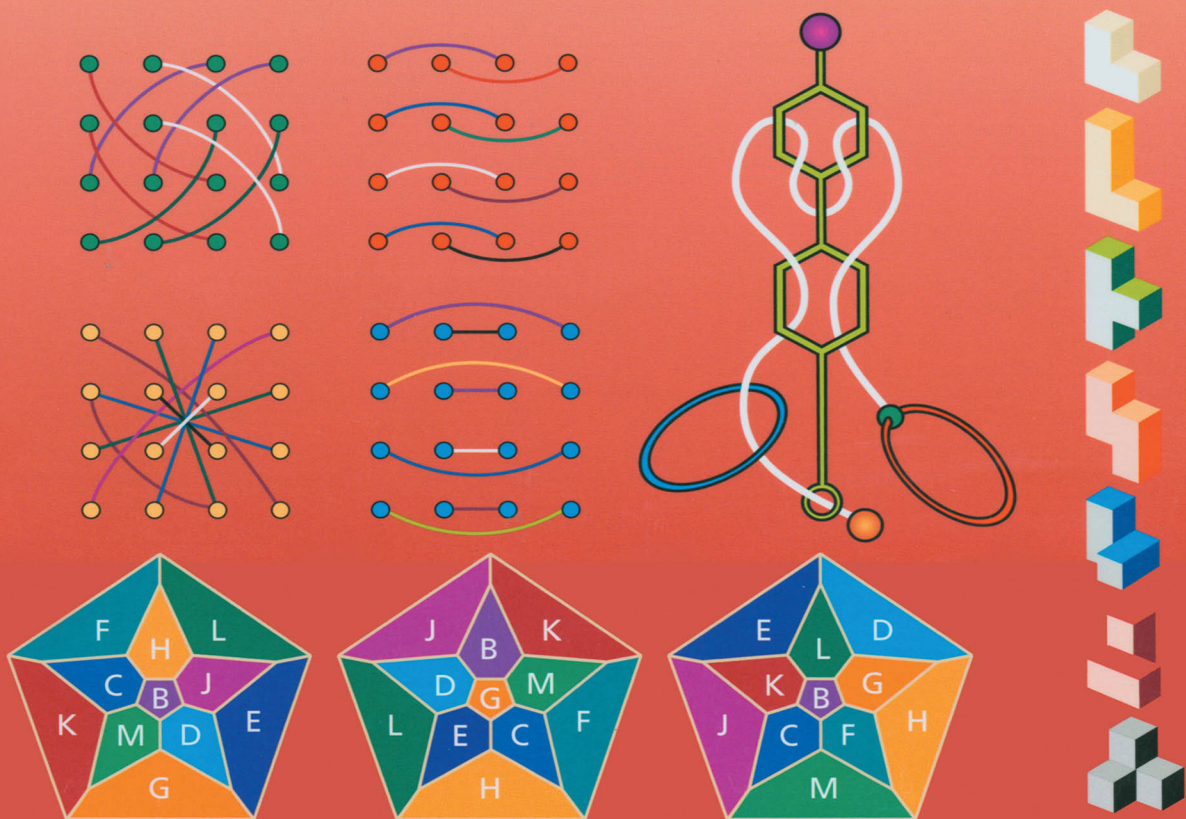


VOLUME 4

S E C O N D E D I T I O N

WINNING WAYS

FOR YOUR MATHEMATICAL PLAYS



ELWYN R. BERLEKAMP • JOHN H. CONWAY • RICHARD K. GUY

 CRC Press
Taylor & Francis Group
AN A K PETERS BOOK

Winning Ways for Your Mathematical Plays, Volume 4



Taylor & Francis

Taylor & Francis Group

<http://taylorandfrancis.com>

Winning Ways

for Your Mathematical Plays



Volume 4, Second Edition

Elwyn R. Berlekamp, John H. Conway, Richard K. Guy



CRC Press

Taylor & Francis Group

Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

AN A K PETERS BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

First issued in hardback 2017

© 2004 by Taylor & Francis Group, LLC
CRC Press is an imprint of Taylor & Francis Group, an Informa business

No claim to original U.S. Government works

ISBN-13: 978-1-56881-144-4 (pbk)
ISBN-13: 978-1-138-42755-6 (hbk)

This book contains information obtained from authentic and highly regarded sources. Reasonable efforts have been made to publish reliable data and information, but the author and publisher cannot assume responsibility for the validity of all materials or the consequences of their use. The authors and publishers have attempted to trace the copyright holders of all material reproduced in this publication and apologize to copyright holders if permission to publish in this form has not been obtained. If any copyright material has not been acknowledged please write and let us know so we may rectify in any future reprint.

