Gravitational Radiation Luminous Black Holes and Gamma-Ray Burst Supernovae

Maurice H.P.M. van Putten

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Black holes and gravitational radiation are two of the most dramatic predictions of general relativity. The quest for rotating black holes – discovered by Roy P. Kerr as exact solutions to the Einstein equations – is one of the most exciting challenges currently facing physicists and astronomers.

Gravitational Radiation, Luminous Black Holes and Gamma-ray Burst Supernovae takes the reader through the theory of gravitational radiation and rotating black holes, and the phenomenology of GRB supernovae. Topics covered include Kerr black holes and the frame-dragging of spacetime, luminous black holes, compact tori around black holes, and black hole–spin interactions. It concludes with a discussion of prospects for gravitational-wave detections of a long-duration burst in gravitational waves as a method of choice for identifying Kerr black holes in the universe.

This book is ideal for a special topics graduate course on gravitational-wave astronomy and as an introduction to those interested in this contemporary development in physics.

Maurice H. P. M. van Putten studied at Delft University of Technology, The Netherlands and received his Ph.D. from the California Institute of Technology. He has held postdoctoral positions at the Institute of Theoretical Physics at the University of California at Santa Barbara, and the Center for Radiophysics and Space Research at Cornell University. He then joined the faculty of the Massachusetts Institute of Technology and became a member of the new Laser Interferometric Gravitational-wave Observatory (MIT-LIGO), where he teaches a special-topic graduate course based on his research.

Professor van Putten's research in theoretical astrophysics has spanned a broad range of topics in relativistic magnetohydrodynamics, hyperbolic formulations of general relativity, and radiation processes around rotating black holes. He has led global collaborations on the theory of gamma-ray burst supernovae from rotating black holes as burst sources of gravitational radiation. His theory describes a unique link between gravitational waves and Kerr black holes, two of the most dramatic predictions of general relativity. Discovery of *triplets* – gamma-ray burst supernovae accompanied by a long-duration gravitational-wave burst – provides a method for calorimetric identification of Kerr black holes in the universe.

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To my parents Anton and Maria,
and
Michael, Pascal, and Antoinette

Contents

Foreword

General relativity is one of the most elegant and fundamental theories of physics, describing the gravitational force with a most awesome precision. When it was first discovered, by Einstein in 1915, the theory appeared to do little more than provide for minute corrections to the older formalism: Newton's law of gravity. Today, more and more stellar systems are discovered, in the far outreaches of the universe, where extreme conditions are suspected to exist that lead to incredibly strong gravitational forces, and where relativistic effects are no longer a tiny perturbation, but they dominate, yielding totally new phenomena. One of these phenomena is gravitational radiation – gravity then acts in a way very similar to what happens with electric and magnetic fields when they oscillate: they form waves that transmit information and energy.

Only the most violent sources emit gravitational waves that can perhaps be detected from the Earth, and this makes investigating such sources interesting. The physics and mathematics of these sources is highly complex.

Maurice van Putten has great expertise in setting up the required physical models and in solving the complicated equations emerging from them. This book explains his methods in dealing with these equations. Not much time is wasted on philosophical questions or fundamental motivations or justifications. The really relevant physical questions are confronted with direct attacks. Of course, we encounter all sorts of difficulties on our way. Here, we ask for practical ways out, rather than indulging on formalities. Different fields of physics are seen to merge: relativity, quantum mechanics, plasma physics, elementary particle physics, numerical analysis and, of course, astrophysics. A book for those who want to get their hands dirty.

Gerard 't Hooft

Acknowledgments

Epigraph to Chapter 11, reprinted with permission from Oxford University Press from *The Mathematical Theory of Black Holes*, by S. Chandrasekhar (1983).

Epigraph to Chapter 16, reprinted with permission from *Gravitation and Cosmology*, by Stephen Weinberg.

Introduction

Observations of gravitational radiation from black holes and neutron stars promise to dramatically transform our view of the universe. This new topic of gravitational-wave astronomy will be initiated with detections by recently commissioned gravitational-wave detectors. These are notably the Laser Interferometric Gravitational wave Observatory LIGO (US), Virgo (Europe), TAMA (Japan) and GEO (Germany), and various bar detectors in the US and Europe.

This book is intended for graduate students and postdoctoral researchers who are interested in this emerging opportunity. The audience is expected to be familiar with electromagnetism, thermodynamics, classical and quantum mechanics. Given the rapid development in gravitational wave experiments and our understanding of sources of gravitational waves, it is recommended that this book is used in combination with current review articles.

This book developed as a graduate text on general relativity and gravitational radiation in a one-semester special topics graduate course at MIT. It started with an invitation of Gerald E. Brown for a *Physics Reports* on gamma-ray bursts. *Why study gamma-ray bursters?* Because they are there, representing the most energetic and relativistic transients in the sky? Or perhaps because they hold further promise as burst sources of gravitational radiation?

