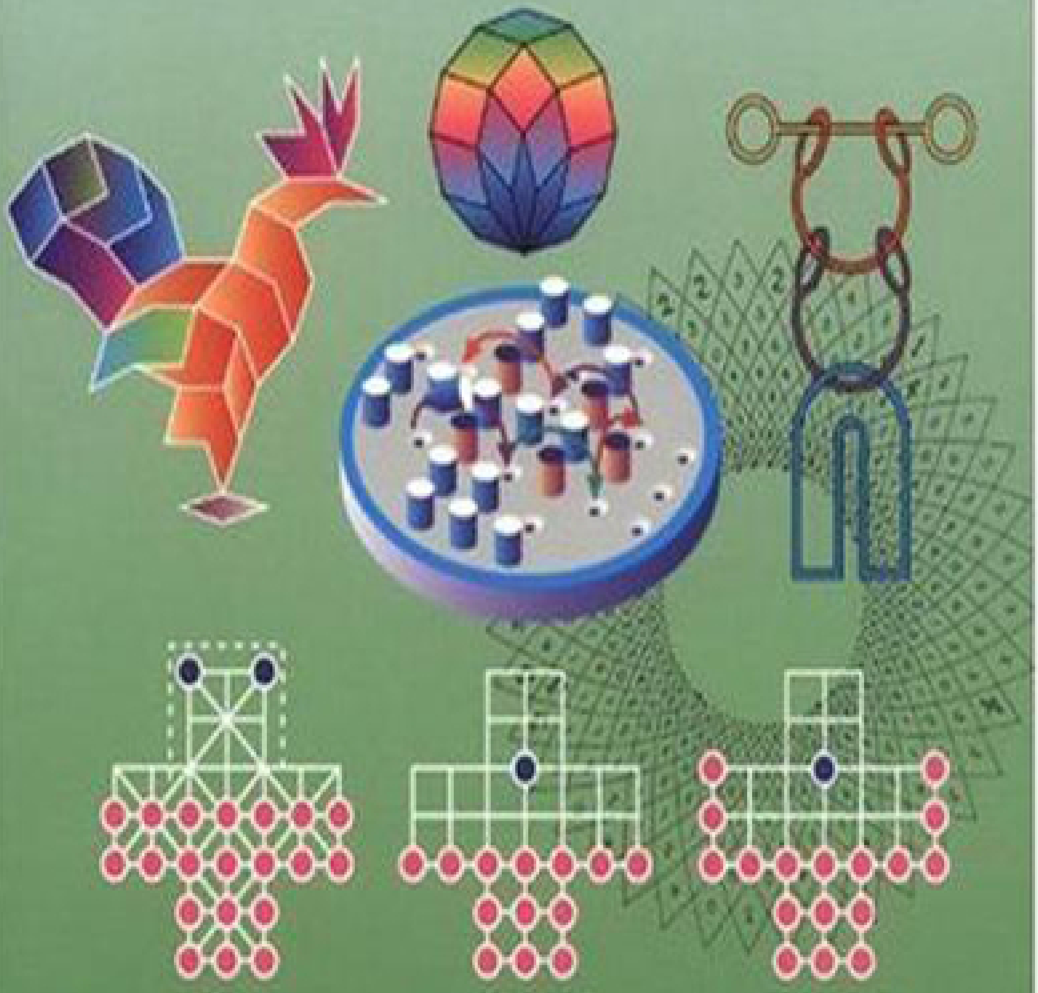
VOLUME 3 SECOND EDITION

WINNING WAYS

FOR YOUR MATHEMATICAL PLAYS



ELWYN R. BERLEKAMP • JOHN H. CONWAY • RICHARD K. GUY

Winning Ways for Your Mathematical Plays, Volume 3

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for Your Mathematical Plays

Volume 3, Second Edition

Elwyn R. Berlekamp, John H. Conway, Richard K. Guy

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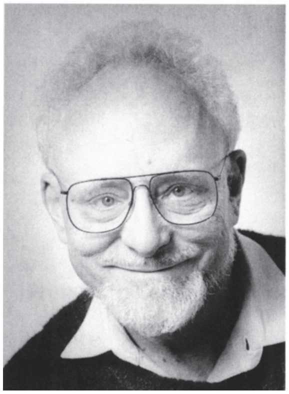
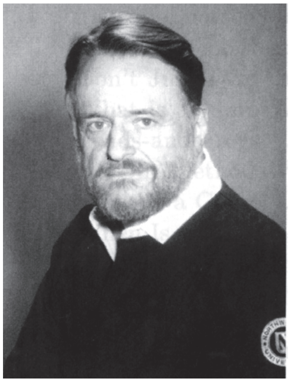
To Martin Gardner

who has brought more mathematics to more millions than anyone else



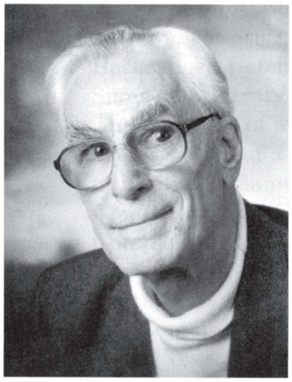
Elwyn Berlekamp was born in Dover, Ohio, on September 6, 1940. He has been Professor of Mathematics and of Electrical Engineering/Computer Science at UC Berkeley since 1971. He has also been active in several technology business ventures. In addition to writing many journal articles and several books, Berlekamp also has 12 patented inventions, mostly dealing with algorithms for synchronization and error correction.

He is a member of the National Academy of Sciences, the National Academy of Engineering, and the American Academy of Arts and Sciences. From 1994 to 1998, he was chairman of the board of trustees of the Mathematical Sciences Research Institute (MSRI).



John H. Conway was born in Liverpool, England, on December 26, 1937. He is one of the preeminent theorists in the study of finite groups and the mathematical study of knots, and has written over 10 books and more than 140 journal articles.

Before joining Princeton University in 1986 as the John von Neumann Distinguished Professor of Mathematics, Conway served as professor of mathematics at Cambridge University, and remains an honorary fellow of Caius College. The recipient of many prizes in research and exposition, Conway is also widely known as the inventor of the Game of Life, a computer simulation of simple cellular "life," governed by remarkably simple rules.



Richard Guy was born in Nuneaton, England, on September 30, 1916. He has taught mathematics at many levels and in many places—England, Singapore, India, and Canada. Since 1965 he has been Professor of Mathematics at the University of Calgary, and is now Faculty Professor and Emeritus Professor. The university awarded him an Honorary Degree in 1991. He was Noyce Professor at Grinnell College in 2000.

He continues to climb mountains with his wife, Louise, and they have been patrons of the Association of Canadian Mountain Guides' Ball and recipients of the A. O. Wheeler award for Service to the Alpine Club of Canada.

