE **2.9**

Show that the sigma-model form of the action of a p-brane, for $p \neq 1$, requires a cosmological constant term.

SOLUTION

Consider a p-brane action of the form

$$S_\sigma = -\frac{T_p}{2}\int d^{p+1}\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X + \Lambda_p \int d^{p+1}\sigma\sqrt{-h}. \qquad (2.16)$$

The equation of motion for the world-volume metric is obtained exactly as in the previous exercise, with the result

$$T_p[\partial_\gamma X \cdot \partial_\delta X - \frac{1}{2}h_{\gamma\delta}(h^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X)] + \Lambda_p h_{\gamma\delta} = 0.$$

This equation is not so easy to solve directly, so let us instead investigate whether it is solved by equating the world-volume metric to the induced metric

$$h_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X. \qquad (2.17)$$

Substituting this ansatz in the previous equation and dropping common factors gives

$$T_p(1 - \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}) + \Lambda_p = 0.$$

Substituting $h^{\alpha\beta}h_{\alpha\beta} = p + 1$, one learns that

$$\Lambda_p = \frac{1}{2}(p-1)T_p. \qquad (2.18)$$

Thus, consistency requires this choice of Λ_p.[2] This confirms the previous result that $\Lambda_1 = 0$ and shows that $\Lambda_p \neq 0$ for $p \neq 1$. Substituting the value of the metric in Eq. (2.17) and the value of Λ_p in Eq. (2.18), one finds that Eq. (2.16) is equivalent classically to Eq. (2.6). For the special case of

2 A different value is actually equivalent, if one makes a corresponding rescaling of $h_{\alpha\beta}$. However, this results in a multiplicative factor in the relation (2.17).

$p = 0$, this reproduces the result in Eq. (2.5) if one makes the identifications $T_0 = m$ and $h_{00} = -m^2 e^2$. □

2.3 String sigma-model action: the classical theory

In this section we discuss the symmetries of the string sigma-model action in Eq. (2.14). This is helpful for writing the string action in a gauge in which quantization is particularly simple.

Symmetries

The string sigma-model action for the bosonic string in Minkowski space-time has a number of symmetries:

- *Poincaré transformations.* These are global symmetries under which the world-sheet fields transform as

$$\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu \qquad \text{and} \qquad \delta h^{\alpha\beta} = 0. \tag{2.19}$$

  Here the constants $a^\mu{}_\nu$ (with $a_{\mu\nu} = -a_{\nu\mu}$) describe infinitesimal Lorentz transformations and b^μ describe space-time translations.
- *Reparametrizations.* The string world sheet is parametrized by two coordinates τ and σ, but a change in the parametrization does not change the action. Indeed, the transformations

$$\sigma^\alpha \to f^\alpha(\sigma) = \sigma'^\alpha \qquad \text{and} \qquad h_{\alpha\beta}(\sigma) = \frac{\partial f^\gamma}{\partial \sigma^\alpha}\frac{\partial f^\delta}{\partial \sigma^\beta} h_{\gamma\delta}(\sigma') \tag{2.20}$$

  leave the action invariant. These local symmetries are also called *diffeomorphisms.* Strictly speaking, this implies that the transformations and their inverses are infinitely differentiable.
- *Weyl transformations.* The action is invariant under the rescaling

$$h_{\alpha\beta} \to e^{\phi(\sigma,\tau)} h_{\alpha\beta} \qquad \text{and} \qquad \delta X^\mu = 0, \tag{2.21}$$

  since $\sqrt{-h} \to e^\phi \sqrt{-h}$ and $h^{\alpha\beta} \to e^{-\phi} h^{\alpha\beta}$ give cancelling factors. This local symmetry is the reason that the energy–momentum tensor is traceless.

Poincaré transformations are global symmetries, whereas reparametrizations and Weyl transformations are local symmetries. The local symmetries can be used to choose a gauge, such as the static gauge discussed earlier, or else one in which some of the components of the world-sheet metric $h_{\alpha\beta}$ are of a particular form.

Gauge fixing

The gauge-fixing procedure described earlier for the point particle can be generalized to the case of the string. In this case the auxiliary field has three independent components, namely

$$h_{\alpha\beta} = \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix}, \tag{2.22}$$

where $h_{10} = h_{01}$. Reparametrization invariance allows us to choose two of the components of h, so that only one independent component remains. But this remaining component can be gauged away by using the invariance of the action under Weyl rescalings. So in the case of the string there is sufficient symmetry to gauge fix $h_{\alpha\beta}$ completely. As a result, the auxiliary field $h_{\alpha\beta}$ can be chosen as

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.23}$$

Actually such a flat world-sheet metric is only possible if there is no topological obstruction. This is the case when the world sheet has vanishing Euler characteristic. Examples include a cylinder and a torus. When a flat world-sheet metric is an allowed gauge choice, the string action takes the simple form

$$S = \frac{T}{2} \int d^2\sigma (\dot{X}^2 - X'^2). \tag{2.24}$$

The string actions discussed so far describe propagation in flat Minkowski space-time. Keeping this requirement, one could consider the following two additional terms, both of which are renormalizable (or super-renormalizable) and compatible with Poincaré invariance,

$$S_1 = \lambda_1 \int d^2\sigma \sqrt{-h} \qquad \text{and} \qquad S_2 = \lambda_2 \int d^2\sigma \sqrt{-h} R^{(2)}(h). \tag{2.25}$$

S_1 is a cosmological constant term on the world sheet. This term is not allowed by the equations of motion (see Exercise 2.8). The term S_2 involves $R^{(2)}(h)$, the scalar curvature of the two-dimensional world-sheet geometry. Such a contribution raises interesting issues, which are explored in the next chapter. For now, let us assume that it can be ignored.

Equations of motion and boundary conditions

Equations of motion

Let us now suppose that the world-sheet topology allows a flat world-sheet metric to be chosen. For a freely propagating closed string a natural choice is an infinite cylinder. Similarly, the natural choice for an open string is an infinite strip. In both cases, the motion of the string in Minkowski space is governed by the action in Eq. (2.24). This implies that the X^μ equation of motion is the wave equation

$$\partial_\alpha \partial^\alpha X^\mu = 0 \qquad \text{or} \qquad \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu = 0. \tag{2.26}$$

Since the metric on the world sheet has been gauge fixed, the vanishing of the energy–momentum tensor, that is, $T_{\alpha\beta} = 0$ originating from the equation of motion of the world-sheet metric, must now be imposed as an additional constraint condition. In the gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$ the components of this tensor are

$$T_{01} = T_{10} = \dot{X} \cdot X' \qquad \text{and} \qquad T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2). \tag{2.27}$$

Using $T_{00} = T_{11}$, we see the vanishing of the trace of the energy–momentum tensor $\mathrm{Tr}T = \eta^{\alpha\beta}T_{\alpha\beta} = T_{11} - T_{00}$. This is a consequence of Weyl invariance, as was mentioned before.

Boundary conditions

In order to give a fully defined variational problem, boundary conditions need to be specified. A string can be either closed or open. For convenience, let us choose the coordinate σ to have the range $0 \leq \sigma \leq \pi$. The stationary points of the action are determined by demanding invariance of the action under the shifts

$$X^\mu \to X^\mu + \delta X^\mu. \tag{2.28}$$

In addition to the equations of motion, there is the boundary term

$$-T \int d\tau \left[X'_\mu \delta X^\mu|_{\sigma=\pi} - X'_\mu \delta X^\mu|_{\sigma=0} \right], \tag{2.29}$$

which must vanish. There are several different ways in which this can be achieved. For an open string these possibilities are illustrated in Fig. 2.5.

