R. BALIAN
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A. MESSIAH
editors

Claude Bloch

SCIENTIFIC WORKS
OEUVRE SCIENTIFIQUE

Volume 1

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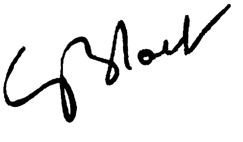

Claude Bloch
1923–1971

Foreword

The untimely death of Claude Bloch leaves us with an unfinished but significant work, composed of published articles, internal reports as well as lectures delivered at various institutes or schools, which are sometimes in unedited form.

The depth of insight, the clarity of ideas, the complete mastery of the tools of mathematics which Claude Bloch possessed to a rare degree permitted him to grasp directly the essential issues and to state the results of his research in a fashion at once simple, elegant and rigorous. These qualities appear as much in his original articles as in the notes of the courses which he had taught. Many of these notes also show an unusual approach presented in a clear and concise style greatly appreciated by all those who had access to them. It seemed to us that to gather the works of Claude Bloch into book form for the benefit of the scientific community would both bring into focus his main lines of thought and insure for this work the recognition that it deserves.

In order to keep this book within reasonable bounds, we have left aside a few minor or redundant writings. An exhaustive bibliography of Claude Bloch's works is listed in the biographical article which opens this volume. The included works follow the chronological order. Rather than trying to regroup the publications by subject matter, we have preferred to keep the order in which they were conceived, to render more apparent the development of Claude Bloch's thoughts. In this way a common thread often shows through the articles on nuclear physics, on statistical mechanics, on black box theories or random ensembles, and any attempt to separate them into distinct categories seemed to us artificial. For the convenience of the reader and in order to guide him if he is especially interested in a particular subject, this table of contents is followed with classification by subjects and a list of lecture notes and monographs.

We are grateful to the Commissariat à l'Energie Atomique and to the First European Nuclear Conference of Aix-en-Provence, dedicated to the memory of Claude Bloch, 1972, for their support which rendered possible this publication.

We wish to thank Mrs. M. Porneuf and her staff for their collaboration in preparing the material for this book.

Saclay, November 15, 1973

R. Balian, C. De Dominicis, V. Gillet, A. Messiah

Préface

La mort prématurée de Claude Bloch nous laisse une oeuvre inachevée mais considérable, faite d'articles publiés, de notes restées à l'état de rapports internes, ainsi que de cours professés dans divers instituts ou écoles et parfois inédits.

La profondeur de vues, la clarté d'idées, la maîtrise complète des outils mathématiques que possédait, à un degré rare, Claude Bloch, lui permettaient d'aller droit à l'essentiel et de donner aux résultats de ses recherches un aspect à la fois simple, dépouillé et rigoureux qui apparaît tant dans ses articles originaux que dans les notes des cours qu'il a professés. Nombre de ces notes contiennent d'ailleurs une approche inhabituelle, couchée dans un style clair et concis, largement apprécié de tous ceux qui y ont eu accès. Il nous est apparu que rassembler au bénéfice de la communauté scientifique les travaux de Claude Bloch en un livre, c'était à la fois en éclairer les lignes directrices et assurer à cette oeuvre le rayonnement qu'elle mérite.

Afin de conserver à cet ouvrage des dimensions raisonnables, nous avons été amenés à renoncer à certains textes mineurs ou redondants. La liste complète des travaux de Claude Bloch se trouve dans la notice biographique qui ouvre ce volume. La présentation des oeuvres retenues suit l'ordre chronologique. Plutôt que d'essayer de classer les publications par grands sujets, nous avons préféré conserver l'ordre dans lequel elles ont été écrites pour mieux faire apparaître le développement de la pensée. C'est ainsi qu'une trame commune transparaît souvent dans des articles relevant de la physique nucléaire, de la mécanique statistique, des théories de boîtes noires ou d'ensembles aléatoires, et toute tentative de les séparer en catégories distinctes nous a semblé artificielle. Pour la commodité du lecteur, et pour le guider s'il est plus particulièrement intéressé par un domaine déterminé, nous faisons suivre cette table des matières d'une classification par sujets ainsi que d'une liste des cours et monographies.

Nous sommes reconnaissants au Commissariat à l'Energie Atomique, ainsi qu'à la Première Conférence Européenne de Physique Nucléaire d'Aix-en-Provence, 1972, dédiée à la mémoire de Claude Bloch, pour leur aide à la publication de cet ouvrage.

Nous remercions Mme M. Porneuf et son groupe de documentation qui en ont assuré la préparation matérielle.

Fait à Saclay, le 15 novembre 1973

R. Balian, C. De Dominicis, V. Gillet, A. Messiah

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CLAUDE BLOCH

18 March 1923 — 29 December 1971

Born in Paris, Claude Bloch was admitted to the Ecole Polytechnique in 1942, graduated first in his class, and entered the Corps des Mines in 1946. Instead of pursuing the career opened to him in the French administration, his devotion for science led him to choose his path in fundamental research.

After the war, physics in France was deeply disorganized and research was dormant. Claude Bloch fought against these difficult conditions. Through self-teaching and mutual training with a group of his contemporaries (including A. Abragam, J. Horowitz and A. Messiah), he succeeded in acquiring a deep knowledge of the foundations of modern physics. He pursued his studies at the Bohr Institute in Copenhagen (1948–1951), where he learned field theory and worked on problems of non-local fields [1–3]). Soon his interest focussed on nuclear physics and during his stay at the California Institute of Technology he made his first contribution to statistical nuclear physics [4, 6]).

On his return to France, he entered the Commissariat à l'Energie Atomique, where he remained throughout his career. He began to lecture at Saclay on topics of theoretical physics and his illuminating lectures had a major influence on the postwar renaissance of French physics. He soon took charge of the theoretical department at Saclay, which he led and developed, drawing to it physicists from many nations and disciplines. Through a spirit of friendly criticism he instilled the group with his own high standards. Much of its present productivity and renown can be traced to his inspiring effort. Throughout the period in which he led this group, he continued to amass, without interruption, a truly impressive collection of scientific works encompassing nuclear physics, the many-body problem, and statistical physics.