Contents

Preface to the Second Edition

In the first edition of *Winning Ways*, which appeared in 1982, we were able to make a rather sharp distinction between those games in Part I, to which the major theory of addition applied directly, and those games in Part 3, which seemed to require more specialized techniques. However, subsequent research by an increasingly large community of combinatorial game theorists has begun to blur this distinction. We now have many more games whose strategies depend both on the general theory of Volume 1 as well as on more specialized results. Introductions to many of these games and some illustrative problems have been added to this new edition. Those that did not readily fit elsewhere can be found in the new Extras to Chapter 22 at the end of this volume. This volume also includes a major revision of the original Chapter 20 on the game of Fox and Geese. Its enhanced variation, Fox-Flocks-Fox, provides compelling illustrations of some of the challenging problems that can now be solved by appropriately combining theories from Volumes 1, 2, and 3 with innovative computing algorithms.

This new edition owes much to the supportive efforts of numerous friends and colleagues, including Noam Elkies, Tom Ferguson, Aviezri Fraenkel, Martin Gardner, Sol Golomb, Al Hales, Greg Kuperberg, Silvio Levy, Donald Knuth, Martin Kutz, Greg Martin, Victor Meally, Richard Nowakowski, Hilarie Orman, Marc Paulhus, Ed Pegg, Michael Reid, Thea van Roode, Katherine Scott, George Sicherman, Aaron Siegel, Neil Sloane, Sally Smith, William Spight, John Tromp, Jonathan Welton, Julian West, David Wilson, and David Wolfe, and to the very professional yet kindly support of our publishers, Alice and Klaus Peters.

Elwyn Berlekamp, University of California, Berkeley
John Conway, Princeton University
Richard Guy, The University of Calgary, Canada

June 23, 2003

Preface to the Original Edition

Does a book need a Preface? What more, after fifteen years of toil, do three talented authors have to add.

We can reassure the bookstore browser, "Yes, this is just the book you want!"

We can direct you, if you want to know quickly what's in the book, to page xx. This in turn directs you to volumes 1,2,3 and 4.

We can supply the reviewer, faced with the task of ploughing through nearly a thousand information-packed pages, with some pithy criticisms by indicating the horns of the polylemma the book finds itself on. It is not an encyclopedia. It is encyclopedic, but there are still too many games missing for it to claim to be complete. It is not a book on recreational mathematics because there's too much serious mathematics in it. On the other hand, for us, as for our predecessors Rouse Ball, Dudeney, Martin Gardner, Kraitchik, Sam Loyd, Lucas, Tom O'Beirne and Fred. Schuh, mathematics itself is a recreation. It is not an undergraduate text, since the exercises are not set out in an orderly fashion, with the easy ones at the beginning. They are there though, and with the hundred and sixty-three mistakes we've left in, provide plenty of opportunity for reader participation. So don't just stand back and admire it, work of art though it is. It is not a graduate text, since it's too expensive and contains far more than any graduate student can be expected to learn. But it does carry you to the frontiers of research in combinatorial game theory and the many unsolved problems will stimulate further discoveries.

We thank Patrick Browne for our title. This exercised us for quite a time. One morning, while walking to the university, John and Richard came up with "Whose game?" but realized they couldn't spell it (there are three tooze in English) so it became a one-line joke on line one of the text. There isn't room to explain all the jokes, not even the fifty-nine private ones (each of our birthdays appears more than once in the book).

Omar started as a joke, but soon materialized as Kimberly King. Louise Guy also helped with proof-reading, but her greater contribution was the hospitality which enabled the three of us to work together on several occasions. Louise also did technical typing after many drafts had been made by Karen McDermid and Betty Teare.

Our thanks for many contributions to content may be measured by the number of names in the index. To do real justice would take too much space. Here's an abridged list of helpers: Richard Austin , Clive Bach, John Beasley, Aviezri Fraenkel, David Fremlin, Solomon Golomb, Steve Grantham, Mike Guy, Dean Hickerson, Hendrick Lenstra, Richard Nowakowski, Anne Scott, David Seal, John Selfridge, Cedric Smith and Steve Tschantz.

No small part of the reason for the assured success of the book is owed to the well-informed and sympathetic guidance of Len Cegielka and the willingness of the staff of Academic Press and of Page Bros. to adapt to the idiosyncrasies of the authors, who grasped every opportunity to modify grammar, strain semantics, pervert punctuation, alter orthography, tamper with traditional typography and commit outrageous puns and inside jokes.

Thanks also the the Isaak Walton Killam Foundation for Richard's Resident Fellowship at The University of Calgary during the compilation of a critical draft, and to the National (Science & Engineering) Research Council of Canada for a grant which enabled Elwyn and John to visit him more frequently than our widely scattered habitats would normally allow.

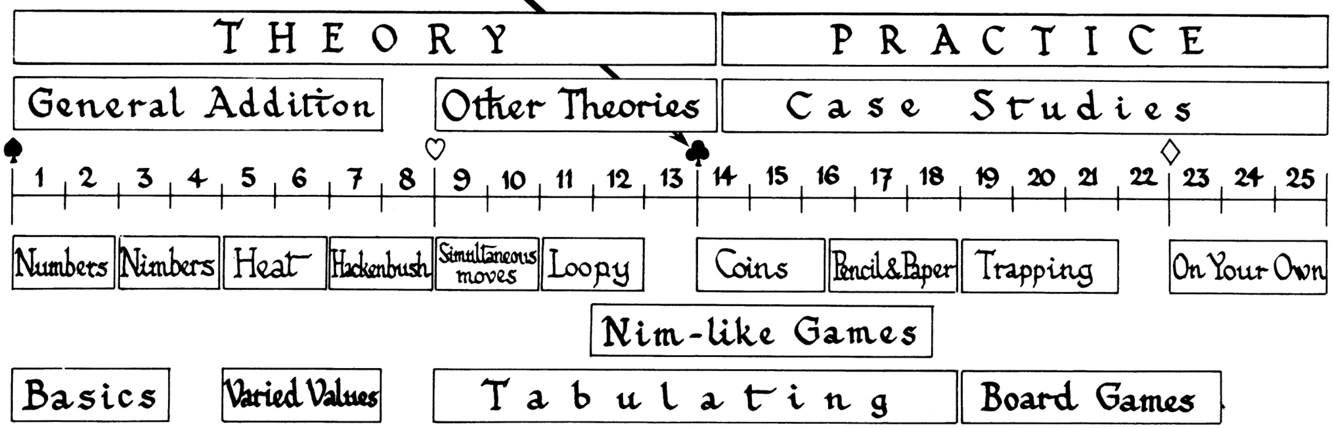
And thank you, Simon!

University of California, Berkeley, CA 94720 Elwyn Berlekamp
University of Cambridge, England, CB2 1SB John Conway
University of Calgary, Canada, T2N 1N4 Richard Guy

You are now here

If you want to know roughly what's elsewhere, turn to the little notes about our four main themes:

There are a number of other connexions between various chapters of the book:



However, you should be able to pick any chapter and read almost all of it without reference to anything earlier, except perhaps the basic ideas at the start of the book.