- *Closed string.* In this case the embedding functions are periodic,

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + \pi, \tau). \tag{2.30}$$

- *Open string with Neumann boundary conditions.* In this case the component of the momentum normal to the boundary of the world sheet vanishes, that is,

$$X'_\mu = 0 \qquad \text{at} \qquad \sigma = 0, \pi. \tag{2.31}$$

If this choice is made for all μ, these boundary conditions respect D-dimensional Poincaré invariance. Physically, they mean that no momentum is flowing through the ends of the string.

- *Open string with Dirichlet boundary conditions.* In this case the positions of the two string ends are fixed so that $\delta X^\mu = 0$, and

$$X^\mu|_{\sigma=0} = X_0^\mu \qquad \text{and} \qquad X^\mu|_{\sigma=\pi} = X_\pi^\mu, \tag{2.32}$$

where X_0^μ and X_π^μ are constants and $\mu = 1, \ldots, D-p-1$. Neumann boundary conditions are imposed for the other $p+1$ coordinates. Dirichlet boundary conditions break Poincaré invariance, and for this reason they were not considered for many years. But, as is discussed in Chapter 6, there are circumstances in which Dirichlet boundary conditions are unavoidable. The modern interpretation is that X_0^μ and X_π^μ represent the positions of Dp-branes. A Dp-brane is a special type of p-brane on which a fundamental string can end. The presence of a Dp-brane breaks Poincaré invariance unless it is space-time filling ($p = D-1$).

Solution to the equations of motion

To find the solution to the equations of motion and constraint equations it is convenient to introduce world-sheet *light-cone coordinates*, defined as

$$\sigma^\pm = \tau \pm \sigma. \tag{2.33}$$

In these coordinates the derivatives and the two-dimensional Lorentz metric take the form

$$\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \qquad \text{and} \qquad \begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.34}$$

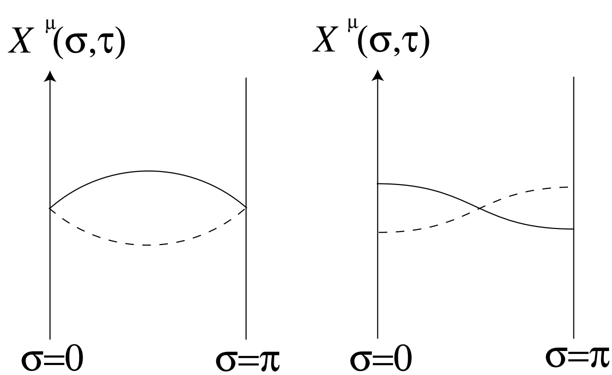

Fig. 2.5. Illustration of Dirichlet (left) and Neumann (right) boundary conditions. The solid and dashed lines represent string positions at two different times.

In light-cone coordinates the wave equation for X^μ is

$$\partial_+\partial_- X^\mu = 0. \tag{2.35}$$

The vanishing of the energy–momentum tensor becomes

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu = 0, \tag{2.36}$$

$$T_{--} = \partial_- X^\mu \partial_- X_\mu = 0, \tag{2.37}$$

while $T_{+-} = T_{-+} = 0$ expresses the vanishing of the trace, which is automatic. The general solution of the wave equation (2.35) is given by

$$X^\mu(\sigma, \tau) = X^\mu_{\rm R}(\tau - \sigma) + X^\mu_{\rm L}(\tau + \sigma), \tag{2.38}$$

which is a sum of *right-movers* and *left-movers*. To find the explicit form of $X_{\rm R}$ and $X_{\rm L}$ one should require $X^\mu(\sigma, \tau)$ to be real and impose the constraints

$$(\partial_- X_{\rm R})^2 = (\partial_+ X_{\rm L})^2 = 0. \tag{2.39}$$

The quantum version of these constraints will be discussed in the next section.

Closed-string mode expansion

The most general solution of the wave equation satisfying the closed-string boundary condition is given by

$$X^\mu_{\rm R} = \frac{1}{2}x^\mu + \frac{1}{2}l_{\rm s}^2 p^\mu(\tau - \sigma) + \frac{i}{2}l_{\rm s}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-2in(\tau-\sigma)}, \tag{2.40}$$

$$X^\mu_{\rm L} = \frac{1}{2}x^\mu + \frac{1}{2}l_{\rm s}^2 p^\mu(\tau + \sigma) + \frac{i}{2}l_{\rm s}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}^\mu_n e^{-2in(\tau+\sigma)}, \tag{2.41}$$

where x^μ is a *center-of-mass* position and p^μ is the total string momentum, describing the free motion of the string center of mass. The exponential terms represent the string excitation modes. Here we have introduced a new parameter, the *string length scale* $l_{\rm s}$, which is related to the string tension T and the open-string Regge slope parameter α' by

$$T = \frac{1}{2\pi\alpha'} \qquad \text{and} \qquad \frac{1}{2}l_{\rm s}^2 = \alpha'. \tag{2.42}$$

The requirement that $X^\mu_{\rm R}$ and $X^\mu_{\rm L}$ are real functions implies that x^μ and p^μ are real, while positive and negative modes are conjugate to each other

$$\alpha^\mu_{-n} = (\alpha^\mu_n)^\star \qquad \text{and} \qquad \widetilde{\alpha}^\mu_{-n} = (\widetilde{\alpha}^\mu_n)^\star\,. \tag{2.43}$$

The terms linear in σ cancel from the sum $X^\mu_{\rm R} + X^\mu_{\rm L}$, so that closed-string boundary conditions are indeed satisfied. Note that the derivatives of the expansions take the form

$$\partial_- X^\mu_{\rm R} = l_{\rm s} \sum_{m=-\infty}^{+\infty} \alpha^\mu_m e^{-2im(\tau-\sigma)} \tag{2.44}$$

$$\partial_+ X^\mu_{\rm L} = l_{\rm s} \sum_{m=-\infty}^{+\infty} \widetilde{\alpha}^\mu_m e^{-2im(\tau+\sigma)}, \tag{2.45}$$

where

$$\alpha^\mu_0 = \widetilde{\alpha}^\mu_0 = \frac{1}{2} l_{\rm s} p^\mu. \tag{2.46}$$

These expressions are useful later. In order to quantize the theory, let us first introduce the canonical momentum conjugate to X^μ. It is given by

$$P^\mu(\sigma,\tau) = \frac{\delta S}{\delta \dot{X}_\mu} = T \dot{X}^\mu. \tag{2.47}$$

With this definition of the canonical momentum, the classical Poisson brackets are

$$\Big[P^\mu(\sigma,\tau), P^\nu(\sigma',\tau)\Big]_{\rm P.B.} = \Big[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\Big]_{\rm P.B.} = 0, \tag{2.48}$$

$$\Big[P^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\Big]_{\rm P.B.} = \eta^{\mu\nu}\delta(\sigma-\sigma'). \tag{2.49}$$

In terms of $\dot{X}^\mu$

$$\Big[\dot{X}^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\Big]_{\rm P.B.} = T^{-1}\eta^{\mu\nu}\delta(\sigma-\sigma'). \tag{2.50}$$

Inserting the mode expansion for X^μ and $\dot{X}^\mu$ into these equations gives the Poisson brackets satisfied by the modes[3]

$$\Big[\alpha^\mu_m, \alpha^\nu_n\Big]_{\rm P.B.} = \Big[\widetilde{\alpha}^\mu_m, \widetilde{\alpha}^\nu_n\Big]_{\rm P.B.} = im\eta^{\mu\nu}\delta_{m+n,0} \tag{2.51}$$

and

$$\big[\alpha^\mu_m, \widetilde{\alpha}^\nu_n\big]_{\rm P.B.} = 0. \tag{2.52}$$

3 The derivation of the commutation relations for the modes uses the Fourier expansion of the Dirac delta function

$$\delta(\sigma-\sigma') = \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} e^{2in(\sigma-\sigma')}.$$