As early as 1953, Claude Bloch became interested in nuclear statistical physics through the problem of nuclear level densities [4, 6]). Using an approach of Bethe based on the evaluation of the grand partition function, he replaced the estimates obtained from a continuum of single-particle levels by more realistic expressions which took shell structure into account. In this manner he obtained a good fit with experimental data on the level density and moment of inertia of ^{20}Ne. As can be seen from this later work, he always kept a fondness for statistical attacks of the nuclear problem [refs. [9, 49–53])]. In 1957, for example, he explained the Lorentzian shape of the strength function near a single-particle resonance by treating as random variables the

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matrix elements of the residual interaction between the particle and the target [15].

In 1953 Claude Bloch first gave his course on nuclear reactions [5]. In teaching it, he was led to restate the various theories and formalisms of this field in a newer and clearer way which contributed to an improved understanding of topics like the optical-model and direct-reaction theories. His outstanding work of that period [14] is the unified presentation of the formal reaction theories of Kapur and Peierls, Wigner and Eisenbud, and Brown and De Dominicis. Their formalisms, which impose boundary conditions in each channel at a finite radius, had rather lengthy presentations, which required successive inversions of various matrices in order to yield the scattering amplitude. The original idea of Claude Bloch was the introduction of an operator, now referred to as Bloch's operator, which is the sum of the Hamiltonian of the system and of a boundary operator. The latter is defined on the surface separating the internal region from the external one. By inverting Bloch's operator, one obtains in a single step the general formal expression of the S-matrix in terms of the parameters of the various reaction theories characterized by the choice of the boundary operator. Thus all the results of the different black box theories are obtained in a unique and simple framework. Furthermore, if H is divided into a model Hamiltonian H_0 and a residual interaction V, the formal expression of the scattering matrix can be developed into powers of V, and yields a method for computing direct reactions in the framework of the resonance theories. For example from his expression Claude Bloch showed how to derive a microscopic expression for the imaginary part of the optical potential. The unified theory of nuclear reactions of Claude Bloch is nowadays the standard presentation of the formal nuclear reaction theories and is adopted in textbooks, lectures and papers by many authors.

At the same time Claude Bloch was completing his work on resonance theory, he began to think about a microscopic theory of nuclear structure. His attention was drawn by the recent publications of Goldstone and Brueckner. After Brueckner's first calculations on nuclear matter, Goldstone had established diagrammatic perturbation expansions for the ground-state energy and wave function of non-degenerate systems. Using this thorough knowledge of field-theory techniques, Claude Bloch then started a pioneering work in the field of the many-body problem which he pursued for several years. He made his first important contribution to the many-body problem by finding a very simple and elegant derivation of Goldstone's expansion [17,20].

An essential feature of this expansion, which is a diagrammatic form of Rayleigh-Schrödinger's perturbation theory, is the fact that its successive terms are proportional to the volume of the system, so that it is well adapted to the study of large systems. However, it applies neither to excited states, nor to degenerate cases, and is thus useless for finite nuclei. Claude Bloch had just studied perturbation theory for degenerate levels [19], which he had restated in a rapidly convergent form, anticipating the now popular Padé approximant method. He noticed that the implicit Brillouin-Wigner type of expansions used in this case had advantages and drawbacks relative to Goldstone's, and, together with J. Horowitz, he worked out a synthesis of the two approaches: the bulk of the nucleus, corresponding to the inner shells, was to be treated by an explicit Rayleigh-Schrödinger perturbation theory, whereas the few nucleons of the last shell were to be described by an implicit equation of the Brillouin–Wigner type. This formalism [21]) may be considered as the foundation of most nuclear structure calculations performed since.

While the techniques of field theory became applicable to the ground state of large systems, perturbation theory remained impracticable and even obscure at non-zero temperature, in spite of an attempt by Matsubara. Claude Bloch contributed greatly to working out such an extension. He established with C. de Dominicis new expansions for the Gibbs potential [16,22–24]), which are now well known, and uses as a starting point for many studies in solid-state physics and in statistical mechanics. The idea consists in showing that independent particle density matrices play in statistical mechanics exactly the same role as the vacuum in field theory, and in particular that averages of products of operators may be calculated in the same way as by Wick's theorem.

This elaboration of a microscopic formalism for dealing with large interacting quantum systems had a major influence on the renewed interest in statistical mechanics over the past ten years. Several applications were soon worked out by Claude Bloch. He was the first to explain why the reaction matrix appearing in nuclear matter calculations is related to the tangent of the phase shift, and not to the phase shift itself, although the latter appears in Bethe and Uhlenbeck's evaluation of the second virial coefficient in statistical mechanics [16,24]). At the time this point was quite obscure. He then explored systematically the various possible resummations of the diagrammatic perturbation expansions at finite temperature. In a classical review article [38]), he showed that the application to the diagrams of the topological theorem of Euler-Poincaré resulted in variational properties of the resummed expressions. Another important application was the extension to quantum systems of the virial expansion established 30 years before by Yvon and Mayer in classical statistical mechanics [31,32]).

In 1963, Claude Bloch returned to the theory of nuclear reactions, with the aim of casting it into the framework of the microscopic theory of nuclear structure to which he had contributed so much. Starting from the 1931 work of U. Fano on the scattering of electrons by atoms, he gave with V. Gillet [41]) the first simple derivation of the scattering amplitudes from the configuration mixing of several continua each limited to one unbound particle. He showed [42]) the feasibility of practical calculations based on the discretization of the energy spectrum of unbound shell-model states. In his lectures at Varenna in 1965 [ref. [45])], he used this microscopic framework to discuss nuclear resonances and the nature of intermediate structures in nuclear spectra. He related their characteristics to those of the underlying quasi-bound states, thus extending the shell-model concepts high into the continuum. Resonances were classified into a hierarchy of states of increasing density and decreasing widths governed by the number and the complexity of the quasi-bound states which could be constructed within the shell model. These ideas and the discretization method gave birth to many developments and they have been applied with success in actual calculations to explain, for example, the fine structure of giant resonances, the cross sections for photonuclear reactions and for inelastic single-particle scattering.