Except as permitted under U.S. Copyright Law, no part of this book may be reprinted, reproduced, transmitted, or utilized in any form by any electronic, mechanical, or other means, now known or hereafter invented, including photocopying, microfilming, and recording, or in any information storage or retrieval system, without written permission from the publishers.

For permission to photocopy or use material electronically from this work, please access www.copyright.com (<http://www.copyright.com/>) or contact the Copyright Clearance Center, Inc. (CCC), 222 Rosewood Drive, Danvers, MA 01923, 978-750-8400. CCC is a not-for-profit organization that provides licenses and registration for a variety of users. For organizations that have been granted a photocopy license by the CCC, a separate system of payment has been arranged.

Trademark Notice: Product or corporate names may be trademarks or registered trademarks, and are used only for identification and explanation without intent to infringe.

Visit the Taylor & Francis Web site at
<http://www.taylorandfrancis.com>

and the CRC Press Web site at
<http://www.crcpress.com>

To Martin Gardner

who has brought more mathematics to more millions than anyone else



Taylor & Francis

Taylor & Francis Group

<http://taylorandfrancis.com>



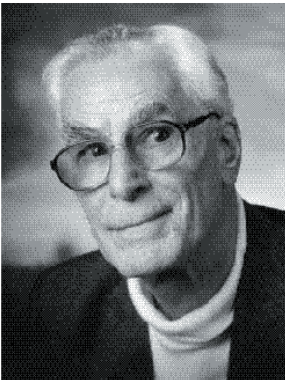
Elwyn Berlekamp was born in Dover, Ohio, on September 6, 1940. He has been Professor of Mathematics and of Electrical Engineering/Computer Science at UC Berkeley since 1971. He has also been active in several technology business ventures. In addition to writing many journal articles and several books, Berlekamp also has 12 patented inventions, mostly dealing with algorithms for synchronization and error correction.

He is a member of the National Academy of Sciences, the National Academy of Engineering, and the American Academy of Arts and Sciences. From 1994 to 1998, he was chairman of the board of trustees of the Mathematical Sciences Research Institute (MSRI).



John H. Conway was born in Liverpool, England, on December 26, 1937. He is one of the preeminent theorists in the study of finite groups and the mathematical study of knots, and has written over 10 books and more than 140 journal articles.

Before joining Princeton University in 1986 as the John von Neumann Distinguished Professor of Mathematics, Conway served as professor of mathematics at Cambridge University, and remains an honorary fellow of Caius College. The recipient of many prizes in research and exposition, Conway is also widely known as the inventor of the Game of Life, a computer simulation of simple cellular “life,” governed by remarkably simple rules.



Richard Guy was born in Nuneaton, England, on September 30, 1916. He has taught mathematics at many levels and in many places—England, Singapore, India, and Canada. Since 1965 he has been Professor of Mathematics at the University of Calgary, and is now Faculty Professor and Emeritus Professor. The university awarded him an Honorary Degree in 1991. He was Noyce Professor at Grinnell College in 2000.

He continues to climb mountains with his wife, Louise, and they have been patrons of the Association of Canadian Mountain Guides' Ball and recipients of the A. O. Wheeler award for Service to the Alpine Club of Canada.