2.4 Canonical quantization

The world-sheet theory can now be quantized by replacing Poisson brackets by commutators

$$[\dots]_{\text{P.B.}} \to i\,[\dots]\,. \tag{2.53}$$

This gives

$$[\alpha_m^\mu, \alpha_n^\nu] = [\widetilde{\alpha}_m^\mu, \widetilde{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \qquad [\alpha_m^\mu, \widetilde{\alpha}_n^\nu] = 0. \tag{2.54}$$

Defining

$$a_m^\mu = \frac{1}{\sqrt{m}}\alpha_m^\mu \qquad \text{and} \qquad a_m^{\mu\dagger} = \frac{1}{\sqrt{m}}\alpha_{-m}^\mu \qquad \text{for} \qquad m > 0, \tag{2.55}$$

the algebra satisfied by the modes is essentially the algebra of raising and lowering operators for quantum-mechanical harmonic oscillators

$$[a_m^\mu, a_n^{\nu\dagger}] = [\widetilde{a}_m^\mu, \widetilde{a}_n^{\nu\dagger}] = \eta^{\mu\nu}\delta_{m,n} \qquad \text{for} \qquad m, n > 0. \tag{2.56}$$

There is just one unusual feature: the commutators of time components have a negative sign, that is,

$$\left[a_m^0, a_m^{0\dagger}\right] = -1. \tag{2.57}$$

This results in negative norm states, which will be discussed in a moment. The spectrum is constructed by applying raising operators on the ground state, which is denoted $|0\rangle$. By definition, the ground state is annihilated by the lowering operators:

$$a_m^\mu|0\rangle = 0 \qquad \text{for} \qquad m > 0. \tag{2.58}$$

One can also specify the momentum k^μ carried by a state $|\phi\rangle$,

$$|\phi\rangle = a_{m_1}^{\mu_1\dagger} a_{m_2}^{\mu_2\dagger} \cdots a_{m_n}^{\mu_n\dagger}|0;k\rangle, \tag{2.59}$$

which is the eigenvalue of the momentum operator p^μ,

$$p^\mu|\phi\rangle = k^\mu|\phi\rangle. \tag{2.60}$$

It should be emphasized that this is *first quantization*, and all of these states (including the ground state) are one-particle states. Second quantization requires *string field theory*, which is discussed briefly at the end of Chapter 3.

The states with an even number of time-component operators have positive norm, while those that are constructed with an odd number of time-component operators have negative norm.[4] A simple example of a negative-norm state is given by

$$a_m^{0\dagger}|0\rangle \qquad \text{with} \qquad \text{norm} \qquad \langle 0|a_m^0 a_m^{0\dagger}|0\rangle = -1, \tag{2.61}$$

where the ground state is normalized as $\langle 0|0\rangle = 1$. In order for the theory to be physically sensible, it is essential that all physical states have positive norm. Negative-norm states in the physical spectrum of an interacting theory would lead to violations of causality and unitarity. The way in which the negative-norm states are eliminated from the physical spectrum is explained later in this chapter.

Open-string mode expansion

The general solution of the string equations of motion for an open string with Neumann boundary conditions is given by

$$X^\mu(\tau,\sigma) = x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m\neq 0} \frac{1}{m}\alpha_m^\mu e^{-im\tau}\cos(m\sigma). \tag{2.62}$$

Mode expansions for other type of boundary conditions are given as homework problems. Note that, for the open string, only one set of modes α_m^μ appears, whereas for the closed string there are two independent sets of modes α_m^μ and $\widetilde{\alpha}_m^\mu$. The open-string boundary conditions force the left- and right-moving modes to combine into standing waves. For the open string

$$2\partial_\pm X^\mu = \dot{X}^\mu \pm X'^\mu = l_s \sum_{m=-\infty}^{\infty} \alpha_m^\mu e^{-im(\tau\pm\sigma)}, \tag{2.63}$$

where, $\alpha_0^\mu = l_s p^\mu$.

Hamiltonian and energy–momentum tensor

As discussed above, the string sigma-model action is invariant under various symmetries.

Noether currents

Recall that there is a standard method, due to Noether, for constructing a conserved current $\mathcal{J}_\alpha$ associated with a global symmetry transformation

$$\phi \to \phi + \delta_\varepsilon \phi, \tag{2.64}$$

4 States that have negative norm are sometimes called *ghosts*, but we reserve that word for the ghost fields that are arise from covariant BRST quantization in the next chapter.

where ϕ is any field of the theory and ε is an infinitesimal parameter. Such a transformation is a symmetry of the theory if it leaves the equations of motion invariant. This is the case if the action changes at most by a surface term, which means that the Lagrangian density changes at most by a total derivative. The Noether current is then determined from the change in the action under the above transformation

$$\mathcal{L} \to \mathcal{L} + \varepsilon \partial_\alpha \mathcal{J}^\alpha. \tag{2.65}$$

When ε is a constant, this change is a total derivative, which reflects the fact that there is a global symmetry. Then the equations of motion imply that the current is conserved, $\partial_\alpha \mathcal{J}^\alpha = 0$. The Poincaré transformations

$$\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu, \tag{2.66}$$

are global symmetries of the string world-sheet theory. Therefore, they give rise to conserved Noether currents. Applying the Noether method to derive the conserved currents associated with the Poincaré transformation of X^μ, one obtains

$$P^\mu_\alpha = T\partial_\alpha X^\mu, \tag{2.67}$$

$$J^{\mu\nu}_\alpha = T\left(X^\mu \partial_\alpha X^\nu - X^\nu \partial_\alpha X^\mu\right), \tag{2.68}$$

where the first current is associated with the translation symmetry, and the second one originates from the invariance under Lorentz transformations.

Hamiltonian

World-sheet time evolution is generated by the Hamiltonian

$$H = \int_0^\pi \left(\dot{X}_\mu P^\mu_0 - \mathcal{L}\right) d\sigma = \frac{T}{2}\int_0^\pi \left(\dot{X}^2 + X'^2\right) d\sigma, \tag{2.69}$$

where

$$P^\mu_0 = \frac{\delta S}{\delta \dot{X}_\mu} = T\dot{X}^\mu, \tag{2.70}$$

was previously called $P^\mu(\sigma,\tau)$. Inserting the mode expansions, the result for the closed-string Hamiltonian is

$$H = \sum_{n=-\infty}^{+\infty} \left(\alpha_{-n}\cdot\alpha_n + \widetilde{\alpha}_{-n}\cdot\widetilde{\alpha}_n\right), \tag{2.71}$$

while for the open string the corresponding expression is

$$H = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \alpha_{-n}\cdot\alpha_n. \tag{2.72}$$

These results hold for the classical theory. In the quantum theory there are ordering ambiguities that need to be resolved.

Energy momentum tensor

Let us now consider the mode expansions of the energy–momentum tensor. Inserting the closed-string mode expansions for $X_{\rm L}$ and $X_{\rm R}$ into the energy–momentum tensor Eqs (2.36), (2.37), one obtains

$$T_{--} = 2\,l_{\rm s}^2 \sum_{m=-\infty}^{+\infty} L_m e^{-2im(\tau-\sigma)} \quad \text{and} \quad T_{++} = 2\,l_{\rm s}^2 \sum_{m=-\infty}^{+\infty} \widetilde{L}_m e^{-2im(\tau+\sigma)}, \tag{2.73}$$

where the Fourier coefficients are the *Virasoro generators*

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \alpha_{m-n}\cdot\alpha_n \qquad \text{and} \qquad \widetilde{L}_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \widetilde{\alpha}_{m-n}\cdot\widetilde{\alpha}_n. \tag{2.74}$$

In the same way, one can get the result for the modes of the energy–momentum tensor of the open string. Comparing with the Hamiltonian, results in the expression

$$\frac{1}{2}H = L_0 + \widetilde{L}_0 = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \left(\alpha_{-n}\cdot\alpha_n + \widetilde{\alpha}_{-n}\cdot\widetilde{\alpha}_n\right), \tag{2.75}$$

for a closed string, while for an open string

$$H = L_0 = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \alpha_{-n}\cdot\alpha_n. \tag{2.76}$$

The above results hold for the classical theory. Again, in the quantum theory one needs to resolve ordering ambiguities.