Quite naturally the problems of intermediate structures brought Claude Bloch to consider the questions raised by the analysis of experimental data at intermediate energies. At that time much interest in the cross-section fluctuations had been stirred up by T. Ericson's work on their statistical aspects. On his part Claude Bloch was looking for a rigorous method of analysing the data so as to separate the simple intermediate structures, which carry the relevant nuclear information, from the statistical background or nuclear noise which modulates them. He solved this problem elegantly [50, 51]) by introducing in the phenomenological analysis of the data the concepts of information theory. His method still remains the only correct one for extracting the positions and widths of the intermediate structures contained in fluctuating cross sections.

Two years ago, Claude Bloch became interested in the theory of nuclear fission. Highly deformed fissioning nuclei are commonly described by a crude model of independent nucleons in a potential well, and the determination of the fission parameters relies on the evaluation of the energy levels of this well as a function of the deformation. The existing calculations showed unexpected shell effects for non-spherical nuclei, and it was a challenge to find an explanation for them. Claude Bloch was thus led to study, together with R. Balian, the distribution of single-particle levels for potentials of arbitrary shape, and in particular the density of eigenmodes in a cavity, in terms of the shape of its boundary. This problem extends in fact far beyond the domain of nuclear physics; for instance to acoustics, to electromagnetic modes in conducting cavities, and to the electrons or phonons of a small metallic grain. The method uses a semi-classical type of approximation for the Green function, from which the level density is readily derived. Although this work was not completed, it led to several important results [54, 56, 59, 62]). On the one hand, the smooth part of the density of eigenmodes was evaluated as an asymptotic expansion, the dominant term of which, proportional to the volume, had been found by H. Weyl at the beginning of the century. The succeeding terms correspond in nuclear physics to surface and curvature corrections in the Weizsäcker mass formula; in electromagnetism, they yield size corrections to Planck's formula for small black bodies. On the other hand, a more detailed study exhibited regularly oscillating terms in the density of eigenvalues. Such oscillations were shown to be related to the closed trajectories of the classical problem associated with the wave equation. Remarkably these density oscillations exist for any potential shape, so that the modes are never distributed at random even when they are dense. This phenomenon explains why shell effects were observed in the computation of single-particle levels for highly deformed nuclei, a fact which in turn is at the origin of the double hump in the fission barrier. Like his previous works, this last contribution of Claude Bloch to physics presents a simple and elegant method, giving a clear explanation of a physical phenomenon and allowing new practical calculations.

Claude Bloch leaves many scientific writings, remarkable for their depth, elegance and clarity. Endowed with a profound knowledge of mathematics and a discerning critical ability, he moved freely from one field to another, discarding the inessential in order to gain a deeper understanding of physical reality. Although he was a theoretician, he did not disdain technical problems. On occasion he turned his hand to accelerator design [7,12]) and nuclear power.

Along with his own scientific work, Claude Bloch devoted a large part of his activity to the development of fundamental research. His lectures at Saclay, presenting ideas often far in advance on his time, shaped an entire generation of young physicists. But, above all, as head of the Department of Theoretical Physics at Saclay, he assembled a group of theorists, drawn from the Commissariat à l'Energie Atomique, the Centre National de la Recherche Scientifique and foreign laboratories, whose researches span the entire field of physics. The innumerable contributions of this group are a result of Claude Bloch's scientific policy: avoid a narrow specialization, submit all work to constant professional criticism, and maintain communications among French and foreign laboratories, through personal contacts, seminars, schools, and conferences. In a spirit of friendly competition between laboratories research will flourish. His influence extended far beyond the limits of his group. He was constantly preoccupied with the progress of fundamental research in France, research whose importance he understood and for whose defence he was always prepared to take the initiative.

One year before he died, his responsibilities were increased to include the direction of the entire physics research program of the CEA. In this new task as in the previous ones, he manifested those qualities of judgement, authority and decision which demonstrated that he knew how to be not only a scientist but a man of action.

All those who knew him closely appreciated his warm sensitivity, often sheltered beneath a certain reserve and apparent coldness. His modesty drove him to discredit the external marks of success, and he frequently expressed this disdain with biting irony. But there was never a young physicist eager to do research who was not the object of his patience, benevolence and help. His loss is deeply felt by all his colleagues, who will keep alive the memory of a cultivated companion, a humanist devoted to progress and a friend who was always there.

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7) Etude de la focalisation intense dans les synchrotrons à protons (with A. Abragam, X. Pottier and I. Solomon), Note CEA 41 (1954)
8) Limites de la polarisation d'un système atomique soumis à un champ radioélectrique (with A. Messiah), J. Phys. Rad. **16** (1955) 785
9) La théorie statistique des réactions nucléaires, J. Phys. Rad. **17** (1956) 510
10) Le taux de production d'énergie par réactions thermonucléaires, Rapport S.P.M. Saclay no. 127 (1956)
11) Le rayonnement d'un gaz ionisé (with C. Schuhl), Rapport S.P.M. Saclay no. 161 (1956)
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13) La non-conservation de la parité et sa signification, L'Âge Nucléaire **3** (1957) 49
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15) La fonction densité (strength function) dans la théorie statistique des réactions nucléaires, Nucl. Phys. **3** (1957) 137
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17) La structure de la matière nucléaire, Interactions nucléaires aux basses énergies et structure des noyaux, in Comptes Rendus du Congrès International de Physique Nucléaire, Paris, 1958 (Dunod, Paris, 1959) p. 243
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20) Sur la détermination de l'état fondamental d'un système de particules, Nucl. Phys. **7** (1958) 451
21) Sur la détermination des premiers états d'un système de fermions dans le cas dégénéré (with J. Horowitz), Nucl. Phys. **8** (1958) 91
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25) The many-body theory, in lectures given at the IV Summer Meeting of Nuclear Physicists, Hercegnovi, 1959, ed. R. V. Popíc (1960) p. 101
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32) Formulation de la mécanique statistique en termes de nombres d'occupation (II) (with R. Balian and C. de Dominicis), Nucl. Phys. **27** (1961) 294

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Ann. of Phys. **63** (1971) 592

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62) Distribution of eigenfrequencies for the wave equation in a finite domain (III). Eigenfrequency density oscillations (with R. Balian), Ann. of Phys., in print

63) Distribution of eigenfrequencies for the wave equation in a finite domain (IV). Casimir effect, to be published

64) Distribution of eigenfrequencies for the wave equation in a finite domain (V). Schrödinger equation, to be published

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Classification of Contents — Classement par Sujets

Cette classification par grands sujets entrainant nécessairement un certain recouvrement, des articles apparaissent sous différentes rubriques.