Our focus is on gravitational radiation powered with rotating black holes – the two most fundamental predictions of general relativity for astronomy (other than cosmology). General relativity is a classical field theory, and we believe it applies to all macroscopic bodies. We do not know whether general relativity is valid down to the Planck scale without modifications at intermediate scales, without any extra dimensions or additional internal symmetries.

Observations of neutron star binaries PSR 1913 + 16 and, more recently, PSR 0737-3039, tell us that gravitational waves exist and carry energy. This discovery is a considerable advance beyond the earlier phenomenology of quasi-static spacetimes in general relativity, such as the deflection of light by the Sun and the orbital precession of Mercury.

Observational evidence of black holes is presently limited to compact stellar mass objects as black hole candidates in soft X-ray transients and their supermassive counterparts at centers of galaxies. Particularly striking is the discovery of compact stellar trajectories in SgrA* in our own galaxy, which reveals a supermassive black hole of a few million solar masses.

Rotating black holes are believed to nucleate in core collapse of massive stars. The exact solution of rotating black holes was discovered by Roy P. Kerr[293]. It shows frame-dragging to be the explicit manifestation of curvature induced by angular momentum. It further predicts a large energy reservoir in rotation in the black hole: its energy content may exceed that in a rapidly rotating neutron star by at least an order of magnitude. While in isolation stellar black holes are stable and essentially nonradiating, in interaction with their environment black holes can become luminous upon emitting angular momentum in various radiation channels.

Essential to the interaction of Kerr black holes with the environment is the Rayleigh criterion. Rotating black holes tend to lower their energy by radiating high specific angular momentum to infinity. In isolation, these radiative processes are suppressed by canonical angular momentum barriers, rendering macroscopic black holes stable. Penrose recognized that, in principle, the rotational energy of a black hole can be liberated by splitting surrounding matter into high and low angular momentum particles[416, 417]. Absorption of low-angular momentum and ejection of high-angular momentum with positive energy to infinity is consistent with the Rayleigh criterion and conservation of mass and angular momentum. These processes are restricted to the so-called *ergosphere*. Black hole spin-induced curvature and curvature coupling to spin combined further give rise to spin–orbit coupling – an effective interaction of black hole spin with angular momentum in an *ergotube* along the axis of rotation. Calorimetry on the ensuing radiation energies promises first-principle evidence for Kerr black holes and, consequently, evidence for general relativity in the nonlinear regime.

While currently observed neutron star binary systems provide us with laboratories to study linearized general relativity, could gamma ray burst supernovae serve a similar role for fully nonlinear general relativity?

Cosmological gamma-ray bursts were accidentally discovered by Vela and Konus satellites in the late 1960s. Their association with supernovae, in its earliest form proposed by Stirling Colgate, has been confirmed by GRB 980425/SN1988bw[224, 536] and GRB030329/SN2003dh[506, 265]. Thus, Type Ib/c supernovae are probably the parent population of long GRBs. It has been appreciated that the observed GRB afterglow emissions represent the dissipation of ultrarelativistic baryon-poor outflows[451, 452], while the associated supernova is strongly aspherical[268] and bright in X-ray line-emissions[17, 432, 613, 434, 454]. These observations further show the time of onset of the gamma-ray burst and the supernova to be the same within observational uncertainties.

This phenomenology reveals a baryon-poor active nucleus as the powerhouse of GRB supernovae in core collapse in massive stars. The only known baryon-free energy source is a rotating black hole. This presents an *energy paradox*: *the rotational energy of a rapidly rotating black hole is orders of magnitude larger than the energy requirements set by the observed radiation energies in GRB supernovae*. A rapidly rotating nucleus formed in core-collapse is relativistically compact and radiative primarily in “unseen” gravitational radiation and MeV-neutrino emissions. These channels provide a new opportunity for probing the inner engine of cosmological GRB supernovae.

The promise of a link between gravitational radiation and black holes in GRB supernovae provides a method for the gravitational wave-dectors LIGO and Virgo to provide first-principle evidence for Kerr black holes in association with a currently known observational phenomenon.

This book consists of three parts: gravitational radiation, waves in astrophysical fluids, and a theory of GRB supernovae from rotating black holes. Chapters 1–7 introduce general relativity and gravitational radiation. Chapters 8–10 discuss fluid dynamical waves in jets and tori around black holes. Gamma-ray burst supernovae are introduced in Chapter 11. A theory of gravitational waves created by GRB supernovae from rotating black holes is discussed in Chapters 12–15. Chapter 16 discusses GRB supernovae as observational opportunities for gravitational wave experiments LIGO and Virgo.