Mass formula for the string

Classically the vanishing of the energy–momentum tensor translates into the vanishing of all the Fourier modes

$$L_m = 0 \qquad \text{for} \qquad m = 0, \pm 1, \pm 2, \ldots \tag{2.77}$$

The classical constraint

$$L_0 = \widetilde{L}_0 = 0, \tag{2.78}$$

can be used to derive an expression for the mass of a string. The relativistic mass-shell condition is

$$M^2 = -p_\mu p^\mu, \tag{2.79}$$

where p_μ is the total momentum of the string. This total momentum is given by

$$p^\mu = T \int_0^\pi d\sigma \dot{X}^\mu(\sigma), \tag{2.80}$$

so that only the zero mode in the mode expansion of $\dot{X}^\mu(\sigma,\tau)$ contributes.

For the open string, the vanishing of L_0 then becomes

$$L_0 = \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n + \frac{1}{2}\alpha_0^2 = \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n + \alpha' p^2 = 0, \tag{2.81}$$

which gives a relation between the mass of the string and the oscillator modes. For the open string one gets the relation

$$M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n. \tag{2.82}$$

For the closed string one has to take the left-moving and right-moving modes into account, and then one obtains

$$M^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty} \left(\alpha_{-n}\cdot\alpha_n + \widetilde{\alpha}_{-n}\cdot\widetilde{\alpha}_n\right). \tag{2.83}$$

These are the mass-shell conditions for the string, which determine the mass of a given string state. In the quantum theory these relations get slightly modified.

The Virasoro algebra

Classical theory

In the classical theory the Virasoro generators satisfy the algebra

$$[L_m, L_n]_{\text{P.B.}} = i(m-n)L_{m+n}. \tag{2.84}$$

The appearance of the Virasoro algebra is due to the fact that the gauge choice Eq. (2.23) has not fully gauge fixed the reparametrization symmetry. Let ξ^α be an infinitesimal parameter for a reparametrization and let Λ be an infinitesimal parameter for a Weyl rescaling. Then residual reparametrization symmetries satisfying

$$\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha = \Lambda \eta^{\alpha\beta}, \tag{2.85}$$

still remain. These are the reparametrizations that are also Weyl rescalings. If one defines the combinations $\xi^\pm = \xi^0 \pm \xi^1$ and $\sigma^\pm = \sigma^0 \pm \sigma^1$, then one finds that Eq. (2.85) is solved by

$$\xi^+ = \xi^+(\sigma^+) \qquad \text{and} \qquad \xi^- = \xi^-(\sigma^-). \tag{2.86}$$

The infinitesimal generators for the transformations $\delta\sigma^\pm = \xi^\pm$ are given by

$$V^\pm = \frac{1}{2}\xi^\pm(\sigma^\pm)\frac{\partial}{\partial\sigma^\pm}, \tag{2.87}$$

and a complete basis for these transformations is given by

$$\xi_n^\pm(\sigma^\pm) = e^{2in\sigma^\pm} \quad n \in \mathbb{Z}. \tag{2.88}$$

The corresponding generators $V_n^\pm$ give two copies of the Virasoro algebra. In the case of open strings there is just one Virasoro algebra, and the infinitesimal generators are

$$V_n = e^{in\sigma^+}\frac{\partial}{\partial\sigma^+} + e^{in\sigma^-}\frac{\partial}{\partial\sigma^-} \quad n \in \mathbb{Z}. \tag{2.89}$$

In the classical theory the equation of motion for the metric implies the vanishing of the energy–momentum tensor, that is, $T_{++} = T_{--} = 0$, which in terms of the Fourier components of Eq. (2.73) is

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \alpha_{m-n}\cdot\alpha_n = 0 \qquad \text{for} \qquad m \in \mathbb{Z}. \tag{2.90}$$

In the case of closed strings, there are also corresponding $\widetilde{L}_m$ conditions.

Quantum theory

In the quantum theory these operators are defined to be normal-ordered, that is,

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha_{m-n}\cdot\alpha_n: . \tag{2.91}$$

According to the normal-ordering prescription the lowering operators always appear to the right of the raising operators. In particular, L_0 becomes

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n. \tag{2.92}$$

Actually, this is the only Virasoro operator for which normal-ordering matters. Since an arbitrary constant could have appeared in this expression, one must expect a constant to be added to L_0 in all formulas, in particular the Virasoro algebra.

Using the commutators for the modes α_m^μ, one can show that in the quantum theory the Virasoro generators satisfy the relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \tag{2.93}$$

where $c = D$ is the space-time dimension. The term proportional to c is a quantum effect. This means that it appears after quantization and is absent in the classical theory. This term is called a *central extension*, and c is called a *central charge*, since it can be regarded as multiplying the unit operator, which when adjoined to the algebra is in the center of the extended algebra.

$SL(2,\mathbb{R})$ subalgebra

The Virasoro algebra contains an $SL(2,\mathbb{R})$ subalgebra that is generated by L_0, L_1 and L_{-1}. This is a noncompact form of the familiar $SU(2)$ algebra. Just as $SU(2)$ and $SO(3)$ have the same Lie algebra, so do $SL(2,\mathbb{R})$ and $SO(2,1)$. Thus, in the case of closed strings, the complete Virasoro algebra of both left-movers and right-movers contains the subalgebra $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) = SO(2,2)$. This is a noncompact version of the Lie algebra identity $SU(2) \times SU(2) = SO(4)$. The significance of this subalgebra will become clear in the next chapter.

Physical states

As was mentioned above, in the quantum theory a constant may need to be added to L_0 to parametrize the arbitrariness in the ordering prescription. Therefore, when imposing the constraint that the zero mode of the energy–momentum tensor should vanish, the only requirement in the case of the open string is that there exists some constant a such that

$$(L_0 - a)|\phi\rangle = 0. \tag{2.94}$$

Here $|\phi\rangle$ is any physical on-shell state in the theory, and the constant a will be determined later. Similarly, for the closed string

$$(L_0 - a)|\phi\rangle = (\widetilde{L}_0 - a)|\phi\rangle = 0. \tag{2.95}$$

Mass operator

The constant a contributes to the mass operator. Indeed, in the quantum theory Eq. (2.94) corresponds to the mass-shell condition for the open string

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n - a = N - a, \tag{2.96}$$

where

$$N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n, \tag{2.97}$$

is called the *number operator*, since it has integer eigenvalues. For the ground state, which has $N = 0$, this gives $\alpha' M^2 = -a$, while for the excited states $\alpha' M^2 = 1 - a,\, 2 - a, \ldots$

For the closed string

$$\frac{1}{4}\alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n - a = \sum_{n=1}^{\infty} \widetilde{\alpha}_{-n} \cdot \widetilde{\alpha}_n - a = N - a = \widetilde{N} - a. \tag{2.98}$$

Level matching

The normal-ordering constant a cancels out of the difference

$$(L_0 - \widetilde{L}_0)|\phi\rangle = 0, \tag{2.99}$$

which implies $N = \widetilde{N}$. This is the so-called *level-matching condition* of the bosonic string. It is the only constraint that relates the left- and right-moving modes.