This classification in subject matter brings in necessarily overlaps, and some articles appear under several headings.

General methods of quantum mechanics — Méthodes générales de la mécanique quantique

Group theory — *Théorie des groupes*

Semi-classical methods — *Méthodes semi-classiques*

Basic perturbation expansions — *Méthodes perturbatives*

Many-body theory and statistical mechanics **Resummed perturbation expansion**

Problème à N-corps et mécanique statistique **Méthodes de resommation**

Nuclear physics **Physique nucléaire**

Nuclear structure *Structure nucléaire*

Nuclear reactions *Réactions nucléaires*

Statistical approaches in nuclear physics *Approches statistiques en physique nucléaire*

Classification of Contents

Distribution of eigenvalues **Distribution des valeurs propres**

Miscellaneous **Divers**

Lecture notes and review articles **Cours et monographies**

Det Kongelige Danske Videnskabernes Selskab
Matematisk-fysiske Meddelelser, bind **27**, nr. 8

Dan. Mat. Fys. Medd. **27**, no. 8 (1952)

ON FIELD THEORIES WITH NON-LOCALIZED INTERACTION

BY

CLAUDE BLOCH

København
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CONTENTS

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Summary.

A relativistic field theory with non-localized interaction is investigated. The field equations are deduced by the variational principle from a Lagrange function containing an interaction term involving a form function. The essential departure from conventional field theory is the lack of causality, or, in other words, the lack of propagation character of the field equations. It is shown, however, that under certain conditions which must be satisfied by the form function this property remains limited to small domains. Similarly there are no continuity equations, but conservation laws hold in the large. The quantization is performed according to an extension of the scheme developed by Yang and Feldman making use of the concept of incoming and outgoing fields. It is shown that this procedure is always consistent with the field equations. Assuming that the field equations can be solved by means of power series expansions it is possible to give rules generalizing Feynman's rules giving all the terms of the expansion of an outgoing operator in terms of the incoming operators. Every term is associated with a doubled graph. An investigation is made of the convergence of the integrals obtained in this way. It is shown that many terms converge automatically as soon as the Fourier transform of the form function is supposed to fall off rapidly at large momenta. Some divergences remain in the higher order terms. They can, however, be removed by assuming that the Fourier transform of the form function has only time-like components. It is finally shown that the gauge invariance requires the addition of a new interaction term in the Lagrange function, corresponding to a sort of exchange current.

1. Introduction.

It has been shown by PEIERLS and MACMANUS[1] that it is possible to introduce a smearing function in a field theory in a Lorentz-invariant way. YUKAWA[2], on the other hand, has proposed a theory involving non-local fields, which, as will be shown later, is equivalent to an ordinary field theory with an interaction containing a form function, if one takes a variation principle as a starting point[3]. These theories cannot be put into a Hamiltonian form and, consequently, have met with some difficulty in quantization. Recently, however, a new treatment of conventional field theory was developed by YANG and FELDMAN[4] and by KÄLLÉN[5], which can immediately be applied to field theories involving smearing functions[6, 7]. It has therefore become possible to build a complete Lorentz-invariant quantized field theory with a non-localized interaction, and it may be worth while to investigate the consistency of such a scheme, and the convergence of the results it yields.

If we take, for simplicity, the example of a nucleon field interacting with a neutral scalar meson field, the scalar non-localized interaction term reads[*]

$$L_i = g \int dx' dx'' dx''' F(x', x'', x''') \, \psi^+ (x') \, u(x'') \, \psi(x'''), \qquad (1,1)$$

where the form function F must be Lorentz invariant and such that contributions to L_i come only from the volume elements for which the three points x', x'', x''' are very near each other. By points near each other is meant points whose coordinates differ by amounts of the order of a characteristic length λ. The interaction (1,1) is Hermitian if the form function satisfies the condi-

[*] In this formula x stands for $x^1, x^2, x^3, x^4 = t$, and dx for $dx^1 dx^2 dx^3 dx^4$. We shall use units such that $\hbar = c = 1$. We shall write ab for the scalar product $\Sigma a_i b^i$, where $a_i = a^i$ for $i = 1, 2, 3$, and $a_4 = -a^4$. The metric tensor $g_{\mu\nu}$ is defined by $g_{\mu\nu} = 0$ if $\mu \neq \nu$, $g_{\mu\mu} = 1$, if $\mu = 1, 2, 3$, and $g_{44} = -1$.

tion $F(x''', x'', x') = F^*(x', x'', x''')$. The introduction of a form function in L_i corresponds to a kind of interaction which has no propagation character, and it is important that such effects should remain limited to small domains. Because of its Lorentz invariance F will remain finite for arbitrarily large distances of the points x', x'', x''' as long as they remain near the light cones of one another. Under certain conditions, however, compensations can occur in the neighborhood of the light cones in such a way that the corresponding volume elements do not contribute appreciably to the integral (1,1)[1]. A quantitative study of this effect will be made in section 2, and the conditions which F must satisfy will be established.

It may be of some interest to show that YUKAWA's non-local field theory leads to an interaction of the form (1,1) with a particular form function F. We may take, for instance, a non-local field U interacting with a conventional field ψ. The field U is a function of two points x' and x''' in space-time[2], and the field equations can be deduced from a variation principle involving the interaction term

$$L_i = g\int dx' dx''' \psi^+(x')(x' \,|\, U \,|\, x''')\,\psi(x''') . \tag{1,2}$$

The field U can be represented by the Fourier integral

$$(x' \,|\, U \,|\, x''') = \int dk\, a(k)\, e^{ikX}\, \delta(kr)\, \delta(r^2 - \lambda^2),$$

where $X = (1/2)(x' + x''')$ and $r = x' - x'''$. If we associate with the field U the local field

$$u(x'') = \int dk\, a(k)\, e^{ikx''},$$

we can write (1,2) in the form of (1,1) with

$$F(x', x'', x''') = (2\pi)^{-4}\int dk\, e^{ik(X-x'')}\, \delta(kr)\, \delta(r^2 - \lambda^2).$$

Detailed investigation of this form function, however, shows that it does not yield convergent self-energies[8].

