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EDITORS: H. A. LAUWERIER AND W. T. KOITER

# **creep and relaxation of nonlinear viscoelastic materials**

**with an introduction to  
linear viscoelasticity**

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MECHANICS

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# CREEP AND RELAXATION OF NONLINEAR VISCOELASTIC MATERIALS

with an  
Introduction to Linear Viscoelasticity

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## PREFACE

This book was started in 1968 with the realization that nonlinear viscoelasticity was an active, developing subject, so that a book at this time could not be a definitive treatise. However, the need for information on creep and relaxation in the nonlinear range for critical designs suggested that a book on the subject would be useful.

Accordingly the present volume was prepared with the object of presenting the material in such a way as to be readable and useful to designers as well as research workers and students. To this end the mathematical background required has been kept to a minimum and supplemented by explanations where it has been necessary to introduce specialized mathematics. Also, appendices have been included to provide sufficient background in Laplace transforms and in step functions.

Chapters 1 and 2 contain an introduction and a historical review of creep. As an aid to the reader a background on stress, strain and stress analysis is provided in Chapters 3 and 4, an introduction to linear viscoelasticity is found in Chapter 5 and linear viscoelastic stress analysis in Chapter 6. While the main thrust of the book is creep and stress relaxation, the chapter on linear viscoelasticity includes oscillatory stress and strain as well. This was included because it is sometimes desirable to have a broader time scale than can be provided by creep tests alone when describing the time-dependent behavior of a material. To this end creep or relaxation and oscillatory tests can be complementary.

It was found necessary for reasonable brevity to employ Cartesian tensor notation. However, at appropriate points the components of stress and strain have been expressed in conventional notation as well. To provide the greatest usefulness for those interested in specific applications, expressions for general results have often been reduced to several common states of stress or strain.

While more than one approach to nonlinear viscoelasticity was being

explored in the literature at the time the book was started, and others have been proposed since, it was decided to treat one approach in depth while only outlining other methods. The approach considered in detail in Chapters 7, 8 and 9 is the multiple integral theory with simplifications to single integrals. Other simplifications considered in some detail are the assumptions of incompressibility and linear compressibility. These topics are developed in Chapters 7, 8 and 9.

The relation between the responses of viscoelastic materials to (a) stress boundary conditions (creep), (b) strain boundary conditions (relaxation) and (c) mixed stress and strain boundary conditions (simultaneous creep and relaxation) are considered in Chapter 10. Chapter 11 treats the problem of the effect of temperature on nonlinear creep—especially variable temperature.

Methods of characterization of kernel functions and examples of experimental results are presented (wherever possible) in Chapters 8, 9 and 11 to assist the reader to obtain a physical feeling about the nonlinear behavior of various materials.

Stress analysis of nonlinear viscoelastic materials presents many difficulties. As a result, the field is not highly developed. In Chapter 12 the stress analysis of nonlinear viscoelastic materials is introduced and several problems of increasing complexity are discussed.

Finally, in deference to the extensive experimental background of the authors, Chapter 13 discusses experimental methods for creep and stress relaxation under combined stress. This chapter considers especially those experimental problems which must be solved properly when reliable experimental results of high precision are required.

In order that the book be as up-to-date as possible at the time of publication a list of “Additional Literature” has been appended to several chapters. This list is presented without citation in the text and (with one exception) without evaluation or comment.

The book was started by the senior author while on sabbatical leave from Brown University as a visiting Professor at the University of Auckland, New Zealand. The quiet office provided by the University of Auckland and the helpful discussions with the staff are much appreciated. The interest of the administration of the Division of Engineering of Brown University in this project is gratefully acknowledged.

The authors are especially indebted to Professor E. H. Lee for reading the completed manuscript and offering many helpful suggestions.