Virasoro generators and physical states

In the quantum theory one cannot demand that the operator L_m annihilates all the physical states, for all $m \neq 0$, since this is incompatible with the Virasoro algebra. Rather, a physical state can only be annihilated by half of the Virasoro generators, specifically

$$L_m|\phi\rangle = 0 \qquad m > 0. \tag{2.100}$$

Together with the mass-shell condition

$$(L_0 - a)|\phi\rangle = 0, \tag{2.101}$$

this characterizes a physical state $|\phi\rangle$. This is sufficient to give vanishing matrix elements of $L_n - a\delta_{n,0}$, between physical states, for all n. Since

$$L_{-m} = L_m^\dagger, \tag{2.102}$$

the hermitian conjugate of Eq. (2.100) ensures that the negative-mode Virasoro operators annihilate physical states on their left

$$\langle\phi|L_m = 0 \qquad m < 0. \tag{2.103}$$

There are no normal-ordering ambiguities in the Lorentz generators[5]

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu \right), \tag{2.104}$$

and therefore they can be interpreted as quantum operators without any quantum corrections. Using this expression, it is possible to check that

$$[L_m, J^{\mu\nu}] = 0, \tag{2.105}$$

which implies that the physical-state condition is invariant under Lorentz transformations. Therefore, physical states must appear in complete Lorentz multiplets. This follows from the fact that, the formalism being discussed here is manifestly Lorentz covariant.

Absence of negative-norm states

The goal of this section is to show that a spectrum free of negative-norm states is only possible for certain values of a and the space-time dimension D. In order to carry out the analysis in a covariant manner, a crucial ingredient is the Virasoro algebra in Eq. (2.93).

In the quantum theory the values of a and D are not arbitrary. For some values negative-norm states appear and for other values the physical Hilbert space is positive definite. At the boundary where positive-norm states turn into negative-norm states, an increased number of zero-norm states appear. Therefore, in order to determine the allowed values for a and D, an effective strategy is to search for zero-norm states that satisfy the physical-state conditions.

Spurious states

A state $|\psi\rangle$ is called *spurious* if it satisfies the mass-shell condition and is orthogonal to all physical states

$$(L_0 - a)|\psi\rangle = 0 \qquad \text{and} \qquad \langle\phi|\psi\rangle = 0, \tag{2.106}$$

where $|\phi\rangle$ represents any physical state in the theory. An example of a spurious state is

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n}|\chi_n\rangle \qquad \text{with} \qquad (L_0 - a + n)|\chi_n\rangle = 0. \tag{2.107}$$

5 J^{ij} generates rotations and J^{i0} generates boosts.

In fact, any such state can be recast in the form

$$|\psi\rangle = L_{-1}|\chi_1\rangle + L_{-2}|\chi_2\rangle \tag{2.108}$$

as a consequence of the Virasoro algebra (e.g. $L_{-3} = [L_{-1}, L_{-2}]$). Moreover, any spurious state can be put in this form. Spurious states $|\psi\rangle$ defined this way are orthogonal to every physical state, since

$$\langle\phi|\psi\rangle = \sum_{n=1}^{\infty}\langle\phi|L_{-n}|\chi_n\rangle = \sum_{n=1}^{\infty}\langle\chi_n|L_n|\phi\rangle^\star = 0. \tag{2.109}$$

If a state $|\psi\rangle$ is spurious *and* physical, then it is orthogonal to all physical states including itself

$$\langle\psi|\psi\rangle = \sum_{n=1}^{\infty}\langle\chi_n|L_n|\psi\rangle = 0. \tag{2.110}$$

As a result, such a state has zero norm.

Determination of a

When the constant a is suitably chosen, a class of zero-norm spurious states has the form

$$|\psi\rangle = L_{-1}|\chi_1\rangle \tag{2.111}$$

with

$$(L_0 - a + 1)|\chi_1\rangle = 0 \quad \text{and} \quad L_m|\chi_1\rangle = 0 \quad m > 0. \tag{2.112}$$

Demanding that $|\psi\rangle$ is physical implies

$$L_m|\psi\rangle = (L_0 - a)|\psi\rangle = 0 \qquad \text{for} \qquad m = 1, 2, \ldots \tag{2.113}$$

The Virasoro algebra implies the identity

$$L_1 L_{-1} = 2L_0 + L_{-1}L_1, \tag{2.114}$$

which leads to

$$L_1|\psi\rangle = L_1L_{-1}|\chi_1\rangle = (2L_0 + L_{-1}L_1)|\chi_1\rangle = 2(a-1)|\chi_1\rangle = 0, \tag{2.115}$$

and hence $a = 1$. Thus $a = 1$ is part of the specification of the boundary between positive-norm and negative-norm physical states.

Determination of the space-time dimension

The number of zero-norm spurious states increases dramatically if, in addition to $a = 1$, the space-time dimension is chosen appropriately. To see this, let us construct zero-norm spurious states of the form

$$|\psi\rangle = \left(L_{-2} + \gamma L_{-1}^2\right)|\widetilde{\chi}\rangle. \tag{2.116}$$

This has zero norm for a certain γ, which is determined below. Here $|\psi\rangle$ is spurious if $|\widetilde{\chi}\rangle$ is a state that satisfies

$$(L_0 + 1)|\widetilde{\chi}\rangle = L_m|\widetilde{\chi}\rangle = 0 \qquad \text{for} \qquad m = 1, 2, \ldots \tag{2.117}$$

Now impose the condition that $|\psi\rangle$ is a physical state, that is, $L_1|\psi\rangle = 0$ and $L_2|\psi\rangle = 0$, since the rest of the constraints $L_m|\psi\rangle = 0$ for $m \geq 3$ are then also satisfied as a consequence of the Virasoro algebra. Let us first evaluate the condition $L_1|\psi\rangle = 0$ using the relation

$$\begin{aligned}\left[L_1, L_{-2} + \gamma L_{-1}^2\right] &= 3L_{-1} + 2\gamma L_0 L_{-1} + 2\gamma L_{-1} L_0 \\ &= (3 - 2\gamma)L_{-1} + 4\gamma L_0 L_{-1}.\end{aligned} \tag{2.118}$$

This leads to

$$L_1|\psi\rangle = L_1\left(L_{-2} + \gamma L_{-1}^2\right)|\widetilde{\chi}\rangle = \left[(3 - 2\gamma)\, L_{-1} + 4\gamma L_0 L_{-1}\right]|\widetilde{\chi}\rangle. \tag{2.119}$$

The first term vanishes for $\gamma = 3/2$ while the second one vanishes in general, because

$$L_0 L_{-1}|\widetilde{\chi}\rangle = L_{-1}(L_0 + 1)|\widetilde{\chi}\rangle = 0. \tag{2.120}$$

Therefore, the result of evaluating the $L_1|\psi\rangle = 0$ constraint is $\gamma = 3/2$. Let us next consider the $L_2|\psi\rangle = 0$ condition. Using

$$\left[L_2, L_{-2} + \frac{3}{2}L_{-1}^2\right] = 13L_0 + 9L_{-1}L_1 + \frac{D}{2} \tag{2.121}$$

gives

$$L_2|\psi\rangle = L_2\left(L_{-2} + \frac{3}{2}L_{-1}^2\right)|\widetilde{\chi}\rangle = \left(-13 + \frac{D}{2}\right)|\widetilde{\chi}\rangle. \tag{2.122}$$

Thus the space-time dimension $D = 26$ gives additional zero-norm spurious states.

Critical bosonic theory

The zero-norm spurious states are unphysical. The fact that they are spurious ensures that they decouple from all physical processes. In fact, all negative-norm states decouple, and all physical states have positive norm. Thus, the complete physical spectrum is free of negative-norm states when the two conditions $a = 1$ and $D = 26$ are satisfied, as is proved in the next section. The $a = 1$, $D = 26$ bosonic string theory is called *critical*, and one says that the *critical dimension* is 26. The spectrum is also free of negative-norm states for $a \leq 1$ and $D \leq 25$. In these cases the theory is called *noncritical*. Noncritical string theory is discussed briefly in the next chapter.

