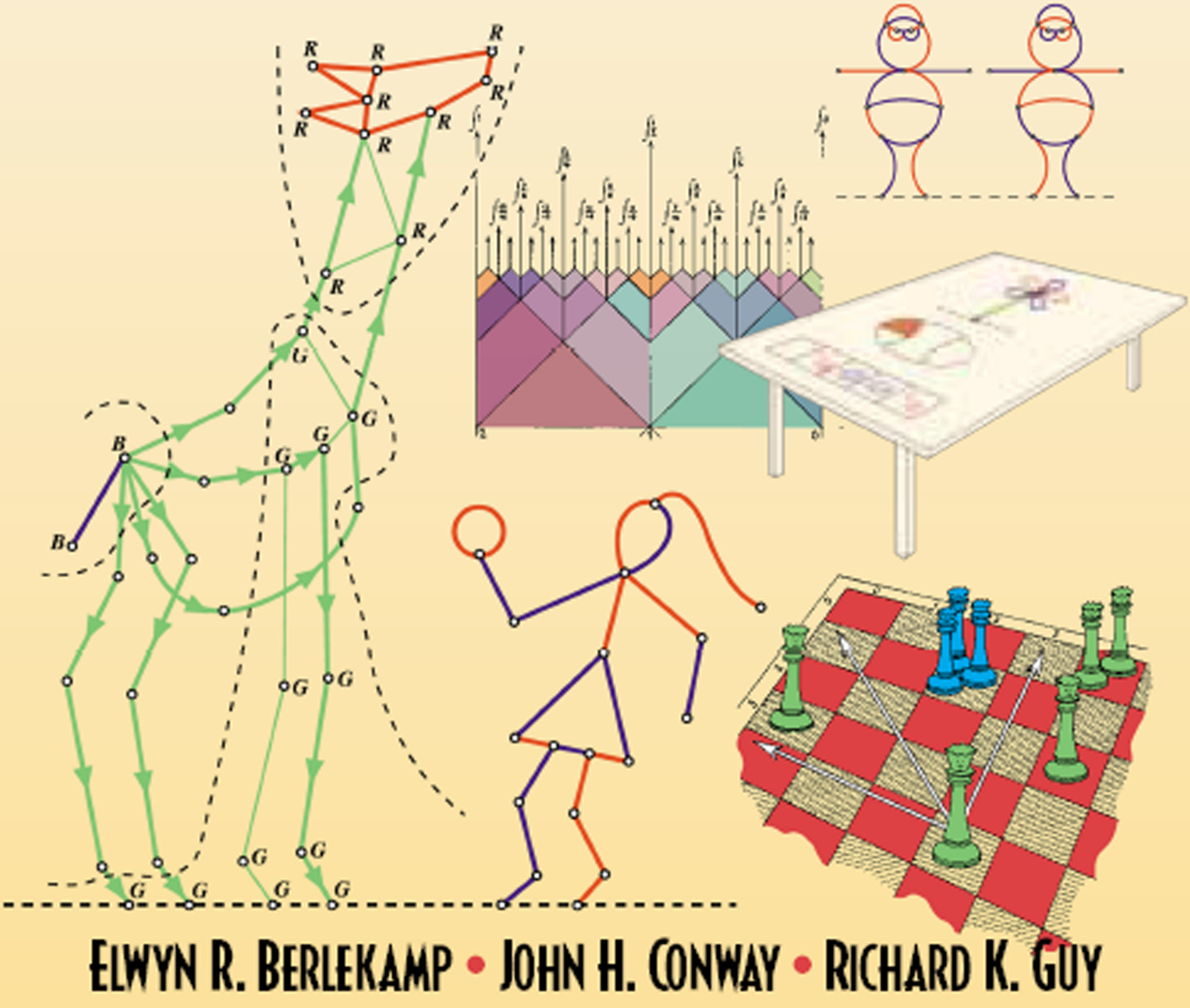
VOLUME 1 SECOND EDITION

WINNING WAYS
FOR YOUR MATHEMATICAL PLAYS



ELWYN R. BERLEKAMP • JOHN H. CONWAY • RICHARD K. GUY

Winning Ways for Your Mathematical Plays, Volume 1

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Winning Ways
for Your Mathematical Plays