Games in Clubs!

To be an Englishman is to belong
to the most exclusive club there is.
Ogden Nash, *England Expects.*

There are lots of games for which the theories we've now developed are useful, and even more for which they're not, and we've grouped them into clubs according to how you play them.

First some games you can play with coins, either by turning them over (Chapter 14) or moving them along strips or about in heaps (Chapter 15).

Then games for which you'll need pencil and paper, perhaps to draw straight lines (Chapter 16), or curved ones (Chapter 17) or merely to do the calculations in Chapter 18.

And for board games we have three case studies in which one player wins by trapping his opponent (Chapters 19, 20, 21) and finally many more which are usually won by the first player to establish some kind of winning configuration (Chapter 22).

-14-

Turn and Turn About

Because I do not hope to turn again
Because I do not hope
Because I do not hope to turn.
T. S. Eliot, *Ash Wednesday*, I.

Open not thine heart to every man, lest he requite thee
with a shrewd turn.
Ecclesiasticus, 8:19.

These games, based on an idea of H. W. Lenstra, are similar in that they all involve turning things over, but we shall see that they call for a variety of strategies.

Turning Turtles

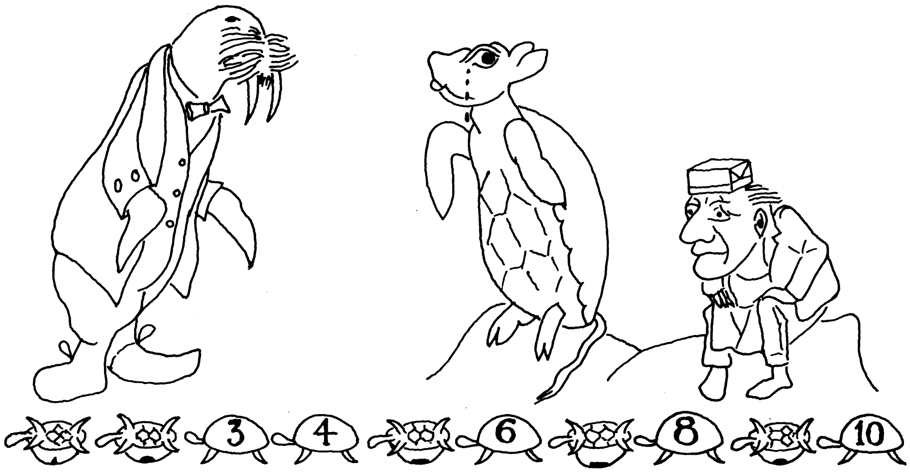


Figure 1. Playing Turning Turtles.

In Fig. 1 the Walrus and the Carpenter are playing a rather cruel game. At each move a player must put one turtle on its back and may also turn over any single turtle to the left of it. This second turtle, unlike the first, may be turned either onto its feet or onto its back. The player wins who turns the last turtle upside-down. Which turtles should the Walrus ($l.$) turn?

Like most readers of this book, he wearily suspects another disguise for Nim. Here only turtles 3, 4, 6, 8 and 10 are on their feet, and since the nim-sum of 3, 4 and 6 is 1, he may turn 10 onto its back and 9 onto its feet, producing 3,4, 6, 8, 9, a $\mathcal{P}$-position since $8 \overset{*}{+} 9 = 1$. The Carpenter ($r.$)responds by turning 8 and 5 producing the position 3, 4, 5, 6, 9 as in Fig. 2.

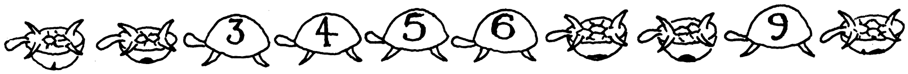


Figure 2. After the Carpenter's Reply.

In Nim there is only one good move from this position—reduce 9 to 4, so as to produce, 3, 4, 4, 5, 6, which, since two equal Nim heaps may be cancelled, is much the same as 3, 5, 6, which the Walrus reaches by turning both 9 and 4 on their backs (Fig. 3).

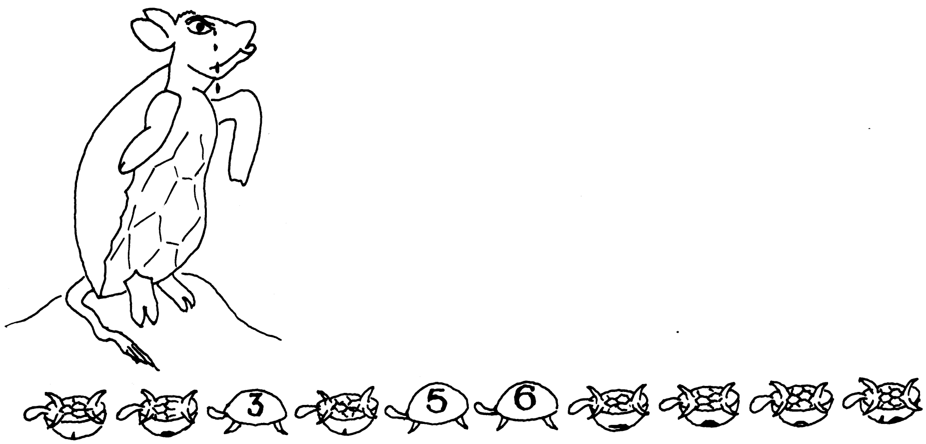


Figure 3. How the Walrus Won.

Nim moves become turtle turns as follows. We reduce a heap to a size not already present by turning one turtle on its back and putting another on its feet, as in the Walrus's opening move. If a heap of the reduced size is already present, we turn two turtles on their backs as in the Walrus's response to the Carpenter's move (cancelling two equal heaps). To eliminate a heap entirely, we merely turn the appropriate turtle. So since 4, 6, 8, 10 is a $\mathcal{P}$-position, the Walrus could have won from Fig. 1 by just turning turtle 3.

Since all our turning games are impartial, they are solved by computing the nim-values, and often may be thought of as heap games in disguise; but many games with interesting theories are more naturally suggested by the turning version.

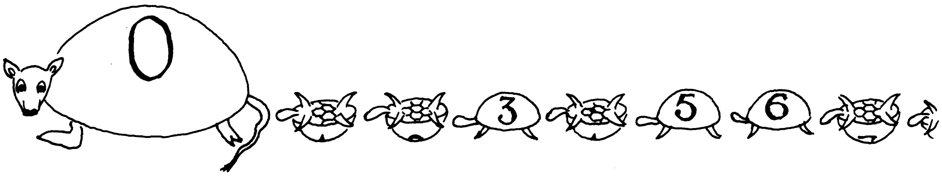


Figure 4. The Mock Turtle Joins in.