The non-localized interaction is also connected with the field theories involving higher order equations considered by several authors and systematically investigated by PAIS and UHLENBECK[9]. The general type of these equations is

$$f(\Box)u(x) = \varrho(x), \tag{1,3}$$

where $\varrho(x)$ is the source of the field. If f is an analytical function it can be factorized and each factor corresponds to a possible mass of the particles described by the field u. Theories with more than one mass, however, should be rejected because they introduce negative energies. The only acceptable equations (1,3) are then of the form

$$e^{f(\Box)}(\Box - m^2)u(x) = \varrho(x). \tag{1,4}$$

The differential operator $e^{f(\Box)}$ has an inverse and we can write (1,4) in the equivalent form

$$(\Box - m^2)u(x) = e^{-f(\Box)}\varrho(x) = \int dx'\, G(x-x')\varrho(x'), \tag{1,5}$$

where

$$G(x) = (2\pi)^{-4}\int dk\, e^{-f(-k^2)}\, e^{ikx}. \tag{1,6}$$

This is the equation which would be obtained with the interaction (1,1) and a form function $F = G(x'' - x')\,\delta(x' - x''')$. The possibility of transforming an equation such as (1,3) into an equation of the form (1,5) shows that one has to eliminate certain types of form functions corresponding to the introduction of particles with different masses and negative energy. If the Fourier transform of G has poles, G can be written

$$G(x) = (2\pi)^{-4}\int dk\, e^{ikx}\, g(-k^2) / \prod_i (k^2 + m_i^2), \tag{1,7}$$

and the equation (1,5) is equivalent to the multi-mass equation

$$\prod_i (-\Box + m_i^2)(\Box - m^2)\, u(x) = g(\Box)\varrho(x).$$

The function (1,7), however, behaves for large x like $|x^2|^{-3/4}$ as does every propagation function (the flux of the square of the function through a given solid angle is independent of the distance). The occurrence of additional masses will then be avoided if it is specified that the form functions should fall off for large x faster than propagation functions.

The field equations are deduced from the Lagrange function by the variation principle. Because of the introduction of a form function in the interaction the field equations are not ordinary differential equations, and the values of the field functions at

$t + dt$ are not simply defined in terms of the values at t. Consequently, the conservation equations do not hold in their differential form. It will be shown, however, that conservation laws hold in the large, in the sense that energy, momentum, angular momentum and electric charge at a time t, before any collision has taken place, are equal to the corresponding quantities after collision.

The quantization can be performed by postulating that the

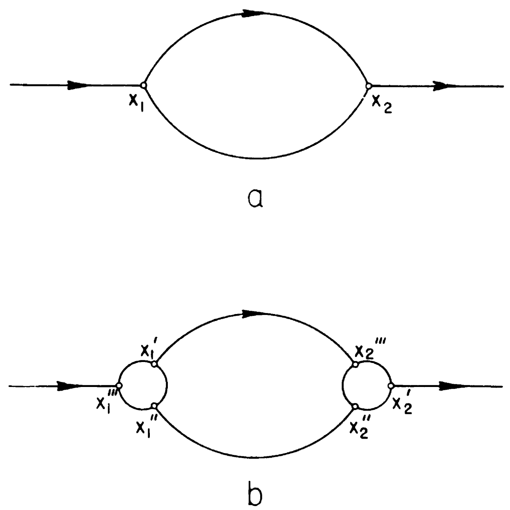

Fig. 1.

asymptotic values of the fields for $t = -\infty$ and $t = +\infty$ (called the incoming and the outgoing fields) satisfy the usual commutation relations of the free fields. It must then be shown that these commutation relations are consistent with the field equations. This can be done by using the fact that the constants of collision: energy, momentum, etc. . . . define the infinitesimal canonical transformations corresponding to the infinitesimal translations etc. . . . The S-matrix is then defined as the matrix which transforms the incoming fields into the outgoing fields.

Any outgoing operator can in principle be computed from the field equations as a power series of the incoming operators. The calculations are simplified by a set of rules similar to FEYNMAN's rules for electrodynamics[10]. These rules are used for an investigation of the convergence of the self-energies to all orders. The way in which convergence results from the introduction of a form function in the interaction can easily be seen on the second order self-energy. The graph corresponding to the conventional field theory is represented on Fig. 1 a. To the lines going from x_1 to x_2 correspond functions of $x_1 - x_2$ which are singular on the light cone, and a divergence arises from the fact that the self-energy integral involves a product of two functions becoming singular at the same points. The small circles on Fig. 1 b correspond to the introduction of form functions $F(x', x'', x''')$, and it is seen that the divergence will disappear if F is a smooth function of $x' - x''$ and $x'' - x'''$. A rigorous treatment requires the use of the energy-momentum space. However, it can already be seen that the convergence of the self-energies of both types of particles requires that F be a smooth function of all three variables.

There is then a little difficulty when the interactions with the electromagnetic field are taken into account since the interaction term (1,1) is not gauge invariant. It will be shown, however, that a supplementary interaction term can be added to (1,1) in such a way that the sum is gauge invariant. This term describes the current due to the jumping of the charge between the points x_1''' and x_1', x_2''' and x_2'.

2. The form functions.

In this section we shall investigate under which conditions the non-localizability of the interaction is limited to dimensions of the order of a given length λ. We consider first a simple case:

A. *Functions of two points*(1). In the conventional theory with a localized interaction the function F is a product of two four-dimensional Dirac functions: $F(x', x'', x''') = \delta(x' - x'')\, \delta(x'' - x''')$. As a first generalization we shall assume that F contains only one four-dimensional Dirac function: $F = \delta(\alpha' x' + \alpha'' x'' + \alpha''' x''')\, G$, where the scalar constants α', α'', α''' satisfy the relation $\alpha' + \alpha'' + \alpha''' = 0$. The factor G can be expressed as a function of two points only, x' and x'', for instance, if $\alpha''' \neq 0$. The invariance under translations and Lorentz transformations requires that G should be a function of $s = (x' - x'')^2$(*).

(*) For $s \leq 0$ the function G can also take two different values for the same value of s depending on whether $x_4' - x_4''$ is positive or negative. We shall come back to this later.