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Volume 1, Second Edition

Elwyn R. Berlekamp, John H. Conway, Richard K. Guy

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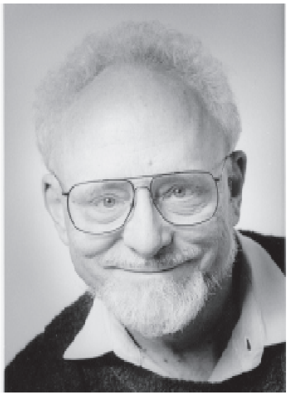
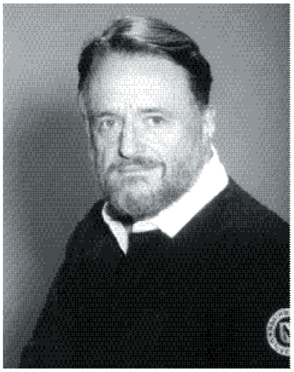
To Martin Gardner

who has brought more mathematics to more millions than anyone else



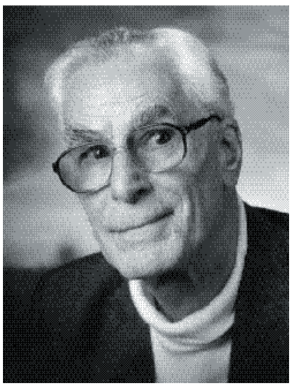
Elwyn Berlekamp was born in Dover, Ohio, on September 6, 1940. He has been Professor of Mathematics and of Electrical Engineering/Computer Science at UC Berkeley since 1971. He has also been active in several technology business ventures. In addition to writing many journal articles and several books, Berlekamp also has 12 patented inventions, mostly dealing with algorithms for synchronization and error correction.

He is a member of the National Academy of Sciences, the National Academy of Engineering, and the American Academy of Arts and Sciences. From 1994 to 1998, he was chairman of the board of trustees of the Mathematical Sciences Research Institute (MSRI).



John H. Conway was born in Liverpool, England, on December 26, 1937. He is one of the preeminent theorists in the study of finite groups and the mathematical study of knots, and has written over 10 books and more than 140 journal articles.

Before joining Princeton University in 1986 as the John von Neumann Distinguished Professor of Mathematics, Conway served as professor of mathematics at Cambridge University, and remains an honorary fellow of Caius College. The recipient of many prizes in research and exposition, Conway is also widely known as the inventor of the Game of Life, a computer simulation of simple cellular "life," governed by remarkably simple rules.



Richard Guy was born in Nuneaton, England, on September 30, 1916. He has taught mathematics at many levels and in many places—England, Singapore, India, and Canada. Since 1965 he has been Professor of Mathematics at the University of Calgary, and is now Faculty Professor and Emeritus Professor. The university awarded him an Honorary Degree in 1991. He was Noyce Professor at Grinnell College in 2000.

He continues to climb mountains with his wife, Louise, and they have been patrons of the Association of Canadian Mountain Guides' Ball and recipients of the A. O. Wheeler award for Service to the Alpine Club of Canada.

Contents

Preface to the Second Edition

It's high time that there was a second edition of *Winning Ways.*

Largely as a result of the first edition, and of John Conway's *On Numbers and Games*, which we are glad to say is also reappearing, the subject of combinatorial games has burgeoned into a vast area, bringing together artificial intelligence experts, combinatorists, and computer scientists, as well as practitioners and theoreticians of particular games such as Go, Chess, Amazons and Konane: games much more interesting to play than the simple examples that we needed to introduce our theory.

Just as the subject of combinatorics was slow to be accepted by many "serious" mathematicians, so, even more slowly, is that of combinatorial games. But now it has achieved considerable maturity and is giving rise to an extensive literature, documented by Aviezri Fraenkel and exemplified by the book *Mathematical Go: Chilling Gets the Last Point* by Berlekamp and Wolfe. Games are fun to play and it's more fun the better you are at playing them.