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July, 1975

William N. Findley  
James S. Lai  
Kasif Onaran

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## CHAPTER 1

### INTRODUCTION

Design of modern high performance machines and structures often must take account of the effect of complex states of stress, strain and environment on the mechanical behavior of different classes of materials. Calculation of the mechanical behavior of a structural member under different conditions of stress or strain and environment requires that the different variables involved be related by means of fundamental equations including the following: (a) The equilibrium equations. These state the relationship among the various stress components at any given point required for equilibrium. (b) The kinematic equations. These express the strain components in terms of displacements which in turn describe the deformation of the body. (c) The compatibility equation. This states the relationship which must exist among the several strain components in order that the strain components in a continuous medium not produce discontinuities. (d) The constitutive equation. This must describe the relationship between stress, strain and time in terms of the material constants for a given material. (e) A set of boundary conditions. These describe the stresses and displacements prescribed at the boundaries. If the material behavior is linear in stress and time independent, then Hooke's law describes the constitutive relationship. A detailed description of equations (a), (b), (c), (d) and (e) for linear, time-independent materials may be found in any book on the theory of elasticity.

In this book emphasis will be placed on discussion of the constitutive equations for time-dependent and nonlinear materials, though an introduction to time-dependent linear behavior is also discussed. Actual materials exhibit a great variety of behavior. However, by means of idealization they can be simplified and classified as follows.

### 1.1 Elastic Behavior

Most materials behave elastically or nearly so under small stresses. As illustrated by the solid curve in Fig. 1.1, an immediate elastic strain response is obtained upon loading. Then the strain stays constant as long

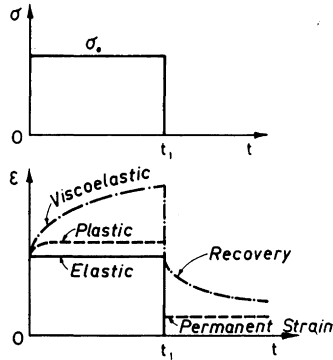


Fig. 1.1. Various Strain Responses to a Constant Load

as the stress is fixed and disappears immediately upon removal of the load. The chief characteristic of elastic strain is reversibility. Most elastic materials are linearly elastic so that doubling the stress in the elastic range doubles the strain.

### 1.2 Plastic Behavior

If the stress is too high the behavior is no longer elastic. The limiting stress above which the behavior is no longer elastic is called the elastic limit. The strain that does not disappear after removal of the stress is called the inelastic strain. In some materials, the strain continues to increase for a short while after the load is fully applied, and then remains constant under a fixed load, but a permanent strain remains after the stress is removed. This permanent strain is called the plastic strain (dashed curves in Fig. 1.1). Plastic strain is defined as time independent although some time dependent strain is often observed to accompany plastic strain.

### 1.3 Viscoelastic Behavior

Some materials exhibit elastic action upon loading (if loading is rapid enough), then a slow and continuous increase of strain at a decreasing rate

is observed. When the stress is removed a continuously decreasing strain follows an initial elastic recovery. Such materials are significantly influenced by the rate of straining or stressing; i.e., for example, the longer the time to reach the final value of stress at a constant rate of stressing, the larger is the corresponding strain. These materials are called viscoelastic (dot-dash curve in Fig. 1.1). Among the materials showing viscoelastic behavior are plastics, wood, natural and synthetic fibers, concrete and metals at elevated temperatures. Since time is a very important factor in their behavior, they are also called time-dependent materials. As its name implies, viscoelasticity combines elasticity and viscosity (viscous flow).

The time-dependent behavior of viscoelastic materials must be expressed by a constitutive equation which includes time as a variable in addition to the stress and strain variables. Even under the most simple loading program, as shown in Fig. 1.1, the shape of the strain-time curve, in this case a creep curve, may be rather complicated. Since time cannot be kept constant, reversed or eliminated during an experiment, the experimental study of the mechanical behavior of such materials is much more difficult than the study of time-independent materials.

Recent developments in technology, such as gas turbines, jet engines, nuclear power plants, and space crafts, have placed severe demands on high temperature performance of materials, including plastics. Consequently the time-dependent behavior of materials has become of great importance.