Mock Turtles

Let the players turn up to three turtles subject only to the condition that the rightmost of these must be turned from his feet onto his back. We may think of this as a game with numbers in which any number may be replaced by 0, 1 or 2 smaller ones. So $\mathcal{G}(n)$ is the least number not of any form

$$0, \mathcal{G}(a), \mathcal{G}(a) \overset{*}{+} \mathcal{G}(b),$$

in which a and b are any numbers less than n.

If we number the positions from 0, we find the nim-values shown in Table 1.

$n =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	...
$\mathcal{G}(n) =$	1	2	4	7	8	11	13	14	16	19	21	22	25	26	28	31	32	35	37	...

Table 1. Nim-values for Mock Turtles.

We see that $\mathcal{G}(n)$ is always $2n$ or $2n+1$, so that its binary expansion is obtained by adjoining a digit 0 or 1 to that of n. Which shall it be?

$n =$	0	1	10	11	100	101	110	111	1000	1001	1010	...
$\mathcal{G}(n) =$	$\underline{1}$	$1\underline{0}$	$10\underline{0}$	$11\underline{1}$	$100\underline{0}$	$101\underline{1}$	$110\underline{1}$	$111\underline{0}$	$1000\underline{0}$	$1001\underline{1}$	$1010\underline{1}$	...

Table 2. The Odious Numbers Revealed.

Table 2 suggests we choose whichever makes the total number of 1-digits *odd*.

Odious and Evil Numbers

Every number is **odious** or **evil** according to the number of 1's in its binary expansion (odious for odd, evil for even). These behave under Nim addition like odd and even numbers under ordinary addition:

$$\begin{aligned} \text{EVIL} \overset{*}{+} \text{EVIL} &= \text{EVIL} = \text{ODIOUS} \overset{*}{+} \text{ODIOUS}, \\ \text{EVIL} \overset{*}{+} \text{ODIOUS} &= \text{ODIOUS} = \text{ODIOUS} \overset{*}{+} \text{EVIL}. \end{aligned}$$

When we compute $\mathcal{G}(n)$ in Mock Turtles, the next odious number is *never* excluded, because the nim-sum of two odious numbers is evil, but smaller evil numbers always *are* excluded.

If $a_1, a_2, \ldots, a_n$ is a $\mathcal{P}$-position in Nim, so that

$$a_1 \overset{*}{+} a_2 \overset{*}{+} \ldots \overset{*}{+} a_n = 0,$$

then for the corresponding odious numbers $\mathcal{G}(a_i)$ in Mock Turtles we shall have

$$G(a_1) \overset{*}{+} G(a_2) \overset{*}{+} \ldots \overset{*}{+} G(a_n) = 0 \text{ or } 1.$$

But if n is even, this nim-sum is evil, and so 0; while if n is odd it is odious, and so 1. The $\mathcal{P}$-positions in Mock Turtles are therefore just those $\mathcal{P}$-positions in Nim for which n is even.

Note that in Mock Turtles we number the turtles from 0. The turtle numbered 0, called the Mock Turtle, must take his turn with the rest and cannot be neglected in the conversion to Nim. To obtain a $\mathcal{P}$-position in Mock Turtles from the Turning Turtles position of Fig. 3, the Mock Turtle must be brought into the game with his four feet on the ground. In Mock Turtles, 3, 5, 6, is *not* a $\mathcal{P}$-position, but 0, 3, 5, 6 is (Fig. 4).

Moebius, Mogul and Gold Moidores

Table 3 shows the nim-values, kindly checked for us on the computer by M.J.T. Guy, for similar games in which we may turn over up to t objects for $t = 1, 2, \ldots$. Because the numbers get much larger than the other nim-values in this book, we have written them in base 8 (octal) notation. Nim-sums of octal numbers may be computed digit by digit thus:

$$\begin{array}{r} 1\,2\,3\,4\,5\,6\,7\,0 \\ 1\,3\,5\,7\,0\,2\,4\,6 \\ \hline 1\,6\,3\,5\,4\,3\,6. \end{array}$$

In the table we have only named the most interesting cases: $t = 3, 5, 7$ and 9. Note that C, E, G and I are the 3rd, 5th, 7th and 9th letters of the alphabet. For convenience, and to avoid cruelty to turtles, the reader may play these games with coins. The coins will show heads or tails according as the turtle is on his feet or on his back, and the rightmost coin that is turned must change from heads to tails.

The Mock Turtle Theorem

Take a $\mathcal{P}$-position in the game for an even value of $t, t = 2m$, and place an extra coin (the Mock Turtle) at the left, whichever way up will ensure an even number of heads. Positions obtained in this way will be called "good" positions for the next odd value of t, $t = 2m + 1$.We assert that the good positions are precisely the $\mathcal{P}$-positions for the game $t = 2m + 1$.

We show first that there is no way of changing from one good position to another by turning at most $2m + 1$ coins. If there were, the number of coins turned would necessarily be even, since the good positions have evenly many heads, and so would actually be at most $2m$. But this would entail a move between two $\mathcal{P}$-positions in the $2m$ game.

	MOCK TURTLES				MOEBIUS		MOGUL		MOIDORES
n	$t=1$	2	3	4	5	6	7	8	9
THE MOCK TURTLE	1		1		1		1		1
1	1	1	2	1	2	1	2	1	2
2	1	2	4	2	4	2	4	2	4
3	1	3	7	4	10	4	10	4	10
4	1	4	10	10	20	10	20	10	20
5	1	5	13	17	37	20	40	20	40
6	1	6	15	20	40	40	100	40	100
7	1	7	16	40	100	77	177	100	200
8	1	10	20	63	147	100	200	200	400
9	1	11	23	100	200	200	400	377	777
10	1	12	25	125	253	400	1000	400	1000
11	1	13	26	152	325	707	1617	1000	2000
12	1	14	31	200	400	1000	2000	2000	4000
13	1	15	32	226	455	1331	2663	4000	10000
14	1	16	34	253	526	1552	3325	7417	17037
15	1	17	37	333	667	1664	3551	10000	20000
16	1	20	40	355	733	2000	4000	20000	40000
17	1	21	43	367	756	2353	4726	31463	63147
18	1	22	45	400	1000	2561	5343	40000	100000
19	1	23	46	427	1056	2635	5472	52525	125253
20	1	24	51	451	1123	3174	6370	65252	152525
21	1	25	52	707	1617	3216	6435	100000	200000
22	1	26	54	1000	2000	3447	7116	113152	226325
23	1	27	57	1031	2063	3722	7644	200000	400000
24	1	30	61	1055	2132	4000	10000	213630	427461
25	1	31	62	1122	2245	10000	20000	263723	547646
26	1	32	64	1203	2407	20000	40000	306136	614274
27	1	33	67	1443	3106	34007	70017	400000	1000000
28	1	34	70	1537	3277	40000	100000	416246	1034515
29	1	35	73	1746	3714	54031	130063	521055	1242133
30	1	36	75	2000	4000	64052	150125	724616	1651435
31	1	37	76	2033	4066	70064	160151	1000000	2000000
32	1	40	100	2056	4134	100000	200000	1023305	2046613
33	1	41	103	2130	4261	114053	230126	1347214	2716431
34	1	42	105	2221	4443	124061	250143	2000000	4000000
35	1	43	106	2465	5153	130035	260072	2027151	4056322
36	1	44	111	2501	5203	144074	310170	2457261	5136542
37	1	45	112	3124	6250	150016	320035	3166444	6355111
38	1	46	114	3512	7225	160047	340116	4000000	10000000
39	1	47	117	4000	10000	174022	370044	4055666	10133554
40	1	50	121	4034	10071	200000	400000	4632577	11465377
41	1	51	122	4045	10113	214301	430603	5251417	12523036
42	1	52	124	4211	10423	224502	451205	7514712	17231625
43	1	53	127	4504	11211	230604	461411	10000000	20000000