The subject has become too big for us to do it justice even in the four-volume work that we now offer. So we've contented ourselves with a minimum of necessary changes to the original text (we are proud that our first formulations have so well withstood the test of time), with additions to the Extras at the ends of the chapters, and with the insertion of many references to guide the more serious student to further reading. And we've corrected some of the one hundred and sixty-three mistakes.

We are delighted that Alice and Klaus Peters have agreed to publish this second edition. Their great experience, and their competent and cooperative staff, notably Sarah Gillis and Kathryn Maier, have been invaluable assets during its production. And of course we are indebted to the rapidly growing band of people interested in the subject. If we mention one name we should mention a hundred; browse through the Index and the References at the end of each chapter. As a start, try *Games of No Chance*, the book of the workshop that we organized a few years ago, and look out for its successor, *More Games of No Chance*, documenting the workshop that took place earlier this year.

Elwyn Berlekamp, University of California, Berkeley
John Conway, Princeton University
Richard Guy, The University of Calgary, Canada

November 3, 2000

Preface

Does a book need a Preface? What more, after fifteen years of toil, do three talented authors have to add. We can reassure the bookstore browser, "Yes, this is just the book you want!" We can direct you, if you want to know quickly what's in the book, to the last pages of this preliminary material. This in turn directs you to Volume 1, Volume 2, Volume 3 and Volume 4.

We can supply the reviewer, faced with the task of ploughing through nearly a thousand information-packed pages, with some pithy criticisms by indicating the horns of the polylemma the book finds itself on. It is not an encyclopedia. It is encyclopedic, but there are still too many games missing for it to claim to be complete. It is not a book on recreational mathematics because there's too much serious mathematics in it. On the other hand, for us, as for our predecessors Rouse Ball, Dudeney, Martin Gardner, Kraitchik, Sam Loyd, Lucas, Tom O'Beirne and Fred. Schuh, mathematics itself is a recreation. It is not an undergraduate text, since the exercises are not set out in an orderly fashion, with the easy ones at the beginning. They are there though, and with the hundred and sixty-three mistakes we've left in, provide plenty of opportunity for reader participation. So don't just stand back and admire it, work of art though it is. It is not a graduate text, since it's too expensive and contains far more than any graduate student can be expected to learn. But it does carry you to the frontiers of research in combinatorial game theory and the many unsolved problems will stimulate further discoveries.

We thank Patrick Browne for our title. This exercised us for quite a time. One morning, while walking to the university, John and Richard came up with "Whose game?" but realized they couldn't spell it (there are three tooze in English) so it became a one-line joke on line one of the text. There isn't room to explain all the jokes, not even the fifty-nine private ones (each of our birthdays appears more than once in the book).

Omar started as a joke, but soon materialized as Kimberley King. Louise Guy also helped with proof-reading, but her greater contribution was the hospitality which enabled the three of us to work together on several occasions. Louise also did technical typing after many drafts had been made by Karen McDermid and Betty Teare.

Our thanks for many contributions to content may be measured by the number of names in the index. To do real justice would take too much space. Here's an abridged list of helpers: Richard Austin, Clive Bach, John Beasley, Aviezri Fraenkel, David Fremlin, Solomon Golomb, Steve Grantham, Mike Guy, Dean Hickerson, Hendrik Lenstra, Richard Nowakowski, Anne Scott, David Seal, John Selfridge, Cedric Smith and Steve Tschantz.

No small part of the reason for the assured success of the book is owed to the well-informed and sympathetic guidance of Len Cegielka and the willingness of the staff of Academic Press and of Page Bros. to adapt to the idiosyncrasies of the authors, who grasped every opportunity to modify grammar, strain semantics, pervert punctuation, alter orthography, tamper with traditional typography and commit outrageous puns and inside jokes.