The time-dependent behavior of materials under a quasi-static state may be studied by means of three types of experiments: creep (including recovery following creep), stress relaxation and constant rate stressing (or straining), although other types of experiments are also available.

## 1.4 Creep

Creep is a slow continuous deformation of a material under constant stress.\* However, creep in general may be described in terms of three different stages illustrated in Fig. 1.2. The first stage in which creep occurs at a decreasing rate is called primary creep; the second, called the secondary stage, proceeds at a nearly constant rate; and the third or tertiary stage occurs at an increasing rate and terminates in fracture.

---

\* Most creep experiments are performed under a constant load even when the cross-section of the specimen changes significantly with time. For small strains, constant load and constant stress experiments are the same.

Total strain  $\varepsilon$  at any instant of time  $t$  in a creep test of a linear material (linearity will be defined later) is represented as the sum of the instantaneous elastic strain  $\varepsilon^e$  and the creep strain  $\varepsilon^c$ ,

$$\varepsilon = \varepsilon^e + \varepsilon^c. \quad (1.1)$$

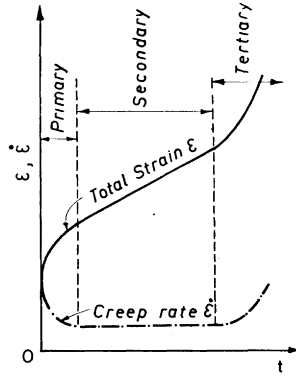


Fig. 1.2. Three Stages of Creep

The strain rate  $\dot{\varepsilon}$  is found by differentiating (1.1) and noting that  $\varepsilon^e$  is a constant:

$$\frac{d\varepsilon}{dt} = \frac{d\varepsilon^c}{dt} = \dot{\varepsilon}. \quad (1.2)$$

### 1.5 Recovery

If the load is removed, a reverse elastic strain followed by recovery of a portion of the creep strain will occur at a continuously decreasing rate.

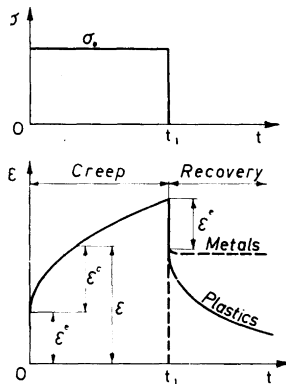


Fig. 1.3. Creep and Recovery of Metals and Plastics

The amount of the time-dependent recoverable strain during recovery is generally a very small part of the time-dependent creep strain for metals, whereas for plastics it may be a large portion of the time-dependent creep strain which occurred (Fig. 1.3). Some plastics may exhibit full recovery if sufficient time is allowed for recovery. The strain recovery is also called delayed elasticity.

### 1.6 Relaxation

Viscoelastic materials subjected to a constant strain will relax under constant strain so that the stress gradually decreases as shown in Fig. 1.4.

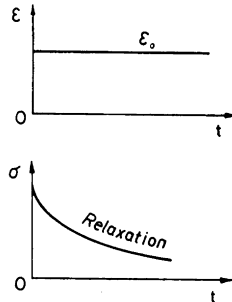


Fig. 1.4. Stress Relaxation at Constant Strain

From a study of the three time-dependent responses of materials explained above, the basic principles governing time-dependent behavior under loading conditions other than those mentioned above, may be established. In actual practice, the stress or strain history may approximate one of those described or a mixture, i.e., creep and relaxation may occur simultaneously under combined loading, or the load or strain history may be a cyclic or random variation.

### 1.7 Linearity

The material is said to be linearly viscoelastic if stress is proportional to strain at a given time, and the linear superposition principle holds. These linear requirements can be stated mathematically in two equations:

$$\varepsilon[c\sigma(t)] = c\varepsilon[\sigma(t)], \quad (1.3)$$

$$\varepsilon[\sigma_1(t) + \sigma_2(t - t_1)] = \varepsilon[\sigma_1(t)] + \varepsilon[\sigma_2(t - t_1)], \quad (1.4)$$