Taylor & Francis

Taylor & Francis Group

<http://taylorandfrancis.com>

Contents

23 Purging Pegs Properly	803
Central Solitaire	804
Dudeney, Bergholt and Beasley	805
Packages and Purges	807
Packages Provide Perfect Panacea	809
The Rule of Two and the Rule of Three	811
Some Pegs Are More Equal Than Others	812
Reiss's 16 Solitaire Position Classes	814
The Continental Board	817
Playing Backwards and Forwards	817
Pagoda Functons	818
The Solitaire Army	821
Managing Your Resources	823
Unproductivity and the Prodigal Son	825
Deficit Accounting and the G.N.P.	826
Accounting for Two-Peg Reversal Problems	826
Forgetting the Order Can Be Useful	827
Beasley's Exit Theorems	829
A Stolid Survivor Problem	829
Another Hard Problem	831
The Spinner	833
Our Fine Finalist	834
Doing the Splits	834
All Soluble One-Peg Problems on the Continental Board	835
The Last Two Moves	835
A 20-Man Solitaire Army	835
Fool's Solitaire, Etc.	835

Beasley Proves Bergholt Is Best	837
The Classical Problems	839
References and Further Reading	840
24 Pursuing Puzzles Purposefully	843
Soma	843
Blocks-in-a-Box	844
Hidden Secrets	844
The Hidden Secrets of Soma	845
Hoffman's Arithmetico-Geometric Puzzle	847
Coloring Three-by-Three-by-Three by Three, Bar Three	848
Wire and String Puzzles	849
The Magic Mirror Method	849
The Barmy Braid	853
The Artful Arrow	854
The Magic Movie Method	854
Party Tricks and Chinese Rings	856
Chinese Rings and the Gray Code	858
The Tower of Hano	861
A Solitaire-Like Puzzle and Some Coin-Sliding Problems	863
The Fifteen Puzzle and the Lucky Seven Puzzle	864
All Other Courses for Point-to-Point	867
The Hungarian Cube-Bűvös Kocka	868
Just How Chaotic Can the Cube Get?	869
Chief Colors and Chief Faces	869
Curing the Cube	871
A: Aloft, Around (Adjust) and About	872
B: Bottom Layer Corner Cubelets	872
C: Central Layer Edge Cubelets	872
D: Domiciling the Top Edge Cubelets	872
E: Exchanging Pairs of Top Corners	874
F: Finishing Flips and Fiddles	874
Explanations	874
Improvements	875
Elena's Elements	876
Are You Partial to Partial Puzzles?	876
Other "Hungarian" Objects	876
A Trio of Sliding Block Puzzles	877
Tactics for Solving Such Puzzles	878
Counting Your Moves	885
Paradoxical Pennies	885
Paradoxical Dice	886
More on Magic Squares	886
The Magic Tesseract	891

Adams's Amazing Magic Hexagon	892
The Great Tantalizer	892
Polyominoes, Polyiamonds and Searching Policy	894
Alan Schoen's Cyclotome	897
Macmahon's Superdominoes	899
Quitominal Dodecahedra	900
The Doomsday Rule	903
... and Easter Easily	905
How Old is the Moon?	907
Jewish New Year (Rosh Hashana)	908
Blocks-in-a-Box	910
The Somap	910
Solutions to the Arithmetico-Geometric Puzzle	913
... and One for "Three" Too!	916
Hares and Tortoises	916
The Lucky Seven Puzzle	916
Top Face Alterations for the Hungarian Cube	917
The Century Puzzle	919
Adams's Amazing Magic Hexagon	919
Flags of the Allies Solution	920
All Hexiamond Solutions Found	920
The Three Quintominal Dodecahedra	921
Answer to Exercise for Experts	921
Where Do the Black Edges of Magmahon Squares Go?	921
Doomsday Answers	922
References and Further Reading	923
25 What is Life?	927
Still Life	929
Life Cycles	930
The Glider and Other Space Ships	931
The Unpredictability of Life	934
Gardens of Eden	938
Life's Problems are Hard!	939
Making a Life Computer	940
When Glider Meets Glider	941
How to Make a not Gate	942
The Eater	943
Gliders Can Build Their Own Guns!	947
The Kickback Reaction	947
Thinning a Glider Stream	947
Building Blocks for Our Computer	948
Auxiliary Storage	951
How We Move Blocks	952

A Little Difficulty 954
Mission Completed—Will Self-Destruct 955
Life Computers Can Reproduce! 958
Genetic Engineering 958
Whither Life? 958
References and Further Reading 959