Thanks also to the Isaak Walton Killam Foundation for Richard's Resident Fellowship at The University of Calgary during the compilation of a critical draft, and to the National (Science & Engineering) Research Council of Canada for a grant which enabled Elwyn and John to visit him more frequently than our widely scattered habitats would normally allow.

And thank you, Simon!

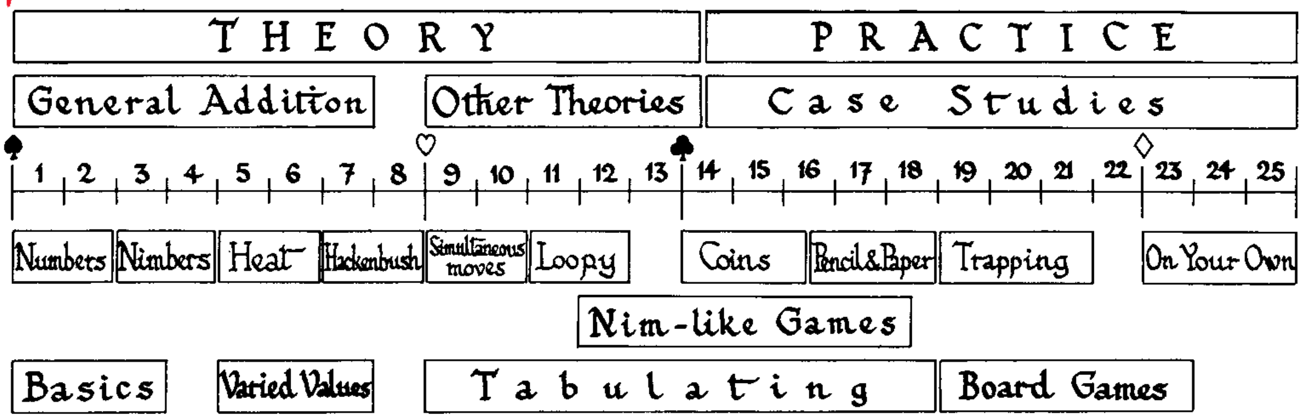
University of California, Berkeley, CA 94720 — *Elwyn Berlekamp*
University of Cambridge, England, CB2 1SB — *John H. Conway*
University of Calgary, Canada, T2N 1N4 — *Richard Guy*

November 1981

You are
now here

If you want to know roughly what's elsewhere,
turn to the little notes about our four main themes:

There are a number of other connexions between various chapters of the book:



However, you should be able to pick any chapter and read almost all of it without reference to anything earlier, except perhaps the basic ideas at the start of the book.

Spade-Work!

Let spades be trumps! she said, and trumps they were.
Alexander Pope, *The Rape of the Lock*, c.iii, I.46.

CECILY: When I see a spade I call it a spade.
GWENDOLEN: I am glad to say I have never seen a spade.
Oscar Wilde, *The Importance of Being Earnest*, II.

Our first few chapters do the spade-work for the rest by telling how to add games together and how to work out their values.

Chapters 1 and 2 introduce these ideas and show that some simple examples have ordinary numbers for values while others don't.

In Chapter 3 you'll see how the special values called nimbers, that arise in the game of Nim, suffice for *all* impartial games, and lots of examples are tackled in Chapter 4.

Chapter 5 has some very small games, and some others which, because they are both big (unlike nimbers) and hot (unlike numbers), really need the theory of Chapter 6.

Finally, Chapter 7 discusses the small games to within an atom or two, and Chapter 8 show how such values arise along with ordinary numbers in the game of Hackenbush.

-1-

Whose Game?

'Begin at the beginning,' the King said, gravely, 'and go on till you come to the end, then stop.'
Lewis Carroll, *Alice in Wonderland*, ch. 12

It is hard if I cannot start some game on these lone heaths.
William Hazlitt, *On Going a Journey*

Who's game for an easy pencil-and-paper (or chalk-and-blackboard) game?