We shall now investigate under which conditions the form factor becomes very small as soon as x' and x'' are not very near one another. More precisely, considering the integral

$$I = \int dx' G(s) f(x'), \tag{2,1}$$

where $f(x')$ is an arbitrary smooth function, it should depend only on the values of $f(x')$ for x' very near x''. A first condition to be fulfilled is that $G(s)$ should fall off very rapidly as $|s|$ becomes much larger than λ^2. This condition, however, is not sufficient as $G(s)$ remains finite for x' near the light cone of x''. Thus, the contribution to I coming from the volume elements which are far from x'', but near the light cone of x'', requires a special investigation.

It is convenient to introduce the point x_0 of the light cone of x'' which is near x' and has the same three first coordinates. We call a the three-dimensional length of $x' - x''$ or $x_0 - x''$. We have $x_0^4 - x''^4 = \varepsilon a$, where ε is $+1$ or -1 depending on whether $x_0^4 - x''^4$ is positive or negative. The distance of x' to the light cone is conveniently defined by $\xi = \varepsilon (x_0^4 - x'^4)$. The relation between s and ξ is

$$s = 2 a \xi - \xi^2. \tag{2,2}$$

It shows that for large a a small variation of ξ corresponds to a large variation of s. As G is very small for large values of $|s|$, it follows that for large a we can expand the function f in powers of ξ around the light cone and extend the integration with respect to x'^4 or ξ from $-\infty$ to $+\infty$.

As we are interested in orders of magnitude only we shall omit all numerical coefficients. The Taylor expansion of f around the light cone reads

$$f(x') = f(\boldsymbol{x}_0, x_0^4 - \varepsilon\xi) \cong \sum_0^\infty \xi^k f_0^k,$$

where $f_0^k = (\partial/\partial x'^4)^k f(x')$ taken at the point $x' = x_0$. Finally, we replace the variable of integration x'^4 by s. We have

$$dx'^4 = ds/2(a-\xi) \cong (ds/a) \sum_0^\infty (\xi/a)^m,$$

and from (2,2) we deduce

$$\xi = a - (a^2 - s)^{1/2} \cong (s/a) \sum_0^\infty (s/a^2)^n.$$

Using all the preceding expansions the contribution to I of the neighborhood of the light cone of x'' can be written

$$\left.\begin{aligned} &\int dx' G(s) f(x') \cong \int d\boldsymbol{x}' ds \sum_0^\infty \xi^{k+m} G(s) f_0^k / a^{m+1} \cong \\ &\int d\boldsymbol{x}' ds \sum_0^\infty (s/a)^{k+m} (s/a^2)^n G(s) f_0^k / a^{m+1} \cong \int d\boldsymbol{x}' \sum_0^\infty M_{m+k} f_0^k / a^{2m+k+1} \end{aligned}\right\} \quad (2,3)$$

where the M_n are the "moments" of the function G defined by

$$M_n = \int_{-\infty}^{+\infty} ds s^n G(s).$$

The formula (2,3) shows how the contributions to I coming from the neighborhood of the light cone of x'' decrease with increasing distance a. Namely, if

$$M_0 = M_1 = M_2 = \cdots = M_{p-1} = 0, \quad M_p \neq 0, \qquad (2,4)$$

the contributions decrease as $(1/a)^{p+1}$. As the volume element $d\boldsymbol{x}'$ is proprotional to $a^2 da$, it is seen that the integral I extended to the whole space-time is convergent for any bounded function f with bounded derivatives if $p \geqslant 3$. The integral $\int dx' G(s)$ is convergent for $p \geqslant 2$.

It should be noted that integrals such as I are usually not absolutely convergent. The convergence is due to cancellations arising within the volume elements which are near the light cone of x''. In calculating such integrals, one must always use a method allowing these cancellations to take place. For instance, one can start by restricting the domain of integration to a finite part of space-time enclosed within a closed surface Σ, and then let Σ go to infinity. It is easily seen that the cancellations which make the integral I convergent will take place if the angle under which Σ cuts the light cone tends nowhere to zero. The possibility of defining in a Lorentz invariant way an integral which is not absolutely convergent clearly comes from the fact that the cancellations making the integral convergent take place within layers along the light cone which become infinitely narrow at infinity.

If for $s \leqslant 0$ the form function takes different values depending on the sign of x^4, it is convenient to write G as a sum of an even function G_+ which is invariant under the substitution $x \rightarrow -x$, and of an odd function G_- which changes sign under the same substitution. It follows from the relativistic invariance that G_- must vanish for $s > 0$. It is easily seen that the functions G_+ and G_- must satisfy the conditions (2,4) independently.

For many calculations it is more appropriate to represent G by a Fourier integral

$$G(x) = \int dk\, e^{ikx} g(k), \tag{2,5}$$

where $g(k)$ is a function of the argument $q = k^2$, and can be represented by a sum of an even function g_+ and an odd function g_-. The Fourier transformation (2,5) gives then G_+ in terms of g_+, and G_- in terms of g_-. We shall now investigate which conditions must be satisfied by g_+ and g_- in order that the corresponding G should be an acceptable form function. This requires a closer investigation of the correspondence between G and g given by (2,5).

The integration in (2,5) with respect to the angular orientation of $\boldsymbol{k}$ yields

$$\begin{aligned} G(x) &= \frac{4\pi}{a}\int_{-\infty}^{+\infty} l\,dl \int_{-\infty}^{+\infty} dk^4 \operatorname{Sin} la\, e^{-ik^4x^4} g(k) \\ &= -\frac{4\pi}{a}\frac{\partial}{\partial a}\int_{-\infty}^{+\infty} dl \int_{-\infty}^{+\infty} dk^4 \operatorname{Cos} la\, e^{-ik^4x^4} g(k) \\ &= -\frac{2\pi}{a}\frac{\partial}{\partial a}\int_{-\infty}^{+\infty} dl \int_{-\infty}^{+\infty} dk^4 e^{i(la - k^4x^4)} g(k), \end{aligned}$$

where $a = |\boldsymbol{x}|$, and $|l| = |\boldsymbol{k}|$. Introducing now the decomposition of G and g into even and odd parts we get

$$\left.\begin{aligned} G_+(s) &= -\frac{2\pi}{a}\frac{\partial}{\partial a}\iint_{-\infty}^{+\infty} dl\, dk^4 e^{i(la - k^4x^4)} g_+(q), \\ G_-(s) &= -\frac{2\pi}{a}\frac{\partial}{\partial a}\iint_{-\infty}^{+\infty} dl\, dk^4 e^{i(la - k^4x^4)} \varepsilon(k^4)\, g_-(q), \end{aligned}\right\}$$

where $s = x^2$, $q = k^2$, and $\varepsilon(k^4) = k^4/|k^4|$. It should be noted that the precise definition of $G_-(s)$ is

$$G_-(x) = G_-(s) \text{ if } x^4 > 0, \quad G_-(x) = -G_-(s) \text{ if } x^4 < 0.$$