The author is greatly indebted to his collaborators and many colleagues for constructive discussions over many years, which made possible this venture into gravitational-wave astronomy: Amir Levinson, Eve C. Ostriker, Gerald E. Brown, Roy P. Kerr, Garry Tee, Gerard ’t Hooft, H. Cheng, S.-T. Yau, Félix Mirabel, Dale A. Frail, Kevin Hurley, Douglas M. Eardley, John Heise, Stirling Colgate, Andy Fabian, Alain Brillet, Rainer Weiss, David Shoemaker, Barry Barish, Kip S. Thorne, Roger D. Blandford, Robert V. Wagoner, E. Sterl Phinney, Jacob Bekenstein, Gary Gibbons, Shrinivastas Kulkarni, Giora Shaviv, Tsvi Piran, Gennadii S. Bisnovatyi-Kogan, Ramesh Narayan, Bohdan Paczyński, Peter Mészáros, Saul Teukolsky, Stuart Shapiro, Edward E. Salpeter, Ira Wasserman, David Chernoff, Yvonne Choquet-Bruhat, Tim de Zeeuw, John F. Hawley, David Coward, Ron Burman, David Blair, Sungeun Kim, Hyun Kyu Lee, Tania Regimbau, Gregory M. Harry, Michele Punturo, Linqing Wen, Stephen Eikenberry, Mark Abramowicz, Michael L. Norman, Valeri Frolov, Donald

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The reader is referred to other texts for more general discussions on stellar structure, compact objects and general relativity, notably: *Gravitation* by C. W. Misner, K. S. Thorne and J. A. Wheeler[382], *Gravitation and Cosmology* by S. Weinberg[587], *The Membrane Paradigm* by K. S. Thorne, R. H. Price & A. MacDonald[534], *Stellar Structure and Evolution* by R. Kippenhahn and A. Weigert[295], *Introduction to General Relativity* by G. 't Hooft[527], *General Relativity* by R. M. Wald[577], *General Relativity* by H. Stephani[509], *Gravitation and Spacetime* by H. C. Ohanian and R. Ruffini[398], *A First Course in General Relativity* by Bernard F. Schutz[485], *Black Holes, White Dwarfs and Neutron Stars* by S. L. Shapiro and S. A. Teukolsky[490], *Black Hole Physics* by V. Frolov and I. D. Novikov[208], *Formation and Evolution of Black holes in the Galaxy* by H. A. Bethe, G. E. Brown and C.-H. Lee[53], and *Analysis, Manifolds and Physics* by Y. Choquet-Bruhat, C. DeWitt-Morette and M. Dillard-Bleick[120].

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Notation

The metric signature $(-, +, +, +)$ is in conformance with Misner, Thorne and Wheeler 1974[382]. The Minkowski metric is given by $\eta_{ab} = \lceil -1, 1, 1, 1 \rfloor$.

Most of the expressions are in geometrical units, except where indicated. In the case of pair creation by black holes (Appendix D), we use mixed geometrical–natural units.

Tensors are written in the so-called abstract index notation in Latin script. Indices from the middle of the alphabet denote spatial coordinates. Four-vectors and p-forms are also indicated in small boldface. Three-vectors are indicated in capital boldface.

The epsilon tensor $\epsilon_{abcd} = \Delta_{abcd}\sqrt{-g}$ is defined in terms of the totally antisymmetric symbol Δ_{abcd} and the determinant g of the metric, where $\Delta_{0123} = 1$ which changes sign under odd permutations.

Tetrad elements are indexed by $\{(e_\mu)^b\}_{\mu=1}^4$, where μ denotes the tetrad index and b denotes the coordinate index.

1

Superluminal motion in the quasar 3C273

"The cowboys have a way of trussing up a steer or a pugnacious bronco which fixes the brute so that it can neither move nor think. This is the hog-tie, and it is what Euclid did to geometry."
Eric Temple Bell (1883–1960), *The Search For Truth* (1934).

General relativity endows spacetime with a causal structure described by observer-invariant *light cones*. This locally incorporates the theory of special relativity: the velocity of light is the same for all observers. Points *inside* a light cone are causally connected with its vertex, while points *outside* the same light cone are out-of-causal contact with its vertex. Light describes null-generators *on* the light cone. This simple structure suffices to capture the kinematic features of special relativity. We illustrate these ideas by looking at relativistic motion in the nearby quasar 3C273.

1.1 Lorentz transformations

Maxwell's equations describe the propagation of light in the form of electromagnetic waves. These equations are linear. The Michelson–Morley experiment[372] shows that the velocity of light is constant, independent of the state of the observer. Lorentz derived the commensurate linear transformation on the coordinates, which leaves Maxwell equations form-invariant. It will be appreciated that form invariance of Maxwell's equations implies invariance of the velocity of electromagnetic waves. This transformation was subsequently rederived by Einstein, based on the stipulation that the velocity of light is the same for any observer. It is non-Newtonian, in that it simultaneously transforms all four spacetime coordinates.