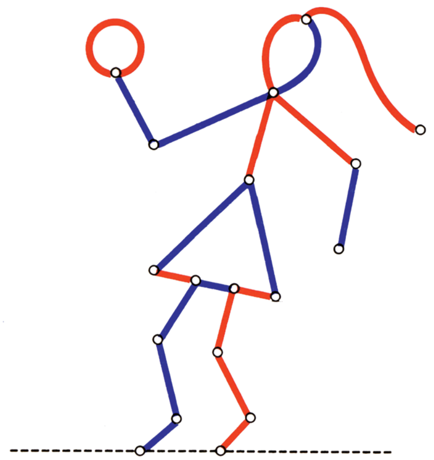


Figure 1. A Blue-Red Hackenbush Picture.

Blue-Red Hackenbush

Blue-Red Hackenbush is played with a picture such as that of Fig. 1. We shall call the two players **Left** and **Right**. Left moves by deleting any bLue edge, together with any edges that are no longer connected to the ground (which is the dotted line in the figure), and Right moves by deleting a Red edge in a similar way. (Play it on a blackboard if you can, because it's easier to rub the edges out.) Quite soon, one of the players will find he can't move because there are no edges of his color in what remains of the picture, and whoever is first trapped in this way is the loser. You must make sure that doesn't happen to you!

Well, what can you do about it? Perhaps it would be a good idea to sit back and watch a game first, to make sure you quite understand the rules of the game before playing with the professionals, so let's watch the effect of a few simple moves. Left might move first and rub out the girl's left foot. This would leave the rest of her left leg dangling rather lamely, but no other edges would actually disappear because every edge of the girl is still connected to the ground through her right leg. But Right at his next move could remove the girl completely, if he so wished, by rubbing out her right foot. Or Left could instead have used his first move to remove the girl's upper arm, when the rest of her arm and the apple would also disappear. So now you really understand the rules, and want to start winning. We think Fig. 1 might be a bit hard for you just yet, so let's look at Fig. 2, in which the blue and red edges are separated into parts that can't interact. Plainly the girl belongs to Left, in some sense, and the boy to Right, and the two players will alternately delete edges of their two people. Since the girl has more edges, Left can survive longer than Right, and can therefore win no matter who starts. In fact, since the girl has 14 edges to the boy's 11, Left ends with at least $14 - 11 = 3$ spare moves, if he chops from the top downwards, and Right can hold him down to this in a similar way.

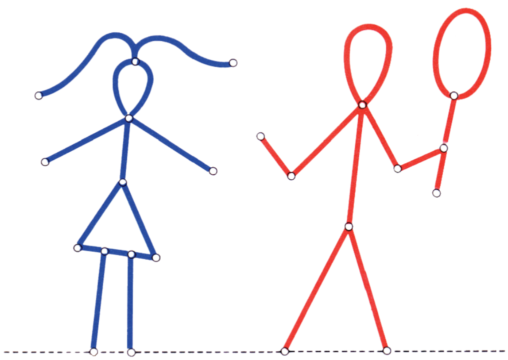


Figure 2. Boy meets Girl.

Tweedledum and Tweedledee in Fig. 3 have the same number of edges each, so that Left is $19 - 19 = 0$ moves ahead. What does this mean? If Left starts, and both players play sensibly from the top downwards, the moves will alternate Left, Right, Left, Right, until each player has made 19 moves, and it will be Left's turn to move when no edge remains. So if Left starts, Left will lose, and similarly if Right starts, Right will lose. So in this **zero position**, whoever starts loses.

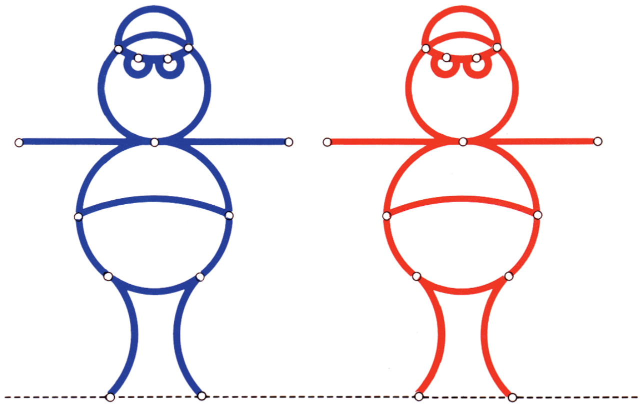


Figure 3. Tweedledum and Tweedledee, about to have a Battle.