A similar definition holds for $g_-(q)$. Replacing the variables of integration l and k^4 by

$$q = l^2 - (k^4)^2, \text{ and } \alpha = \frac{l-k^4}{a-x^4} \text{ if } a \neq x^4, \quad \text{or } \alpha = \frac{l+k^4}{a+x^4} \text{ if } a = x^4,$$

we get after some simple manipulations

$$G_+(s) = -\frac{\pi}{a}\frac{\partial}{\partial a}\iint\limits_{-\infty}^{+\infty}\frac{d\alpha\, dq}{\alpha}\,\varepsilon(\alpha)\,e^{(i/2)(\alpha s+q/\alpha)}\,g_+(q),$$
$$G_-(s) = -\frac{\pi}{a}\frac{\partial}{\partial a}\iint\limits_{-\infty}^{+\infty}\frac{d\alpha\, dq}{\alpha}\,e^{(i/2)(\alpha s+q/\alpha)}\,g_-(q).$$

As $\frac{1}{a}\frac{\partial}{\partial a} = 2\frac{\partial}{\partial s}$, we can also write

$$G_+(s) = -i\pi\iint\limits_{-\infty}^{+\infty} d\alpha\, dq\,\varepsilon(\alpha)\,e^{(i/2)(\alpha s+q/\alpha)}\,g_+(q),$$
$$G_-(s) = -i\pi\iint\limits_{-\infty}^{+\infty} d\alpha\, dq\, e^{(i/2)(\alpha s+q/\alpha)}\,g_-(q)^{(*)}. \quad (2,6)$$

The formulas (2,6) may be interpreted in the following manner. G is obtained from g by three successive transformations:

a) the Fourier transformation

$$\varphi(\beta) = \int_{-\infty}^{+\infty} dq\, e^{(i/2)\beta q}\, g(q),$$

b) the transformation

$$\psi_+(\alpha) = \varepsilon(\alpha)\varphi_+(1/\alpha), \text{ or } \psi_-(\alpha) = \varphi_-(1/\alpha),$$

c) the Fourier transformation

$$G(s) = -i\pi\int_{-\infty}^{+\infty} d\alpha\, e^{(i/2)\alpha s}\,\psi(\alpha).$$

(*) In this formula $g_-(q) = 0$ for $q > 0$. It follows then from Cauchy's theorem applied to the integration with respect to α that $G_-(s) = 0$ for $s > 0$.

We shall use this decomposition to find the properties of g sufficient that G defined by (2,5) shall be an acceptable form function.

The conditions which must be satisfied by G are:

$$\left.\begin{array}{l} 1) \text{ to be continuous;} \\ 2) \text{ to go to zero as } |s| \to \infty \text{, for instance as } (1/s)^k; \\ 3) \int_{-\infty}^{+\infty} ds\, s^n G(s) = 0, \text{ for } n = 0, 1, \ldots, p-1. \end{array}\right\} \quad (2,7)$$

As regards the condition 2), one might require that $G(s)$ should tend to zero much faster than we assume here, exponentially for instance. It seems, however, natural to require only that G behaves for large s in such a way that its integral over the whole space-time is convergent. The condition 2) with $k \geq 3$ is then sufficient. The convergence of the moments involved in the condition 3) requires in fact $k \geq p + 1$(*). The condition assumed here seems natural in view of the fact that the contribution to the integral (2,1) coming from the neighborhood of the light cone never decreases faster than an inverse power of the distance.

Sufficient conditions for $\psi(\alpha)$ corresponding to (2,7) are that ψ must have k continuous derivatives such that

$$\left.\begin{array}{ll} 1) \int_{-\infty}^{+\infty} d\alpha\, |\psi(\alpha)| < \infty; & \\ 2) \int_{-\infty}^{+\infty} d\alpha\, |\psi^{(n)}(\alpha)| < \infty, & \text{for } n = 1, 2, \ldots, k^{(**)}; \\ 3) \psi^{(n)}(0) = 0, & \text{for } n = 0, 1, \ldots, p-1. \end{array}\right\} \quad (2,8)$$

The derivatives of ψ (with respect to α) are given in terms of the derivatives of φ (with respect to β) by the formula

$$\psi^{(n)}(\alpha) \simeq \sum_{m=1}^{m=n} \beta^{m+n} \varphi^{(m)}(\beta), \quad (2,9)$$

where $\beta = 1/\alpha$, and where the numerical coefficients have been

(*) The expansion (2,3) requires the existence of moments of all orders i. e. an exponential decrease of G for large s. The whole argument, however, can be carried through by means of limited asymptotic expansions only. The condition we have assumed is then sufficient.

(**) Here we make use of a well-known theorem on the aysmptotic value of Fourier integrals; see for instance, S. Bochner, Fouriersche Integrale, Chelsea Publishing Co., New York, p. 11.

omitted. The presence in the relation between φ_+ and ψ_+ of the factor $\varepsilon(\alpha)$ which has a discontinuous variation at $\alpha = 0$ does not modify the equation (2,9) since the function ψ and all its derivatives involved here vanish at $\alpha = 0$ (this follows from 3 for the $p - 1$ first derivatives and for the derivatives of the order $p, p + 1, \ldots, k$ from the behavior of $\varphi^{(m)}$ at infinity as indicated below).