The results can be expressed geometrically, by introducing the notion of light cones. Suppose we have a beacon that produces a single flash of light in all directions. This flash creates an expanding shell. We can picture this in a spacetime diagram by plotting the cross-section of this shell with the x-axis as a function of time – two diagonal and straight lines in an inertial setting (neglecting gravitational effects or accelerations). The interior of the light cone corresponds to points interior to the shell. These points can be associated with the centre of the shell by particles moving slower than the speed of light. The interior of the light cone is hereby causally connected to its vertex. The exterior of the shell is out-of-causal contact with the vertex of the light cone. This causal structure is local to the vertex of each light cone, illustrated in Figure (1.1).

Light-cones give a geometrical description of causal structure which is observer-invariant by invariance of the velocity of light, commonly referred to as "covariance". Covariance of a light cone gives rise to a linear transformation of the spacetime coordinates of two observers, one with a coordinate frame $K(t, x)$ and the other with a coordinate frame $K'(t', x')$. We may insist on coincidence of K and K' at $t = t' = 0$, and use geometrical units in which $c = 1$, whereby

$$\mathrm{sign}(x^2 - t^2) = \mathrm{sign}(x'^2 - t'^2). \tag{1.1}$$

The negative (positive) sign in (1.1) corresponds to the interior (exterior) of the light cone. The light cone itself satisfies

$$x^2 - t^2 = x'^2 - t'^2 = 0. \tag{1.2}$$

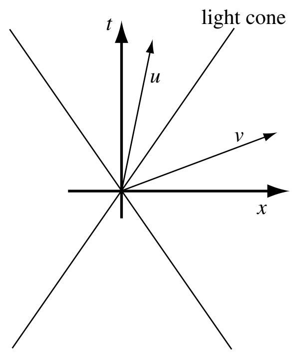

Figure 1.1 The local causal structure of spacetime is described by a light cone. Shown are the future and the past light cone about its vertex at the origin of a coordinate system (t, x). Vectors u within the light cone are timelike ($x^2 - t^2 < 0$); vectors v outside the light cone are spacelike ($x^2 - t^2 > 0$). By invariance of the velocity of light, this structure is the same for all observers. The linear transformation which leaves the signed distance $s^2 = x^2 - t^2$ invariant is the Lorentz transformation – a four-dimensional transformation of the coordinates of the frame of an observer.

A linear transformation between the coordinate frames of two observers which preserves the local causal structure obtains through Einstein's invariant distance

$$s^2 = -x^2 + t^2. \tag{1.3}$$

This generalizes Eqns (1.1) and (1.2). Remarkably, this simple ansatz recovers the Lorentz transformation, derived earlier by Lorentz on the basis of invariance of Maxwell's equations. The transformation in the invariant

$$x^2 - t^2 = x'^2 - t'^2 \tag{1.4}$$

can be inferred from rotations, describing the invariant $x^2 + y^2 = x'^2 + y'^2$ in the (x, y)-plane, as the hyperbolic variant

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\lambda & -\sinh\lambda \\ -\sinh\lambda & \cosh\lambda \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}. \tag{1.5}$$

The coordinates $(t, 0)$ in the observer's frame K correspond to the coordinates (t', x') in the frame K', such that

$$-\frac{x'}{t'} = \tanh\lambda. \tag{1.6}$$

This corresponds to a velocity $v = \tanh\lambda$ in terms of the "rapidity" λ of K' as seen in K. The matrix transformation (1.4) can now be expressed in terms of the relative velocity v,

$$t' = \Gamma(t - vx), \quad x' = \Gamma(x - vt), \tag{1.7}$$

where

$$\Gamma = \frac{1}{\sqrt{1 - v^2}} \tag{1.8}$$

denotes the Lorentz factor of the observer with three-velicity v.

The trajectory in spacetime traced out by an observer is called a world-line, e.g. that of K along the t-axis or the same observer as seen in K' following (1.8). The above shows that the tangents to world-lines – four-vectors – are connected by Lorentz transformations. The Lorentz transformation also shows that $v = 1$ is the limiting value for the relative velocity between observers, corresponding to a Lorentz factor Γ approaching infinity.

Minkowski introduced the world-line $x^b(\tau)$ of a particle and its tangent according to the velocity four-vector

$$u^b = \frac{dx^b}{d\tau}. \tag{1.9}$$

Here, we use a normalization in which τ denotes the eigentime,

$$u^2 = -1. \tag{1.10}$$

At this point, note the Einstein summation rule for repeated indices:

$$u^b u_b = \Sigma_{b=0}^3 u^b u_b = \eta_{ab} u^a u^b \tag{1.11}$$

in the Minkowski metric

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.12}$$

The Minkowski metric extends the Euclidian metric of a Cartesian coordinate system to four-dimensional spacetime. By (1.10) we insist

$$(u^x)^2 + (u^y)^2 + (u^z)^2 - (u^t)^2 = -1, \tag{1.13}$$

where $u^b = (u^t, u^x, u^y, u^z)$. In one-dimensional motion, it is often convenient to use the hyperbolic representation

$$u^b = (u^t, u^x, 0, 0) = (\cosh\lambda, \sinh\lambda, 0, 0) \tag{1.14}$$

in terms of λ, whereby the particle obtains a Lorentz factor $\Gamma = \cosh\lambda$ and a three-velocity

$$v = \frac{dx}{dt} = \frac{dx/d\tau}{dt/d\tau} = \frac{u^x}{u^t} = \tanh\lambda. \tag{1.15}$$

The Minkowski velocity four-vector u^b hereby transforms according to a Lorentz transformation ($d\tau$ is an invariant in (1.9)). We say that u^b is a covariant vector, and that the normalization $u^2 = -1$ is a Lorentz invariant, also known as a scalar.