Exercises

Exercise **2.10**

Find the mode expansion for angular-momentum generators $J^{\mu\nu}$ of an open bosonic string.

Solution

Using the current in Eq. (2.68),

$$J^{\mu\nu} = \int_0^\pi J_0^{\mu\nu} d\sigma = T \int_0^\pi (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu) d\sigma.$$

Now

$$X^\mu(\tau, \sigma) = x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m \neq 0} \frac{1}{m} \alpha_m^\mu e^{-im\tau} \cos(m\sigma),$$

$$\dot{X}^\mu(\tau, \sigma) = l_s^2 p^\mu + l_s \sum_{m \neq 0} \alpha_m^\mu e^{-im\tau} \cos(m\sigma),$$

and $T = 1/(\pi l_s^2)$. A short calculation gives

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i \sum_{m=1}^\infty \frac{1}{m} \Big(\alpha_{-m}^\mu \alpha_m^\nu - \alpha_{-m}^\nu \alpha_m^\mu \Big).$$

□

2.5 Light-cone gauge quantization

As discussed earlier, the bosonic string has residual diffeomorphism symmetries, even after choosing the gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$, which consist of all the conformal transformations. Therefore, there is still the possibility of making an additional gauge choice. By making a particular noncovariant gauge choice, it is possible to describe a Fock space that is manifestly free of negative-norm states and to solve explicitly all the Virasoro conditions instead of imposing them as constraints.

Let us introduce light-cone coordinates for space-time[6]

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}). \tag{2.123}$$

Then the D space-time coordinates X^μ consist of the null coordinates $X^{\pm}$ and the $D-2$ transverse coordinates X^i. In this notation, the inner product of two arbitrary vectors takes the form

$$v \cdot w = v_\mu w^\mu = -v^+ w^- - v^- w^+ + \sum_i v^i w^i. \tag{2.124}$$

Indices are raised and lowered by the rules

$$v^- = -v_+, \qquad v^+ = -v_-, \qquad \text{and} \qquad v^i = v_i. \tag{2.125}$$

Since two coordinates are treated differently from the others, Lorentz invariance is no longer manifest when light-cone coordinates are used.

What simplification can be achieved by using the residual gauge symmetry? In terms of $\sigma^{\pm}$ the residual symmetry corresponds to the reparametrizations in Eq. (2.86) of each of the null world-sheet coordinates

$$\sigma^{\pm} \to \xi^{\pm}(\sigma^{\pm}). \tag{2.126}$$

These transformations correspond to

$$\widetilde{\tau} = \frac{1}{2}\left[\xi^+(\sigma^+) + \xi^-(\sigma^-)\right], \tag{2.127}$$

$$\widetilde{\sigma} = \frac{1}{2}\left[\xi^+(\sigma^+) - \xi^-(\sigma^-)\right]. \tag{2.128}$$

This means that $\widetilde{\tau}$ can be an arbitrary solution to the free massless wave equation

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right)\widetilde{\tau} = 0. \tag{2.129}$$

6 It is convenient to include the $\sqrt{2}$ factor in the definition of space-time light-cone coordinates while omitting it in the definition of world-sheet light-cone coordinates.

Once $\widetilde{\tau}$ is determined, $\widetilde{\sigma}$ is specified up to a constant.

In the gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$, the space-time coordinates $X^\mu(\sigma, \tau)$ also satisfy the two-dimensional wave equation. The light-cone gauge uses the residual freedom described above to make the choice

$$X^+(\widetilde{\sigma}, \widetilde{\tau}) = x^+ + l_{\mathrm{s}}^2 p^+ \widetilde{\tau}. \tag{2.130}$$

This corresponds to setting

$$\alpha_n^+ = 0 \qquad \text{for} \qquad n \neq 0. \tag{2.131}$$

In the following the tildes are omitted from the parameters $\widetilde{\tau}$ and $\widetilde{\sigma}$.

When this noncovariant gauge choice is made, there is a risk that a quantum-mechanical anomaly could lead to a breakdown of Lorentz invariance. So this needs to be checked. In fact, conformal invariance is essential for making this gauge choice, so it should not be surprising that a *Lorentz anomaly* in the light-cone gauge approach corresponds to a *conformal anomaly* in a covariant gauge that preserves manifest Lorentz invariance.

The light-cone gauge has eliminated the oscillator modes of X^+. It is possible to determine the oscillator modes of X^-, as well, by solving the Virasoro constraints $(\dot{X} \pm X')^2 = 0$. In the light-cone gauge these constraints become

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{2p^+ l_{\mathrm{s}}^2} (\dot{X}^i \pm X^{i\prime})^2. \tag{2.132}$$

This pair of equations can be used to solve for X^- in terms of X^i. In terms of the mode expansion for X^-, which for an open string is

$$X^- = x^- + l_{\mathrm{s}}^2 p^- \tau + i l_{\mathrm{s}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^- e^{-in\tau} \cos n\sigma, \tag{2.133}$$

the solution is

$$\alpha_n^- = \frac{1}{p^+ l_{\mathrm{s}}} \left(\frac{1}{2} \sum_{i=1}^{D-2} \sum_{m=-\infty}^{+\infty} :\alpha_{n-m}^i \alpha_m^i: - a\delta_{n,0} \right). \tag{2.134}$$

Therefore, in the light-cone gauge it is possible to eliminate both X^+ and X^- (except for their zero modes) and express the theory in terms of the transverse oscillators. Thus a critical string only has transverse excitations, just as a massless particle only has transverse polarization states. The convenient feature of the light-cone gauge in Eq. (2.130) is that it turns the Virasoro constraints into linear equations for the modes of X^-.

Mass-shell condition

In the light-cone gauge the open-string mass-shell condition is

$$M^2 = -p_\mu p^\mu = 2p^+p^- - \sum_{i=1}^{D-2} p_i^2 = 2(N-a)/l_{\rm s}^2, \tag{2.135}$$

where

$$N = \sum_{i=1}^{D-2}\sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i. \tag{2.136}$$

Let us now construct the physical spectrum of the bosonic string in the light-cone gauge.

In the light-cone gauge all the excitations are generated by acting with the transverse modes α_n^i. The first excited state, given by $\alpha_{-1}^i|0;p\rangle$, belongs to a $(D-2)$-component vector representation of the rotation group $SO(D-2)$ in the transverse space. As a general rule, Lorentz invariance implies that physical states form representations of $SO(D-1)$ for massive states and $SO(D-2)$ for massless states. Therefore, the bosonic string theory in the light-cone gauge can only be Lorentz invariant if the vector state $\alpha_{-1}^i|0;p\rangle$ is massless. This immediately implies that $a=1$.

Having fixed the value of a, the next goal is to determine the space-time dimension D. A heuristic approach is to compute the normal-ordering constant appearing in the definition of L_0 directly. This constant can be determined from

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{n=-\infty}^{+\infty} \alpha_{-n}^i\alpha_n^i = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n=-\infty}^{+\infty} :\alpha_{-n}^i\alpha_n^i: + \frac{1}{2}(D-2)\sum_{n=1}^{\infty} n. \tag{2.137}$$

The second sum on the right-hand side is divergent and needs to be regularized. This can be achieved using ζ-function regularization. First, one considers the general sum

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \tag{2.138}$$

which is defined for any complex number s. For $\mathrm{Re}(s) > 1$, this sum converges to the Riemann zeta function $\zeta(s)$. This zeta function has a unique analytic continuation to $s=-1$, where it takes the value $\zeta(-1) = -1/12$. Therefore, after inserting the value of $\zeta(-1)$ in Eq. (2.137), the result for the additional term is

$$\frac{1}{2}(D-2)\sum_{n=1}^{\infty} n = -\frac{D-2}{24}. \tag{2.139}$$

Using the earlier result that the normal-ordering constant a should be equal to 1, one gets the condition

$$\frac{D-2}{24} = 1, \tag{2.140}$$

which implies $D = 26$. Though it is not very rigorous, this is the quickest way to determined the values of a and D. The earlier analysis of the no-negative-norm states theorem also singled out $D = 26$. Another approach is to verify that the Lorentz generators satisfy the Lorentz algebra, which is not manifest in the light-cone gauge. The nontrivial requirement is

$$[J^{i-}, J^{j-}] = 0. \tag{2.141}$$

Once the α_n^- oscillators are eliminated, J^{i-} becomes cubic in transverse oscillators. The algebra is rather complicated, but the bottom line is that the commutator only vanishes for $a = 1$ and $D = 26$. Other derivations of the critical dimension are presented in the next chapter.