The Tweedledum and Tweedledee Argument

In Fig. 4, we have swapped a few edges about so that Tweedledum and Tweedledee both have some edges of each color. But since we turn the new Dum into the new Dee exactly by interchanging blue with red, neither player seems to have any advantage. Is Fig. 4 still a zero position in the same sense that whoever starts loses? Yes, for the player second to move can copy any of his opponent's moves by simply chopping the corresponding edge from the other twin. If he does this throughout the game, he is sure to win, because he can never be without an available move. We shall often find games for which an argument like this gives a good strategy for one of the two players—we shall call it the **Tweedledum and Tweedledee Argument** (or **Strategy**) from now on.

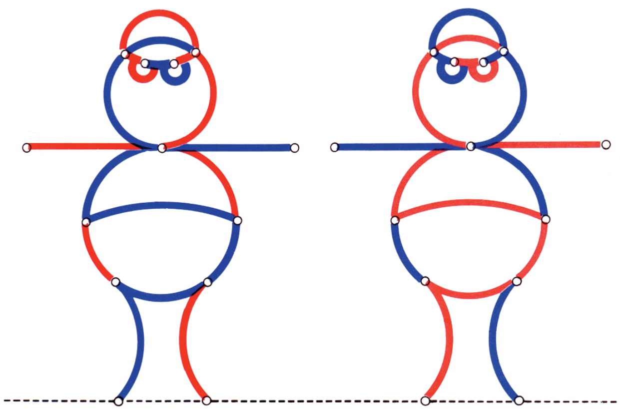


Figure 4. After their first Battle: Ready for the Next?

The main difficulty in playing Blue-Red Hackenbush is that your opponent might contrive to steal some of your moves by cutting out of the picture a large number of edges of your color. But there are several cases when even though the picture may look very complicated, you can be sure that he will be unable to do this. Figure 5 shows a simple example. In this little dog, each player's edges are connected to the ground via other edges of his own color. So if he chops these in a suitable order, each player can be sure of making one move for each edge of his own color, and plainly he can't hope for more. The value of Fig. 5 is therefore once again determined by counting edges—it is $9 - 7 = 2$ moves for Left. In pictures like this, the correct chopping order is to take first those edges whose path to the ground via your own color has most edges—this makes sure you don't isolate any of your edges by chopping away any of their supporters. Thus in Fig. 5 Left would be extremely foolish to put the blue edges of the neck and head at risk by removing the dog's front leg; for then Right could arrange that after only 2 moves the 5 blue edges here would have vanished.

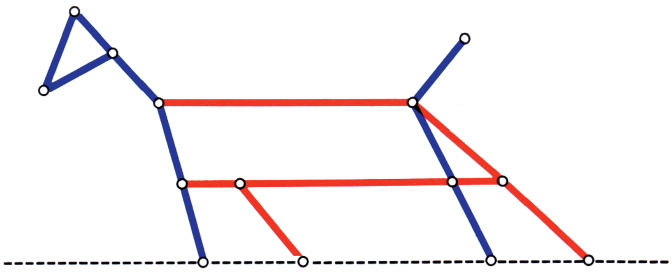


Figure 5. A Dog with Leftward Leanings.

How Can You Have Half a Move?

But these easy arguments won't suffice for all Hackenbush positions. Perhaps the simplest case of failure is the two-edge "picture" of Fig. 6(a). Here if Left starts, he takes the bottom edge and wins instantly, but if Right starts, necessarily taking the top edge, Left can still remove the bottom edge and win. So Left can win no matter who starts, and this certainly sounds like a positive advantage for Left. Is it as much as a 1-move advantage? We can try counterbalancing it by putting an extra red edge (which counts as a 1-move advantage for Right) on the ground, getting Fig. 6(b). Who wins now?