To summarize, Einstein concluded on the basis of Maxwell's equations that spacetime exhibits an invariant causal structure in the form of an observer-invariant light cone at each point of spacetime. Points inside the light cone are causally connected to its vertex, and points outside are out-of-causal contact with its vertex. This structure is described by the Minkowski line-element

$$s^2 = x^2 + y^2 + z^2 - t^2, \tag{1.16}$$

which introduces a Lorentz-invariant signed distance in four-dimensional spacetime (t, x, y, z) following (1.12). In attributing the causal structure as a property intrinsic to spacetime, Einstein proposed that *all* physical laws and physical observables are observer-independent, i.e. obey invariance under Lorentz transformations. This invariance is the principle of his theory of special relativity. Galileo's picture of spacetime corresponds to the limit of slow motion or, equivalently, the singular limit in which the velocity of light approaches infinity – back to Euclidean geometry and Newton's picture of spacetime.

1.2 Kinematic effects

In Minkowski spacetime, rapidly moving objects give rise to apparent kinematic effects, representing the intersections of their world-lines with surfaces Σ_t of constant time in the laboratory frame K. In a two-dimensional spacetime diagram (x, t), Σ_t corresponds to horizontal lines parallel to the x-axis.

Consider an object moving uniformly with Lorentz factor Γ as shown in Figure (1.2) such that its world-line – a straight line – intersects the origin. The *lapse* in eigentime τ in the motion of the object from Σ_0 to Σ_t is given by

$$\tau = \int_0^t \frac{ds}{dt}\, dt = \sqrt{-(t, vt)^2} = t\sqrt{1-v^2}, \tag{1.17}$$

or

$$\frac{\tau}{t} = \frac{1}{\Gamma}. \tag{1.18}$$

Moving objects have a smaller lapse in eigentime between two surfaces of constant time, relative to the static observer in the laboratory frame. Rapidly moving elementary particles hereby appear with enhanced decay times. This effect is known as *time-dilation*.

The distance between two objects moving uniformly likewise depends on their common Lorentz factor Γ as seen in the laboratory frame K, as shown in

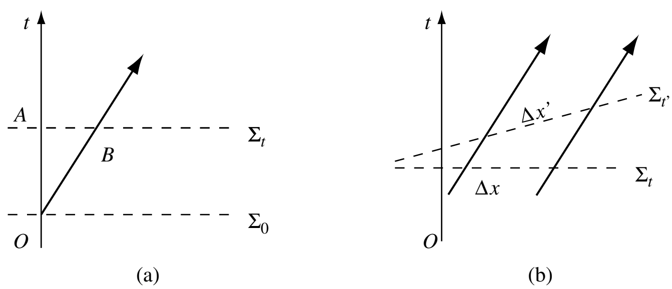

Figure 1.2 (*a*) Time dilation is described by the lapse in eigentime of a moving particle (arrow) between two surfaces of constant time Σ_0 and Σ_t in the laboratory frame K. The distance beteen these to surfaces in K is t, corresponding to O and A. The lapse in eigentime is t/Γ upon intersecting Σ_0 at O and Σ_t at B, where Γ is the Lorentz factor of the particle. Moving clocks hereby run slower. (*b*) The distance between two parallel world-lines (arrows) is the distance between their points of intersection with surfaces of constant time: Σ_t in K and $\Sigma_{t'}$ in the comoving frame K'. According to the Lorentz transformation, $\Delta x = \Delta x'/\Gamma$, showing that moving objects appear shortened and, in the ultrarelativistic case, become so-called "pancakes."

Figure (1.2). According to (1.7), the distance Δx between them as seen in K is related to the distance $\Delta x'$ as seen in the comoving frame K' by

$$\Delta x = \Delta x'/\Gamma. \tag{1.19}$$

Hence, the distance between two objects in uniform motion appears reduced as seen in the laboratory frame. This effect is known as the *Lorentz contraction*. Quite generally, an extended blob moving relativistically becomes a "pancake" as seen in the laboratory frame.

1.3 Quasar redshifts

Quasars are highly luminous and show powerful one-sided jets. They are now known to represent the luminous center of some of the active galaxies. These centers are believed to harbor supermassive black holes.