Table 3. These Nim-values Are in Octal (base 8), *not* Decimal.

It remains to show that from any bad position in the $2m+1$ game there is a move to some good position. If the position is bad because it corresponds to an $\mathcal{N}$-position in the $2m$ game, there is a move in that game to some $\mathcal{P}$-position, and, by turning the Mock Turtle if necessary, we obtain a move to a good position in the $2m+1$ game. The other bad positions correspond to $\mathcal{P}$-positions in the $2m$ game, but have an odd number of heads. In this case, by turning over the rightmost head, we obtain a position that gives an $\mathcal{N}$-position in the $2m$ game. We can now turn over at most $2m$ further coins to make this a $\mathcal{P}$-position and then, if necessary to obtain a good position, also turn the Mock Turtle. We have turned at most $2m+2$ coins in all, but since we started with an odd number of heads and finished with an even number, we have in fact turned over at most $2m+1$ coins, and so have made a legal move in the $2m+1$ game.

This result is equivalent to the statement:

> Every nim-value for the $2m+1$ game is an odious number, and the corresponding value for the $2m$ game is obtained by dropping the final binary digit.

THE MOCK TURTLE THEOREM

Why Moebius?

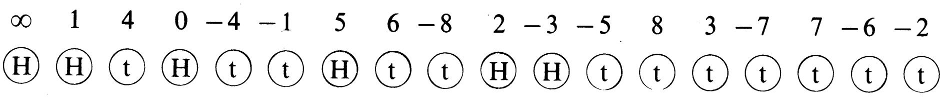


Figure 5. Moebius Labels Make $\mathcal{P}$-positions Easy to Find.

When restricted to 18 coins, the $\mathcal{P}$-positions of the game with $t = 5$ possess a remarkable symmetry. To see this, name the heads of a position by the numbers shown in Fig. 5. For example, the $\mathcal{P}$-position with heads in just the first 6 places is $\infty, 0, \pm 1, \pm 4$. In this notation $\mathcal{P}$-positions remain $\mathcal{P}$-positions when their numbers are increased by any fixed amount, modulo 17, leaving ∞ unchanged. Adding 1 to the numbers $\infty, 0, \pm 1, \pm 4$ we find $\infty, 1, 2, 0, 5, -3$, so that the position displayed in Fig. 5 is another $\mathcal{P}$-position. The 15 positions shown in Table 4 yield a total of $15 \times 17 = 255$ $\mathcal{P}$-positions in this way. It is also true that a $\mathcal{P}$-position remains a $\mathcal{P}$-position if we interchange heads and tails in every place. The positions with all tails or all heads are therefore both $\mathcal{P}$-positions, giving $2 \times 255 + 2 = 512$ $\mathcal{P}$-positions in all, distributed as follows:

Number of heads	0	6	8	10	12	18
Number of $\mathcal{P}$-positions	1	102	153	153	102	1.

6 heads				*8 heads*			
$\infty, 0$	± 1	± 4		$\infty, 0$	± 1	± 5	± 7
$\infty, 0$	± 2	± 8		$\infty, 0$	± 2	± 3	± 7
± 1	± 3	± 6		$\infty, 0$	± 3	± 4	± 6
± 2	± 5	± 6		$\infty, 0$	± 5	± 6	± 8
± 4	± 5	± 7		± 1	± 2	± 4	± 8
± 3	± 7	± 8		± 1	± 2	± 3	± 5
				± 2	± 4	± 6	± 7
				± 3	± 4	± 5	± 8
				± 1	± 6	± 7	± 8

Table 4. The $\mathcal{P}$-positions for Moebius.

Dropping the Mock Turtle (at ∞) we find that the $\mathcal{P}$-positions for the game $t = 4$ on 17 coins are distributed:

Number of heads	0	5	6	7	8	9	10	11	12	17
Number of $\mathcal{P}$-positions	1	34	68	68	85	85	68	68	34	1.

We can also double the numbers (modulo 17) of any $\mathcal{P}$-position to give another. Thus $\infty, 0, 1, 2, -3, 5$ of Fig. 5 becomes $\infty, 0, 2, 4, -6, -7$. We can invert them modulo 17; since $1/2 = -8$, $1/3 = 6$ and $1/5 = 7$, Fig. 5 inverts into $0, \infty, 1, -8, -6, 7$. In fact we can make any transformation (modulo 17)

$$x \to \frac{ax+b}{cx+d}, \quad ad - bc = 1.$$

Since these are known as the Möbius transformations, we have named our game after that distinguished mathematician.

Mogul

On 24 coins the game for $t = 7$ displays even more symmetries. The $\mathcal{P}$-positions among the first 24 places are distributed as follows:

Number of heads	0	8	12	16	24
Number of $\mathcal{P}$-positions	1	759	2576	759	1.

Figure 6 enables us to find the 759 $\mathcal{P}$-positions with just 8 heads, or equally those with 8 tails. In either case the set of 8 places involved is called an **octad**. In Fig. 6 there are 35 **pictures** and each picture shows the 24 places colored in six sets of four (the 6 colors used are black, white, star, circle, plus and dot). Any two sets of 4 (any two colors) in the same picture make an octad: in particular this gives every octad with just 4 places in the last pair of (black and white) rows, and this pair of rows themselves form an octad. By interchanging this last pair of rows with the first pair, or the middle pair, of the same picture, we can now find all the octads, since it can be shown that these pairs of rows form octads and that every other octad meets at least one of them in just 4 places.