It is seen from (2,9) that we must assume that φ has k continuous derivatives. From the condition 1) it follows that φ must be such that

$$\int_{-\infty}^{+\infty} \frac{d\beta}{\beta^2} \left| \varphi(\beta) \right| < \infty .$$

This condition is satisfied if we assume that φ is bounded for $\beta = \pm \infty$, is regular at $\beta = 0$, and that $\varphi(0) = \varphi'(0) = 0$. Finally, it is easily seen that the conditions 2) are satisfied if we assume that $\varphi^{(m)}(\beta)$ $(m = 1, 2, \ldots, k)$ behaves at infinity as $(1/\beta)^{m+k}$. The conditions 3) are then automatically satisfied. From the relations $\varphi(0) = \varphi'(0) = 0$ it follows that

$$\int_{-\infty}^{+\infty} dq g(q) = \int_{-\infty}^{+\infty} dq q g(q) = 0 . \qquad (2,10)$$

On the other hand,

$$\varphi^{(m)}(\beta) \cong \int_{-\infty}^{+\infty} dq e^{(i/2)\beta q} q^m g(q);$$

and this function behaves at infinity as $(1/\beta)^{m+k}$ if $q^m g(q)$ has $m + k$ continuous derivatives absolutely integrable from $-\infty$ to $+\infty$ (Bochner loc. cit.). As m takes the values $1, 2, \ldots, k$ we are led to the following conditions:

$g(q)$ is continuous and has $2k$ continuous derivatives;
$g^{(n)}(q)$ $(n = 0, 1, \ldots, 2k)$ goes to zero as $q \to \pm\infty$ faster than $(1/q)^{k+1}$. (2,11)

The condtitions (2,10 and 11) are sufficient to insure that $G(s)$ satisfies (2,7). The function $g_-(q)$ vanishes for $q > 0$. It follows then from the continuity of the $2k$ first derivatives that

$$g_-^{(n)}(0) = 0, \text{ for } n = 0, 1, \ldots, 2k. \qquad (2,12)$$

The conditions (2, 10 and 11) allow us to choose functions $g(q)$ which vanish outside a certain interval. In such a case, $g(q)$ and its $2k$ first derivatives must vanish at the ends of the interval.

The function $G_-(s)$ can be expressed in terms of the usual function D of field theory. Performing in (2,5) first the integration at q constant, and then the integration over q, we get indeed

$$G_-(s) = (2\pi)^3 i \int_{-\infty}^{0} dq\, g_-(q) D(s, q), \tag{2,13}$$

where $D(s, q)$ is the function corresponding to the mass $\sqrt{-q}$. If we assume that $g_+(q)$ is different from zero only if $q < 0$ (which implies that $g_+^{(n)}(0) = 0$, for $n = 0, 1, \ldots, 2k$), we can, similarly, express $G_+(s)$ in terms of $D^{(1)}$(11):

$$G_+(s) = (2\pi)^3 \int_{-\infty}^{0} dq\, g_+(q) D^{(1)}(s, q). \tag{2,14}$$

The expressions (2,13 and 14) are identical with those used in the theory of regularization(12). The relations (2,10) also belong to the latter theory. They express the condition that the singularities of the functions D and $D^{(1)}$ at $s = 0$ should not appear in $G(s)$. The conditions (2,11), however, are in contradiction with the limiting process used in the idealistic renormalization, or with the introduction of a discrete set of masses. Consequently, the behavior for large s of a form function is essentially different from that of a regularized function.

It may be noted, finally, that the transformations considered in this section are special cases of the Fourier-Bessel transformation(13). The transformation of the odd functions, for instance, can be written

$$r G_-(r) = 2 i \pi^2 \int_{0}^{+\infty} \varkappa\, d\varkappa\, J_1(\varkappa r)\, \varkappa\, g_-(\varkappa),$$

where $r = \sqrt{-s}$, and $\varkappa = \sqrt{-q}$.

B. *Functions of three points.* As it was shown in section 1, the form function should actually be a smooth function of all three variables. It will then be a function of the invariants(*)

$$s = (x'' - x''')^2, \qquad t = (x''' - x')^2, \qquad u = (x' - x'')^2.$$

(*) These invariants are not entirely independent. No triangle x', x'', x''' exists if s, t and u are negative and if $s^2 + t^2 + u^2 - 2st - 2tu - 2us < 0$.

First of all, the form function should fall off rapidly as s, t, or u becomes large (strictly speaking, one could also require that F falls off as any of two only of the quantities s, t, u becomes large). There are, however, large triangles x', x'', x''' for which s, t and u are small. The contribution to an integral such as

$$I = \int dx'\, dx''\, dx'''\, F f(x', x'', x''') \qquad (2,15)$$

coming from such triangles can be investigated by the same method as for the functions of two variables. Let a, b and c be the lengths of the space parts of $x'' - x'''$, $x''' - x'$ and $x' - x''$, respectively, and suppose that $a \geqslant b \geqslant c$. For a large triangle a and b at least will be large compared with λ. Let x_0 and x_1 be the points of the light cone of x''' which are near x' and x'', and have the same space coordinates. We have

$$x_1^4 - x'''^4 = \varepsilon a, \qquad x_0^4 - x'''^4 = \varepsilon b, \qquad \varepsilon = \pm 1.$$

Introducing the distances of x' and x'' to the light cone of x''' by

$$\xi = \varepsilon (x_1^4 - x''^4), \qquad \eta = \varepsilon (x_0^4 - x'^4), \qquad (2,16)$$

we have

$$s = 2a\xi - \xi^2, \qquad t = 2b\eta - \eta^2. \qquad (2,17)$$

Again, we can carry out the integration in I with respect to x'^4 and x''^4 using Taylor expansions around the light cone of x'''. It is then convenient to replace the variables x'^4 and x''^4 by s and t with the help of (2,16) and (2,17). An additional complication comes from the fact that u is now a function of s and t since all three quantities are functions of x'^4 and x''^4 (or ξ and η). It is readily found that

$$u = c^2 - (a-b)^2 + (a-b)\left(\frac{s}{a} - \frac{t}{b}\right) + (a-b)\left(\frac{\xi^2}{a} - \frac{\eta^2}{b}\right) - (\xi - \eta)^2,$$

which shows that when the triangle becomes large the quantity $q = c^2 - (a-b)^2$ must remain finite. It is one of the parameters which define the way in which the triangle is increasing. As other parameter we can take $a/b = \mu$, and we have then

$$u = (1-\mu)(t - s/\mu) + q + \ldots,$$