The archetype quasar is 3C273 at a redshift of $z = 0.158$. The redshift is defined as the relative increase in the wavelength of a photon coming from the source, as seen in the observer's frame: if λ_0 denotes the rest wavelength in the frame of the quasar, and λ denotes the wavelength in the observer's frame, we may write

$$1 + z = \frac{\lambda}{\lambda_0}. \tag{1.20}$$

The quasar 3C273 shows a relative increase in wavelength by about 16%. This feature is achromatic: it applies to any wavelength.

We can calculate z in terms of the three-velocity v with which the quasar is receding away from us. Consider a single period of the photon, as it travels a distance λ_0 in the rest frame. The null-displacement (λ_0, λ_0) *on* the light cone (in geometrical units) corresponds by a Lorentz transformation to

$$\Gamma(\lambda_0 + v\lambda_0), \quad \Gamma(\lambda_0 + v\lambda_0). \tag{1.21}$$

Note the plus sign in front of $v\lambda$ for a receding velocity of the quasar relative to the observer. The observer measures a wavelength

$$\lambda = \lambda_0\Gamma(1 + v) = \lambda_0\sqrt{\frac{1+v}{1-v}}. \tag{1.22}$$

It is instructive also to calculate the redshift factor z in terms of a redshift in energy. Let p_a denote the four-momentum of the photon, which satisfies $p^2 = 0$ as it moves along a null-trajectory on the light cone. Let also u^a and v^a denote the velocity four-vectors of of the quasar and that of the observer, respectively. The energies of the photon satisfy

$$\epsilon_0 = -p_a u^a, \quad \epsilon = -p_a v^a. \tag{1.23}$$

The velocity four-vectors u^a and v^a are related by a Lorentz transformation

$$v^a = \Lambda_a^b u^b, \quad \tanh\lambda = -v \tag{1.24}$$

in the notation of (1.5). It follows that

$$\epsilon = -p_a \Lambda_c^a u^c = -\eta_{ab} p^a \Lambda_c^a u^c. \tag{1.25}$$

This is a *scalar* expression, in view of complete contractions over all indices. We can evaluate it in any preferred frame. Doing so in the frame of the quasar, we have $p^a = \epsilon_0(1,1)$ and $u^b = (1,0)$. Hence, the energy in the observer's frame satisfies

$$\epsilon = \epsilon_0(\cosh\lambda - \sinh\lambda) = \epsilon_0\sqrt{\frac{1-v}{1+v}}. \tag{1.26}$$

Together, (1.22) and (1.26) obey the relationship $\epsilon = 2\pi/\lambda$, where $\epsilon_0\lambda_0 = 2\pi$.

1.4 Superluminal motion in 3C273

The quasar 3C273 is a variable source. It ejected a powerful synchrotron emitting blob of plasma in 1977, shown in Figure (1.3)[412]. In subsequent years, the angular displacement of this blob was monitored. Given the distance to 3C273 (based on cosmological expansion, as described by the Hubble constant), the velocity projected on the sky was found to be

$$v_\perp = (9.6 \pm 0.8) \times c. \tag{1.27}$$

An elegant geometrical explanation is in terms of a relativistically moving blob, moving close to the line-of-sight towards the observer, given by R. D. Blandford, C. F. McKee and M. J. Rees[65]. Consider two photons emitted from the blob moving towards the observer at consecutive times. Because the second photon is emitted while the blob has moved closer to the observer, it requires less travel time to reach the observer compared with the preceding photon. This gives the blob the appearance of rapid motion. We can calculate this as follows, upon neglecting the relative motion between the observer and the quasar. (The relativistic motion of the ejecta is much faster than that of the quasar itself.)

Consider the time-interval Δt_e between the emission of the two photons. The associated time-interval Δt_r between the times of receiving these two photons is reduced by the distance $D_{||} = v\cos\theta\Delta t_\mathrm{e}$ along the line-of-sight traveled by the blob:

$$\Delta t_\mathrm{r} = \Delta t_\mathrm{e} - v\cos\theta\Delta t_\mathrm{e}, \tag{1.28}$$