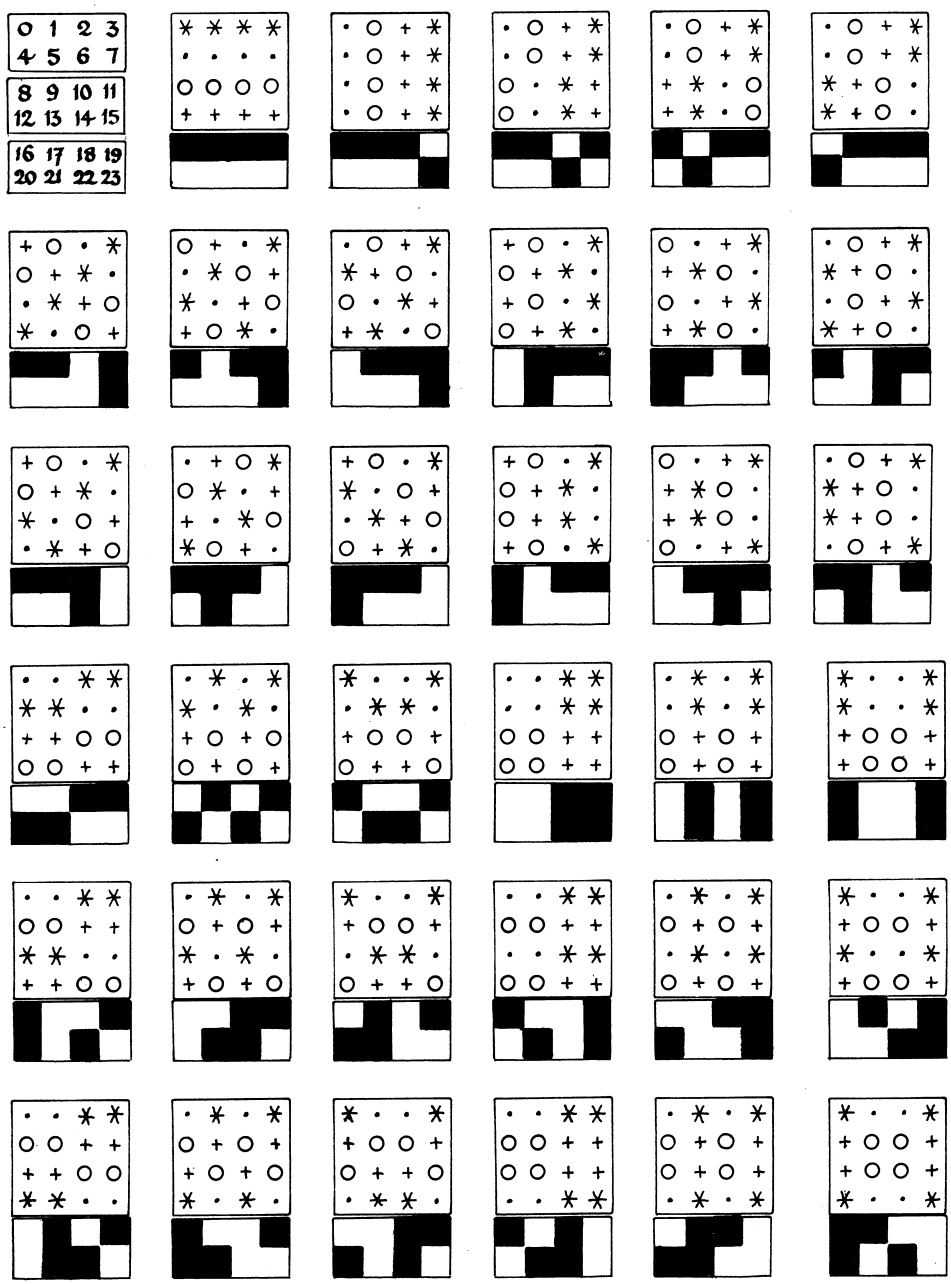


Figure 6. Curtis's Miracle Octad Generator.

This Miracle Octad Generator, or MOG, is due to R. T. Curtis, but we have modified it slightly for the Mogul player's convenience. Various regular features of its arrangement make it easy for the practised user to locate the unique octad containing any five given places. It seems to be the case that the winner in 24-place Mogul need never play into a 12-head $\mathcal{P}$-position.

Motley

This is the game in which any number of coins may be turned. When well played it lasts at most one move, since we can turn all the heads to tails instantly! The nim-values are the powers of 2:

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, \ldots$$

so, when played with several rows, Motley is yet another disguise for Nim; the heads in a row are binary digits 1 in the number of beans in the corresponding Nim-heap.

Twins, Triplets, Etc.

We can also play the game **Twins**, in which we must turn *exactly* two coins, or **Triplets**, in which we turn exactly three, etc. The nim-value sequence for the game in which we turn exactly t coins consists of $t-1$ zeros followed by the nim-value sequence for the game in which we turn *at most* t coins. Thus the nim-values for Triplets are

$$0, 0, 1, 2, 4, 7, 8, 11, 13, 14, 16, 19, 21, 22, 25, \ldots.$$

We may think of the first $t-1$ coins as $t-1$ Mock Turtles which may be used to fill out our move to its proper complement of turns.

The Ruler Game

If the coins we turn must be *consecutive* but are otherwise unrestricted (except that the rightmost coin must be turned from heads to tails), then the nim-values are computed by the rule:

$$\mathcal{G}(n) = \text{mex} \left\{ \begin{array}{l} 0 \\ \mathcal{G}(n-1) \\ \mathcal{G}(n-1) \overset{*}{+} \mathcal{G}(n-2) \\ \mathcal{G}(n-1) \overset{*}{+} \mathcal{G}(n-2) \overset{*}{+} \mathcal{G}(n-3) \\ \ldots\ldots. \end{array} \right\},$$

and are found to be reminiscent of Dividing Rulers (Fig. 7 of Chapter 13, Vol. 2).

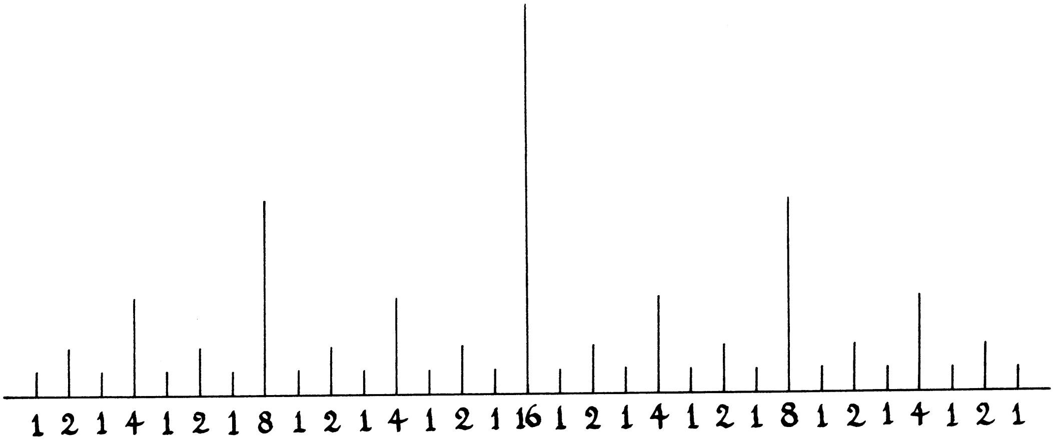


Figure 7. Nim-values for the Ruler Game.