Preface to the Second Edition

In the first edition of *Winning Ways*, which appeared in 1982, we were able to make a rather sharp distinction between those games in Part I, to which the major theory of addition applied directly, and those games in Part 3, which seemed to require more specialized techniques. However, subsequent research by an increasingly large community of combinatorial game theorists has begun to blur this distinction. We now have many more games whose strategies depend both on the general theory of Volume 1 as well as on more specialized results. Introductions to many of these games and some illustrative problems have been added to this new edition. Those that did not readily fit elsewhere can be found in the new Extras to Chapter 22 at the end of this volume. This volume also includes a major revision of the original Chapter 20 on the game of Fox and Geese. Its enhanced variation, Fox-Flocks-Fox, provides compelling illustrations of some of the challenging problems that can now be solved by appropriately combining theories from Volumes 1, 2, and 3 with innovative computing algorithms.

This new edition owes much to the supportive efforts of numerous friends and colleagues, including Noam Elkies, Tom Ferguson, Aviezri Fraenkel, Martin Gardner, Sol Golomb, Al Hales, Greg Kuperberg, Silvio Levy, Donald Knuth, Martin Kutz, Greg Martin, Victor Meally, Richard Nowakowski, Hilarie Orman, Marc Paulhus, Ed Pegg, Michael Reid, Thea van Roode, Katherine Scott, George Sicherman, Aaron Siegel, Neil Sloane, Sally Smith, William Spight, John Tromp, Jonathan Welton, Julian West, David Wilson, and David Wolfe, and to the very professional yet kindly support of our publishers, Alice and Klaus Peters.

Elwyn Berlekamp, University of California, Berkeley
John Conway, Princeton University
Richard Guy, The University of Calgary, Canada

June 23, 2003

Preface to the Original Edition

Does a book need a Preface? What more, after fifteen years of toil, do three talented authors have to add.

We can reassure the bookstore browser, “Yes, this is just the book you want!”

We can direct you, if you want to know quickly what’s in the book, to page xx. This in turn directs you to volumes 1,2,3 and 4.

We can supply the reviewer, faced with the task of ploughing through nearly a thousand information-packed pages, with some pithy criticisms by indicating the horns of the polylemma the book finds itself on. It is not an encyclopedia. It is encyclopedic, but there are still too many games missing for it to claim to be complete. It is not a book on recreational mathematics because there’s too much serious mathematics in it. On the other hand, for us, as for our predecessors Rouse Ball, Dudeney, Martin Gardner, Kraitchik, Sam Loyd, Lucas, Tom O’Beirne and Fred. Schuh, mathematics itself is a recreation. It is not an undergraduate text, since the exercises are not set out in an orderly fashion, with the easy ones at the beginning. They are there though, and with the hundred and sixty-three mistakes we’ve left in, provide plenty of opportunity for reader participation. So don’t just stand back and admire it, work of art though it is. It is not a graduate text, since it’s too expensive and contains far more than any graduate student can be expected to learn. But it does carry you to the frontiers of research in combinatorial game theory and the many unsolved problems will stimulate further discoveries.

We thank Patrick Browne for our title. This exercised us for quite a time. One morning, while walking to the university, John and Richard came up with “Whose game?” but realized they couldn’t spell it (there are three tooze in English) so it became a one-line joke on line one of the text. There isn’t room to explain all the jokes, not even the fifty-nine private ones (each of our birthdays appears more than once in the book).

Omar started as a joke, but soon materialized as Kimberly King. Louise Guy also helped with proof-reading, but her greater contribution was the hospitality which enabled the three of us to work together on several occasions. Louise also did technical typing after many drafts had been made by Karen McDermid and Betty Teare.

Our thanks for many contributions to content may be measured by the number of names in the index. To do real justice would take too much space. Here’s an abridged list of helpers: Richard Austin, Clive Bach, John Beasley, Aviezri Fraenkel, David Fremlin, Solomon Golomb, Steve Grantham, Mike Guy, Dean Hickerson, Hendrik Lenstra, Richard Nowakowski, Anne Scott, David Seal, John Selfridge, Cedric Smith and Steve Tschantz.