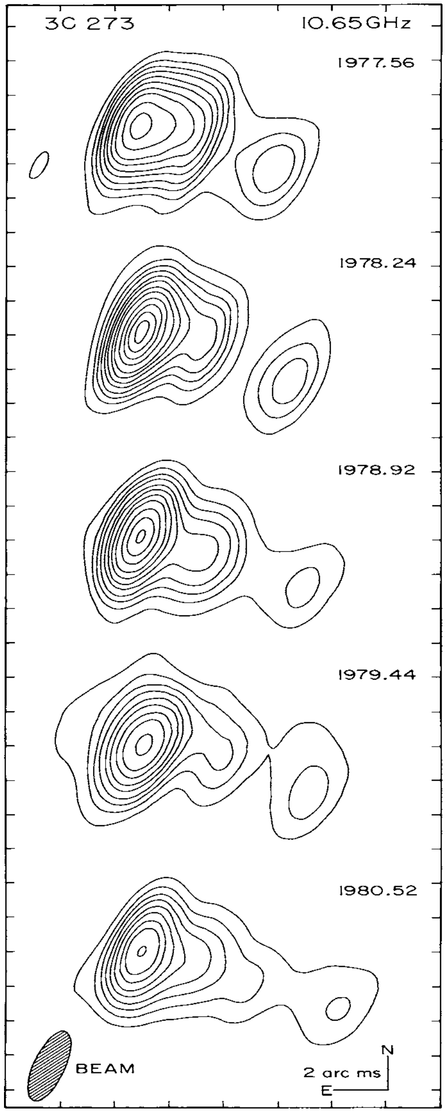

Figure 1.3 A Very Large Baseline Interferometry (VLBI) contour map of five epochs on an ejection event in the quasar 3C273 in the radio (10.65 GHz). (Reprinted by permission from the authors and *Nature*, Pearson, T. J. *et al.*, *Nature*, 280, 365. ©1981 Macmillan Publishers Ltd.)

where θ denotes the angle between the velocity of the blob and the line-of-sight. The projected distance on the celestial sphere is $D_\perp = \Delta t_e v \sin\theta$. The projected velocity on the sky is, therefore,

$$v_\perp = \frac{D_\perp}{\Delta t_r} = \frac{v \sin\theta}{1 - v\cos\theta}. \tag{1.29}$$

Several limits can be deduced. The maximal value of the apparent velocity $v_\perp$ is

$$v_\perp = v\Gamma. \tag{1.30}$$

Thus, an observed value for $v_\perp$ gives a minimal value of the three-velocity and Lorentz factor

$$v = \frac{v_\perp}{\sqrt{1+v_\perp^2}}, \quad \Gamma = \sqrt{1+v_\perp^2}. \tag{1.31}$$

Similarly, an observed value for $v_\perp$ gives rise to a maximal angle θ upon setting $v = 1$. With (1.27), we conclude that the blob has a Lorentz factor $\Gamma \geq 10$.

1.5 Doppler shift

The combined effects of redshift and projection are known as Doppler shift. Consider harmonic wave-motion described by $e^{i\phi}$. The phase ϕ is a scalar, i.e. it is a Lorentz invariant. For a plane wave we have $\phi = k_a x^a = \eta_{ab} k^a x^b$ in terms of the wave four-vector k^a. Thus, k^a is a four-vector and transforms accordingly. A photon moving towards an observer with angle θ to the line-of-sight has $k^x = \epsilon \cos\theta$ for an energy $k^0 = \epsilon$. By the Lorentz transformation, the energy in the source frame with velocity v is given by

$$k'^0 = \Gamma(k^0 - vk^1), \tag{1.32}$$

so that

$$\epsilon' = \Gamma\epsilon(1 - v\cos\theta). \tag{1.33}$$

The result can be seen also by considering the arrival times of pulses emitted at the beginning and the end of a period of the wave. If T' and T denote the period, in the source and in the laboratory frame, respectively, then $2\pi = \epsilon' T' = \epsilon'(T/\Gamma)$. The two pulses have a difference in arrival times $\Delta t = T(1 - v\cos\theta)$ and the energy in the observer's frame becomes

$$\epsilon = \frac{2\pi}{\Delta t} = \frac{\epsilon'}{\Gamma(1 - v\cos\theta)}. \tag{1.34}$$

This is the same as (1.33).

1.6 Relativistic equations of motion

Special relativity implies that all physical laws obey the same local causal structure defined by light cones. This imposes the condition that the world-line of any particle through a point remains inside the local light cone. This is a geometrical description of the condition that all physical particles move with velocities less than (if massive) or equal to (if massless) the velocity of light.

Newton's laws of motion for a particle of mass m are given by the three equations

$$F_i = m\frac{d^2x_i(t)}{dt^2} \quad (i = 1, 2, 3). \tag{1.35}$$

We conventionally use Latin indices from the middle of the alphabet to denote spatial components i, corresponding to (x, y, z). The velocity $dx_i(t)/dt$ is unbounded in response to a constant forcing (m is a constant), and we note that (1.35) consists of merely three equations motion. It follows that (1.35) does not satisfy causality, and is not Lorentz-invariant.

Minkowski's world-line x^b of a particle is generated by a tangent given by the velocity four-vector (1.9). Here, we use a normalization in which τ denotes the eigentime, (1.10). We consider the Lorentz-invariant equations of motion

$$f^b = \frac{dp^b}{d\tau}, \tag{1.36}$$

where

$$p^b = mu^b = (E, P^i) \tag{1.37}$$

denotes the particle's four-momentum in terms of its energy, conjugate to the time-coordinate t, and three-momentum, conjugate to the spatial coordinates x^i. There is one Lorentz invariant:

$$p^2 = -m^2, \tag{1.38}$$

which is an integral of motion of (1.36). The forcing in (1.36) is subject to the orthogonality condition $f^b p_b = 0$, describing orthogonality to its world-line.