Analysis of the spectrum

Having determined the preferred values $a = 1$ and $D = 26$, one can now determine the spectrum of the bosonic string.

The open string

At the first few mass levels the physical states of the open string are as follows:

- For $N = 0$ there is a tachyon $|0; k\rangle$, whose mass is given by $\alpha' M^2 = -1$.
- For $N = 1$ there is a vector boson $\alpha_{-1}^i |0; k\rangle$. As was explained in the previous section, Lorentz invariance requires that it is massless. This state gives a vector representation of $SO(24)$.
- $N = 2$ gives the first states with positive (mass)^2. They are

  $$\alpha_{-2}^i |0; k\rangle \qquad \text{and} \qquad \alpha_{-1}^i \alpha_{-1}^j |0; k\rangle, \tag{2.142}$$

  with $\alpha' M^2 = 1$. These have 24 and $24 \cdot 25/2$ states, respectively. The total number of states is 324, which is the dimensionality of the symmetric traceless second-rank tensor representation of $SO(25)$, since $25 \cdot 26/2 - 1 = 324$. So, in this sense, the spectrum consists of a single massive spin-two state at this mass level.

All of these states have a positive norm, since they are built entirely from the transverse modes, which describe a positive-definite Hilbert space. In the light-cone gauge the fact that the negative-norm states have decoupled is made manifest. All of the massive representations can be rearranged in complete $SO(25)$ multiplets, as was just demonstrated for the first massive level. Lorentz invariance of the spectrum is guaranteed, because the Lorentz algebra is realized on the Hilbert space of transverse oscillators.

The number of states

The total number of physical states of a given mass is easily computed. For example, in the case of open strings, it follows from Eqs (2.135) and (2.136) with $a = 1$ that the number of physical states d_n whose mass is given by $\alpha' M^2 = n - 1$ is the coefficient of w^n in the power-series expansion of

$$\operatorname{tr} w^N = \prod_{n=1}^{\infty} \prod_{i=1}^{24} \operatorname{tr} w^{\alpha_{-n}^i \alpha_n^i} = \prod_{n=1}^{\infty} (1 - w^n)^{-24}. \tag{2.143}$$

This number can be written in the form

$$d_n = \frac{1}{2\pi i} \oint \frac{\operatorname{tr} w^N}{w^{n+1}} dw. \tag{2.144}$$

The number of physical states d_n can be estimated for large n by a saddle-point evaluation. Since the saddle point occurs close to $w = 1$, one can use the approximation

$$\operatorname{tr} w^N = \prod_{n=1}^{\infty} (1 - w^n)^{-24} \sim \exp\left(\frac{4\pi^2}{1 - w}\right). \tag{2.145}$$

This is an approximation to the modular transformation formula

$$\eta(-1/\tau) = (-i\tau)^{1/2} \eta(\tau) \tag{2.146}$$

for the Dedekind eta function

$$\eta(\tau) = e^{i\pi\tau/12} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n \tau}\right), \tag{2.147}$$

as one sees by setting $w = e^{2\pi i \tau}$. Then one finds that, for large n,

$$d_n \sim \text{const.}\, n^{-27/4} \exp(4\pi\sqrt{n}). \tag{2.148}$$

The exponential factor can be rewritten in the form $\exp(M/M_0)$ with

$$M_0 = (4\pi\sqrt{\alpha'})^{-1}. \tag{2.149}$$

The quantity M_0 is called the *Hagedorn temperature*. Depending on details that go beyond present considerations, it is either a maximum possible temperature or else the temperature of a phase transition.

The closed string

For the case of the closed string, there are two sets of modes (left-movers and right-movers), and the level-matching condition must be taken into account. The spectrum is easily deduced from that of the open string, since closed-string states are tensor products of left-movers and right-movers, each of which has the same structure as open-string states. The mass of states in the closed-string spectrum is given by

$$\alpha' M^2 = 4(N-1) = 4(\widetilde{N}-1). \tag{2.150}$$

The physical states of the closed string at the first two mass levels are as follows:

- The ground state $|0;k\rangle$ is again a tachyon, this time with

$$\alpha' M^2 = -4. \tag{2.151}$$

- For the $N=1$ level there is a set of $24^2 = 576$ states of the form

$$|\Omega^{ij}\rangle = \alpha_{-1}^i \widetilde{\alpha}_{-1}^j |0;k\rangle, \tag{2.152}$$

  corresponding to the tensor product of two massless vectors, one left-moving and one right-moving. The part of $|\Omega^{ij}\rangle$ that is symmetric and traceless in i and j transforms under $SO(24)$ as a massless spin-two particle, the *graviton*. The trace term $\delta_{ij}|\Omega^{ij}\rangle$ is a massless scalar, which is called the *dilaton*. The antisymmetric part $|\Omega^{ij}\rangle - |\Omega^{ji}\rangle$ transforms under $SO(24)$ as an antisymmetric second-rank tensor. Each of these three massless states has a counterpart in superstring theories, where they play fundamental roles that are discussed in later chapters.

Homework Problems

Problem **2.1**

Consider the following classical trajectory of an open string

$$\begin{aligned} X^0 &= B\tau, \\ X^1 &= B\ \cos(\tau)\cos(\sigma), \\ X^2 &= B\ \sin(\tau)\cos(\sigma), \\ X^i &= 0, \qquad i > 2, \end{aligned}$$

and assume the conformal gauge condition.

(i) Show that this configuration describes a solution to the equations of motion for the field X^μ corresponding to an open string with Neumann boundary conditions. Show that the ends of this string are moving with the speed of light.

(ii) Compute the energy $E = P^0$ and angular momentum J of the string. Use your result to show that

$$\frac{E^2}{|J|} = 2\pi T = \frac{1}{\alpha'}.$$

(iii) Show that the constraint equation $T_{\alpha\beta} = 0$ can be written as

$$(\partial_\tau X)^2 + (\partial_\sigma X)^2 = 0, \qquad \partial_\tau X^\mu \partial_\sigma X_\mu = 0,$$

and that this constraint is satisfied by the above solution.

PROBLEM **2.2**

Consider the following classical trajectory of an open string

$$\begin{aligned} X^0 &= 3A\tau, \\ X^1 &= A\,\cos(3\tau)\cos(3\sigma), \\ X^2 &= A\,\sin(a\tau)\cos(b\sigma), \end{aligned}$$

and assume the conformal gauge.

(i) Determine the values of a and b so that the above equations describe an open string that solves the constraint $T_{\alpha\beta} = 0$. Express the solution in the form

$$X^\mu = X_L^\mu(\sigma^-) + X_R^\mu(\sigma^+).$$

Determine the boundary conditions satisfied by this field configuration.

(ii) Plot the solution in the (X^1, X^2)-plane as a function of τ in steps of $\pi/12$.

(iii) Compute the center-of-mass momentum and angular momentum and show that they are conserved. What do you obtain for the relation between the energy and angular momentum of this string? Comment on your result.