If the coins are numbered starting from 1, $\mathcal{G}(n)$ is just the highest power of 2 dividing n.

Circumscribed Games

We can play any of these games under the additional restriction that the coins to be turned may not be too far apart. Thus in **Mock Turtle Fives** we may turn *up to three* of five consecutive coins. In **Triplet Fives** we turn *exactly three* out of five consecutive coins. In **Ruler Fives** we may turn 1, 2, 3, 4 or 5 consecutive coins. The nim-values for these three games are:

Mock Turtle Fives: 1 2 4 7 8 1 2 4 7 8 1 2 4 7 8 1 2 4 7 8 ...
Triplet Fives: 0 0 1 2 4 0 0 1 2 4 0 0 1 2 4 0 0 1 2 4 ...
Ruler Fives: 1 2 1 4 1 2 1 4 1 2 1 4 1 2 1 4 1 2 1 4 ...

These are parts of general patterns. Thus, **Moebius Nineteens**, for example, would have the first 19 values of Moebius repeated indefinitely. This happens for all the above games except the Ruler game; Ruler Fours, Sixes and Sevens have the same values as Ruler Fives, while Ruler Eights to Fifteens all have nim-values:

1 2 1 4 1 2 1 8 1 2 1 4 1 2 1 8 1 2 1 4 1 2 1 8 1 2 1 4 1 2 1 8 1....

Turnips (or Ternups)

This game has a richer theory, but it is a great pity that the full theory is only needed by people wealthy enough to play with a very large number of coins. The move is to turn over any three equally spaced coins, the rightmost going from heads to tails as usual. Numbering from 0 we find that the nim-values for 0 to 100 are:

0-8	0	0	1	0	0	1	2	2	1			
9-17	0	0	1	0	0	1	2	2	1			
18-26	4	4	1	4	4	1	2	2	1			
27-35	0	0	1	0	0	1	2	2	1			
36-44	0	0	1	0	0	1	2	2	1			
45-53	4	4	1	4	4	1	2	2	1			
54-62	7	7	1	7	7	1	2	2	1			
63-71	7	7	1	7	7	1	2	2	1			
72-80	4	4	1	4	4	1	2	2	1			
81-89	0	0	1	0	0	1	2	2	1			
90-100	0	0	1	0	0	1	2	2	1	4	4	...

Table 5. The Nim-values for Turnips.

To find $\mathcal{G}(n)$ in general, we expand n in base 3:

				n in ternary					$\mathcal{G}(n)$	
$\phi = 0$ or 1	...	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	0	
$? = 0, 1$ or 2	...	?	?	?	?	?	?	2	1	the
	...	?	?	?	?	?	2	ϕ	2	odious
	...	?	?	?	?	2	ϕ	ϕ	4	numbers
	...	?	?	?	2	ϕ	ϕ	ϕ	7	in
	...	?	?	2	ϕ	ϕ	ϕ	ϕ	8	order
	...	?	2	ϕ	ϕ	ϕ	ϕ	ϕ	11	
	...	...	...	...	...	...	...	...	...	...

In words, $\mathcal{G}(n) = 0$ if the ternary expansion of n has no 2-digit, but is the kth odious number if the last 2-digit is in the kth place from the right, when we call n a k-**number**. The numbers n whose ternary expansions have no 2-digit will be called **empty numbers**.

To see all this, note that $\mathcal{G}(n)$ is the mex of all the numbers

$$\mathcal{G}(n-\delta) \overset{*}{+} \mathcal{G}(n-2\delta) \text{ for } \delta = 1, 2, \ldots .$$

We show first that the putative value for $\mathcal{G}$(n) is not one of these numbers, or equivalently that

$$\mathcal{G}(n) \overset{*}{+} \mathcal{G}(n-\delta) \overset{*}{+} \mathcal{G}(n-2\delta) \neq 0.$$

Since the nim-sum of three odious numbers is odious, this will be true unless one of

$$\mathcal{G}(n), \mathcal{G}(n-\delta), \mathcal{G}(n-2\delta)$$

is zero and the other two coincide. But if the last non-zero ternary digit ($x = 1$ or 2) of δ is in the kth place, the expansions of n, $n-\delta$, $n-2\delta$ look like:

	? ? ? $x_{(k)}$ 0 0 0 $0_{(j)}$ 0 0		? ? ? $x_{(k)}$ 0 0 0 0 0
δ:	? ? ? x 0 0 0 0 0 0		? ? ? x 0 0 0 0 0
n	? ? ? 0 ? ? ? 2 ϕ ϕ	or	? ? ? 0 ϕ ϕ ϕ ϕ ϕ
$n-\delta$	? ? ? 1 ? ? ? 2 ϕ ϕ		? ? ? 1 ϕ ϕ ϕ ϕ ϕ
$n-2\delta$	? ? ? 2 ? ? ? 2 ϕ ϕ		? ? ? 2 ϕ ϕ ϕ ϕ ϕ

according as n has or has not a 2-digit in some j place, $j < k$. In the first case the three putative nim-values are all the jth odious number, and in the second exactly one of them is the kth odious number, so they cannot have zero nim-sum.

Now we know from our analysis of Mock Turtles that each odious number is the first number not the nim-sum of two or fewer earlier ones. It suffices to show that if n is a k-number, we can choose δ so as to make $n-\delta$ and $n-2\delta$ i- and j-numbers or empty, for any i and j less than k. The subtraction sums in Table 6 show how to do this.