No small part of the reason for the assured success of the book is owed to the well-informed and sympathetic guidance of Len Cegiela and the willingness of the staff of Academic Press and of Page Bros. to adapt to the idiosyncrasies of the authors, who grasped every opportunity to modify grammar, strain semantics, pervert punctuation, alter orthography, tamper with traditional typography and commit outrageous puns and inside jokes.

Thanks also to the the Isaak Walton Killam Foundation for Richard's Resident Fellowship at The University of Calgary during the compilation of a critical draft, and to the National (Science & Engineering) Research Council of Canada for a grant which enabled Elwyn and John to visit him more frequently than our widely scattered habitats would normally allow.

And thank you, Simon!

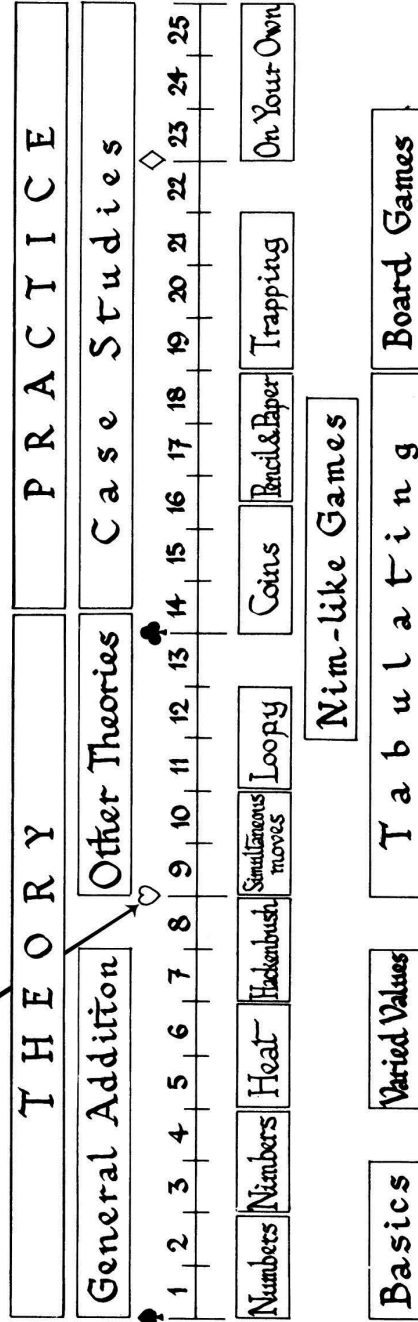
University of California, Berkeley, CA 94720 Elwyn Berlekamp
University of Cambridge, England, CB2 1SB John Conway
University of Calgary, Canada, T2N 1N4 Richard Guy

You are
now here

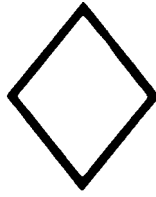
If you want to know roughly what's elsewhere,
turn to the little notes about our four main themes:

- Adding Games ... ♠ ... page 1
- Bending the Rules ... ♥ ... page 255
- Case Studies ... ♣ ... page 427
- Doing It Yourself ... ♦ ... page 695

There are a number of other connexions between various chapters of the book:



However, you should be able to pick any chapter and read almost all of it without reference to anything earlier, except perhaps the basic ideas at the start of the book.



Solitaire Diamonds!

Twinkle, twinkle little star,
How I wonder what you are!
Up above the world so high
Like a diamond in the sky!
Jane Taylor, *The Star*.

We are all in the dumps, For diamonds are trumps;
The kittens are gone to St. Paul's.
the babies are bit, the Moon's in a fit,
And the houses are built without walls.
Nursery Rhyme

If you've followed everything in *Winning Ways* so far, you're probably finding it hard to get people to play with you, so you will need something to do on your own. Here are our favorite solitaire diamonds:

The classical game of Peg Solitaire, treated by old and new methods in Chapter 23.
A host of puzzles, pastimes and other party tricks in Chapter 24.
And finally, every automaton will enjoy playing the notorious game of Life (Chapter 25).