PROBLEM **2.3**

Compute the mode expansion of an open string with Neumann boundary conditions for the coordinates $X^0, \ldots, X^{24}$, while the remaining coordinate X^{25} satisfies the following boundary conditions:

(i) Dirichlet boundary conditions at both ends

$$X^{25}(0,\tau) = X_0^{25} \quad \text{and} \quad X^{25}(\pi,\tau) = X_\pi^{25}.$$

What is the interpretation of such a solution? Compute the conjugate momentum P^{25}. Is this momentum conserved?

(ii) Dirichlet boundary conditions on one end and Neumann boundary conditions at the other end

$$X^{25}(0,\tau) = X_0^{25} \quad \text{and} \quad \partial_\sigma X^{25}(\pi,\tau) = 0.$$

What is the interpretation of this solution?

PROBLEM **2.4**

Consider the bosonic string in light-cone gauge.

(i) Find the mass squared of the following on-shell open-string states:

$$\begin{aligned} |\phi_1\rangle &= \alpha_{-1}^i|0;k\rangle, & |\phi_2\rangle &= \alpha_{-1}^i\alpha_{-1}^j|0;k\rangle, \\ |\phi_3\rangle &= \alpha_{-3}^i|0;k\rangle, & |\phi_4\rangle &= \alpha_{-1}^i\alpha_{-1}^j\alpha_{-2}^k|0;k\rangle. \end{aligned}$$

(ii) Find the mass squared of the following on-shell closed-string states:

$$|\phi_1\rangle = \alpha_{-1}^i\tilde{\alpha}_{-1}^j|0;k\rangle, \qquad |\phi_2\rangle = \alpha_{-1}^i\alpha_{-1}^j\tilde{\alpha}_{-2}^k|0;k\rangle.$$

(iii) What can you say about the following closed-string state?

$$|\phi_3\rangle = \alpha_{-1}^i\tilde{\alpha}_{-2}^j|0;k\rangle$$

PROBLEM **2.5**

Use the mode expansion of an open string with Neumann boundary conditions in Eq. (2.62) and the commutation relations for the modes in Eq. (2.54) to check explicitly the equal-time commutators

$$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = [P^\mu(\sigma,\tau), P^\nu(\sigma',\tau)] = 0,$$

while

$$[X^\mu(\sigma,\tau), P^\nu(\sigma',\tau)] = i\eta^{\mu\nu}\delta(\sigma-\sigma').$$

Hint: The representation $\delta(\sigma-\sigma') = \frac{1}{\pi}\sum_{n\in\mathbb{Z}}\cos(n\sigma)\cos(n\sigma')$ might be useful.

PROBLEM **2.6**

Exercise 2.10 showed that the Lorentz generators of the open-string world sheet are given by

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu \right).$$

Use the canonical commutation relations to verify the Poincaré algebra

$$[p^\mu, p^\nu] = 0,$$

$$[p^\mu, J^{\nu\sigma}] = -i\eta^{\mu\nu} p^\sigma + i\eta^{\mu\sigma} p^\nu,$$

$$[J^{\mu\nu}, J^{\sigma\lambda}] = -i\eta^{\nu\sigma} J^{\mu\lambda} + i\eta^{\mu\sigma} J^{\nu\lambda} + i\eta^{\nu\lambda} J^{\mu\sigma} - i\eta^{\mu\lambda} J^{\nu\sigma}.$$

PROBLEM **2.7**

Exercise 2.10 derived the angular-momentum generators $J^{\mu\nu}$ for an open bosonic string. Derive them for a closed bosonic string.

PROBLEM **2.8**

The open-string angular momentum generators in Exercise 2.10 are appropriate for covariant quantization. What are the formulas in the case of light-cone gauge quantization.

PROBLEM **2.9**

Show that the Lorentz generators commute with all Virasoro generators,

$$[L_m, J^{\mu\nu}] = 0.$$

Explain why this implies that the physical-state condition is invariant under Lorentz transformations, and states of the string spectrum appear in complete Lorentz multiplets.

PROBLEM **2.10**

Consider an on-shell open-string state of the form

$$|\phi\rangle = \left(A\alpha_{-1} \cdot \alpha_{-1} + B\alpha_0 \cdot \alpha_{-2} + C(\alpha_0 \cdot \alpha_{-1})^2 \right) |0; k\rangle,$$

where A, B and C are constants. Determine the conditions on the coefficients A, B and C so that $|\phi\rangle$ satisfies the physical-state conditions for $a = 1$ and arbitrary D. Compute the norm of $|\phi\rangle$. What conclusions can you draw from the result?

PROBLEM **2.11**

The open-string states at the $N = 2$ level were shown in Section 2.5 to form a certain representation of $SO(25)$. What does this result imply for the spectrum of the closed bosonic string at the $N_{\mathrm{L}} = N_{\mathrm{R}} = 2$ level?

PROBLEM **2.12**

Construct the spectrum of open and closed strings in light-cone gauge for level $N = 3$. How many states are there in each case? Without actually doing it (unless you want to), describe a strategy for assembling these states into irreducible $SO(25)$ multiplets.

PROBLEM **2.13**

We expect the central extension of the Virasoro algebra to be of the form

$$[L_m, L_n] = (m - n)L_{m+n} + A(m)\delta_{m+n,0},$$

because normal-ordering ambiguities only arise for $m + n = 0$.

(i) Show that if $A(1) \neq 0$ it is possible to change the definition of L_0, by adding a constant, so that $A(1) = 0$.

(ii) For $A(1) = 0$ show that the generators L_0 and $L_{\pm 1}$ form a closed subalgebra.

PROBLEM **2.14**

Derive an equation for the coefficients $A(m)$ defined in the previous problem that follows from the Jacobi identity

$$[[L_m, L_n], L_p] + [[L_p, L_m], L_n] + [[L_n, L_p], L_m] = 0.$$

Assuming $A(1) = 0$, prove that $A(m) = (m^3 - m)A(2)/6$ is the unique solution, and hence that the central charge is $c = 2A(2)$.

PROBLEM **2.15**

Verify that the Virasoro generators in Eq. (2.91) satisfy the Virasoro algebra. It is difficult to verify the central-charge term directly from the commutator. Therefore, a good strategy is to verify that $A(1)$ and $A(2)$ have the correct values. These can be determined by computing the ground-state matrix element of Eq. (2.93) for the cases $m = -n = 1$ and $m = -n = 2$.

3

Conformal field theory and string interactions

The previous chapter described the free bosonic string in Minkowski space-time. It was argued that consistency requires the dimension of space-time to be $D = 26$ (25 space and one time). Even then, there is a tachyon problem. When interactions are included, this theory might not have a stable vacuum. The justification for studying the bosonic string theory, despite its deficiencies, is that it is a good warm-up exercise before tackling more interesting theories that do have stable vacua. This chapter continues the study of the bosonic string theory, covering a lot of ground rather concisely.

One important issue concerns the possibilities for introducing more general backgrounds than flat 26-dimensional Minkowski space-time. Another concerns the development of techniques for describing interactions and computing scattering amplitudes in perturbation theory. We also discuss a quantum field theory of strings. In this approach field operators create and destroy entire strings. All of these topics exploit the conformal symmetry of the world-sheet theory, using the techniques of conformal field theory (CFT). Therefore, this chapter begins with an overview of that subject.

3.1 Conformal field theory

Until now it has been assumed that the string world sheet has a Lorentzian signature metric, since this choice is appropriate for a physically evolving string. However, it is extremely convenient to make a Wick rotation $\tau \to -i\tau$, so as to obtain a world sheet with Euclidean signature, and thereby make the world-sheet metric $h_{\alpha\beta}$ positive definite. Having done this, one can introduce complex coordinates (in local patches)

$$z = e^{2(\tau - i\sigma)} \quad \text{and} \quad \bar{z} = e^{2(\tau + i\sigma)} \tag{3.1}$$