	... ? ? $2_{(k)}$ ϕ ϕ $1_{(j)}$ ϕ ϕ $1_{(i)}$ ϕ ϕ		ϕ 2 ϕ ϕ ϕ 2 ϕ 2 ϕ ϕ $2_{(k)}$ ϕ ϕ
δ :	2 0 0 1 0 0	δ	1 0 0 0 1 0 1 0 0 1 0 0
n :	... ? ? 2 ϕ ϕ 1 ϕ ϕ 1 ϕ ϕ	n :	ϕ 2 ϕ ϕ ϕ 2 ϕ 2 ϕ ϕ 2 ϕ ϕ
$n-\delta$:	... ? ? ? ? ? 2 ϕ ϕ 0 ϕ ϕ	$n-\delta$:	ϕ 1 ϕ ϕ ϕ 1 ϕ 1 ϕ ϕ 1 ϕ ϕ
$n-2\delta$:	... ? ? ? ? ? ? ? ? 2 ϕ ϕ	$n-2\delta$:	ϕ 0 ϕ ϕ ϕ 0 ϕ 0 ϕ ϕ 0 ϕ ϕ
δ :	1 0 0 1 0 0		ϕ 2 ϕ ϕ ϕ 2 ϕ $2_{(k)}$ ϕ ϕ $1_{(i)}$ ϕ ϕ
n :	... ? ? 2 ϕ ϕ 0 ϕ ϕ 1 ϕ ϕ	n :	ϕ 2 ϕ ϕ ϕ 2 ϕ 2 ϕ ϕ 1 ϕ ϕ
$n-\delta$:	... ? ? ? ? ? 2 ϕ ϕ 0 ϕ ϕ	$n-\delta$:	? ? ? ? ? ? ? ? ? ? 2 ϕ ϕ
$n-2\delta$:	... ? ? ? ? ? ? ? ? 2 ϕ ϕ	$n-2\delta$:	0 0 0 0 0 0 ϕ 0 0 0 ϕ ϕ
δ :	1 2 2 2 0 0		
n :	... ? ? 2 ϕ ϕ 1 ϕ ϕ 0 ϕ ϕ	n :	ϕ 2 ϕ ϕ ϕ 2 ϕ 2 ϕ ϕ 0 ϕ ϕ
$n-\delta$:	... ? ? ? ? ? 2 ϕ ϕ 1 ϕ ϕ	$n-\delta$:	? ? ? ? ? ? ? ? ? ? 2 ϕ ϕ
$n-2\delta$:	... ? ? ? ? ? ? ? ? 2 ϕ ϕ	$n-2\delta$:	0 0 0 0 0 0 ϕ 0 0 1 ϕ ϕ
δ :	0 2 2 2 0 0		**In the last two cases above, the first ϕ of**
n :	... ? ? 2 ϕ ϕ 0 ϕ ϕ 0 ϕ ϕ		**the last line is whichever of 0 or 1 makes**
$n-\delta$:	... ? ? ? ? ? 2 ϕ ϕ 1 ϕ ϕ		**$n-2\delta$ have the same parity as n. Then δ**
$n-2\delta$:	... ? ? ? ? ? ? ? ? 2 ϕ ϕ		**can be found from n and $n-2\delta$.**

Table 6. How to Make $n-\delta$, $n-2\delta$ into i- and j-numbers, or Empty.

Grunt

In a move of this game one must turn over four symmetrically arranged coins of which the first must be the leftmost coin of the game and the last must be turned from heads to tails. Numbering from 0 the restriction is that we turn numbers

$$0, a, n-a \text{ and } n,\ 0 < a < \frac{1}{2}n,$$

and we find the nim-values:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	...
$\mathcal{G}(n)$	0	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4	3	0	4	3	....

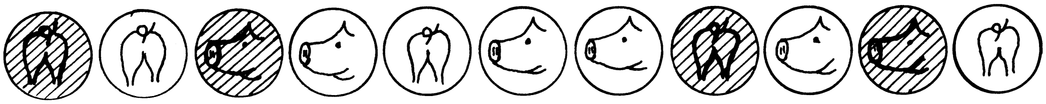


Figure 8. A Winning Move in Grunt.

Since $\mathcal{G}(0) = 0$, $\mathcal{G}(n)$ can more easily be computed as the mex of all numbers of the form

$$\mathcal{G}(a) \overset{*}{+} \mathcal{G}(n-a), 0 < a < \frac{1}{2}n,$$

and so the game is a disguise for Grundy's Game (see Chapter 4, Vol. 1) in which any heap may be split into two smaller heaps of different sizes.

Sym

As an example where the nim-values display no recognizable pattern, let us turn over any symmetrically arranged set of coins, not necessarily including the leftmost coin, number 0. We find, with thanks to Donald Knuth for the last four values

n	=	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20 ...
$\mathcal{G}(n)$	=	1	2	4	3	6	7	8	16	18	25	32	11	64	31	128	10	256	5	512	28	1024 ....

The reader can also try to solve the game **Sympler** in which the leftmost coin *is* to be included in the symmetrical set of coins turned.

Two-Dimensional Turning Games

All our one-dimensional games were played with the restriction that the rightmost coin to be turned was to be changed from heads to tails. In the two-dimensional games the corresponding requirement is that the most "south-easterly" coin which is turned must go from heads to tails. In such games we'll write $\mathcal{G}(a,b)$ for the value of a coin in row a and column b.

Acrostic Twins

We start with a very simple game. The move is to turn two coins which must either be in the same row or in the same column. The typical entry in the nim-value table is therefore the least number not appearing earlier in the same row or column, and we find Table 7. So we see that Acrostic Twins defines nim-addition:

$$\mathcal{G}(a,b) = a \overset{*}{+} b.$$

Turning Corners

This is a much more interesting game. The move is to turn over the four corners of any rectangle with horizontal and vertical sides. The nim-values can be computed using

$$\mathcal{G}(a,b) = \text{mex}\left\{\mathcal{G}(a',b) \overset{*}{+} \mathcal{G}(a,b') \overset{*}{+} (a',b')\right\},$$

where a' and b' are any numbers respectively less than a and b (see Fig. 9). Table 8 gives values for a and b less than 16.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

Table 7. How to Play Acrostic Twins.

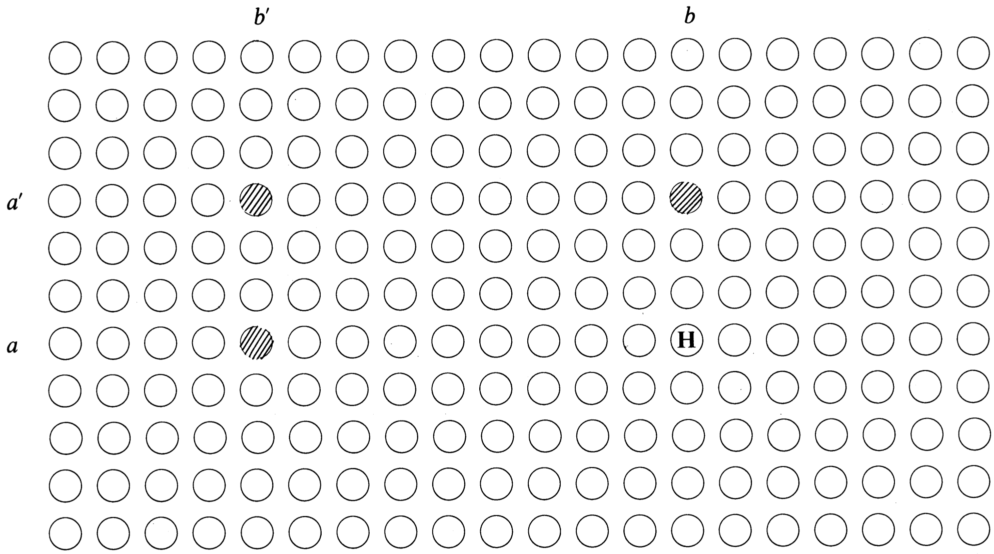


Figure 9. A Typical Move in Turning Corners.