The non-relativistic limit corresponding to small three-velocities v in (1.38) gives

$$E = \sqrt{m^2 + P^2} \simeq m + \frac{1}{2}mv^2. \tag{1.39}$$

We conclude that E represents the sum of the Newtonian kinetic energy and the mass of the particle. This indicates that m (i.e. mc^2) represents rest mass-energy of a particle. As demonstrated by nuclear reactions, rest mass-energy can be released in other forms of energy, and notably so in radiation. In general, it is important to note that energy is the time-component of a four-vector, and that it transforms accordingly.

Exercises

1. Maxwell's equations in a vacuum are the first-order linear equations $\nabla \times \mathbf{H} = -\partial_t \epsilon_0 \mathbf{E}$ and $\nabla \times \mathbf{E} = \partial_t \mu_0 \mathbf{H}$ on the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$, where ϵ_0 and μ_0 denote the electric permittivity and magnetic permeability of vacuum. Show that both $\mathbf{E}$ and $\mathbf{H}$ satisfy the wave-equation $\nabla^2\{\mathbf{E}, \mathbf{H}\} = 0$ in terms of the d'Alembertian
$$\nabla^2 = -c^{-2}\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2, \tag{1.40}$$
where $c = (\epsilon_0\mu_0)^{-1/2}$ denotes the velocity of light.
2. *Simple wave* solutions $\mathbf{E} = \Phi(k_a x^a)$ to (1.40) are plane-wave solutions satisfing the characteristic equation
$$k^a k^b \eta_{ab} = 0, \tag{1.41}$$
where k^a denotes the wave-vector and η_{ab} denotes the Minkowski metric (1.12). Verify that the null-surface (1.41) describes a cone in spacetime with vertex at the origin. Coordinate transformations which leave the Minkowski metric explicitly in the form (1.12), and hence the d'Alembertian in the form (1.40), are the Lorentz transformations. The postulate that c is constant hereby introduces Lorentz transformations between different observers. Verify geometrically that the Lorentz transformations form a group.
3. Obtain explicitly the product of two Lorentz transformations representing boosts along the x-axis with velocities v and w.
4. Derive the general class of infinitesimal Lorentz transformations for (1.16), consisting of small boosts and rotations. What is their dimensionality and do they commute?
5. Consider two world-lines with velocity four-vectors u^b and v^b which intersect at $\mathcal{O}$. In the wedge product
$$(\mathbf{u} \wedge \mathbf{m})^{ab} = u^a m^b - u^b m^a \tag{1.42}$$

we may assume without loss of generality $m^c u_c = 0$. Show that

$$\mathbf{n} = (\mathbf{u} \wedge \mathbf{m}) \cdot \mathbf{v} \tag{1.43}$$

produces a vector field n^b such that $n^c v_c = 0$. If u^b and v^b represent a boost between two observers, show that n^b are related by the same if (n^b, u^b, v^b) are coplanar. In this event, (1.43) represents a finite Lorentz transformation between two four-vectors m^b and n^b.

6. An experimentalist emits a photon of energy ϵ onto a mirror, which moves rapidly towards the observer with Lorentz factor Γ. What is the energy of the reflected photon received by the observer? (This is the mechanism of inverse Compton scattering, raising photon energies by moving charged particles below the Klein-Nishina limit[468].)

7. Generalize the results of Section 1.4 by including the redshift factor of the quasar.

8. Consider a radiation front moving towards the observer with Lorentz factor Γ. If the front is time variable on a timescale $\delta\tau$ in the comoving frame, what is the observed timescale of variability?

2

Curved spacetime and SgrA*

"When writing about transcendental issues, be transcendentally clear."
René Descartes (1596–1650), in G. Simmons, *Calculus Gems.*

General relativity extends Newton's theory of gravitation, by taking into account a local causal structure described by coordinate-invariant light cones. This proposal predicts some novel features around stars. Ultimately, it predicts black holes as fundamental objects and gravitational radiation.

It was Einstein's great insight to consider Lorentz invariance of Maxwell's equations as a property of spacetime. All physical laws hereby are subject to one and the same causal structure. To incorporate gravitation, he posed a local equivalence between gravitation and acceleration. This introduces the concept of freely falling observers in the limit of zero acceleration and described by geodesic motion.

The accelerated motion of the proverbial Newton's apple freely falling in the gravitation field is fundamental to gravitation. The weight of the apple when hanging on the tree or in Newton's hand is exactly equal to the body force when accelerated by hand at the same acceleration as that imparted by the gravitational field in free-fall. The mass of the apple as measured by its "weight" is unique whether gravitational or inertial.

Rapidly moving objects show kinematic effects in accord with special relativity. These effects may be attributed to the associated kinetic energies. In the Newtonian limit, the gravitational field may be described in terms of a potential energy. Kinetic energy and potential energy are interchangeable subject to conservation of total energy. Kinematic effects can hereby be attributed equivalently to a particle's kinetic energy or drop in potential energy. When viewed from the tree, *the apple in Newton's hand looks more red and flat than those still hanging in the tree.*