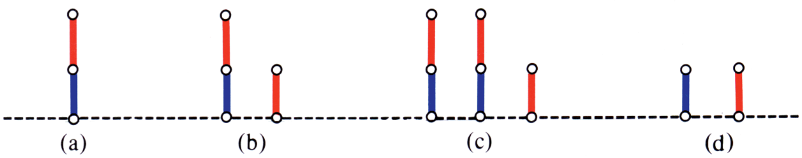


Figure 6. What do we mean by Half a Move?

If Right starts, he should take the higher of his two red edges, since this is clearly in danger. Then when Left removes his only blue edge, Right can still move and win. If Left starts, his only possible move still leaves Right a free edge, and so Right still wins. So this time, it is Right that wins, whoever starts, and Left's positive advantage of Fig. 6(a) has now been overwhelmed by adding the free move for Right. We can say that Left's advantage in Fig. 6(a), although positive, was strictly less than an advantage of one free move. Will it perhaps be one-half of a move?

We test this in Fig. 6(c), made up of *two* copies of Fig. 6(a) with just *one* free move for Right added, since if we are correct $\frac{1}{2} + \frac{1}{2}$ for Left will exactly balance 1 for Right. Who wins Fig. 6(c)? Left has essentially only one kind of move, leading to a picture like Fig. 6(b), which we know Right wins. On the other hand, if Right starts sensibly by taking either of his two threatened edges, Left will move to a picture like Fig. 6(d) and win after Right's next move. If Right has used up his free move at the outset, Left's reply would take us to Fig. 6(a), which we know he wins.

We've just shown that Right wins if Left starts and Left wins if Right starts, so that Fig. 6(c) is a zero game. This seems to show that *two* copies of Fig. 6(a) behave just like *one* free move for Left, in that together they exactly counterbalance a free move for Right. So it's really quite sensible to regard Fig. 6(a) as being a half-move's advantage for Left.

Putting Right's red edge partly under Left's control made Fig. 6(a) worse for him than Fig. 6(d). So perhaps Fig. 7(a) should be worth less to Right than Fig. 7(b) in which Right's edge is threatened by only one of Left's?

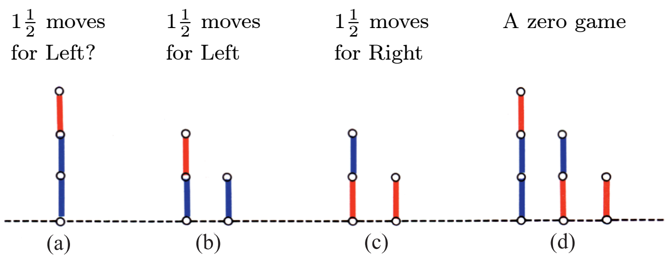


Figure 7. Is Right's Edge even more under Left's Control?

We are asking whether Fig. 7(a) is worth exactly $1\frac{1}{2}$ moves to Left like Fig. 7(b). We can test this by adding $1\frac{1}{2}$ free moves for Right to Fig. 7(a). Since Fig. 7(c) is the opposite of Fig. 7(b), we produce the required allowance by adjoining it to Fig. 7(a), giving Fig. 7(d).

Who wins this complicated little pattern? Here each player has just one risky edge partly in control of his opponent, and if a player starts by taking his risky edge, his opponent can remove the other, leaving two unfettered moves each. If instead he takes the edge just below his opponent's risky edge, the opponent can do likewise, now leaving just one free move each. The only other starting move for Left is stupid since it leaves only red edges touching the ground and indeed Right can win with a move to spare.

What about Right's remaining move? Since this is to remove the isolated red edge, it *must* be stupid, for surely it would be better to take the middle red edge and so demolish a blue edge at the same time? And indeed Left's reply of chopping the middle edge of the chain of three proves perfectly adequate. So *every* first move loses, and once again the game is what we called a zero game. This seems to show that contrary to our first guess, Figs. 7(a) and 7(b) confer exactly the same advantage to Left, namely one and a half free moves.

... And Quarter Moves?

In Fig. 8(a), Right's topmost edge is partly under Left's control, but also partly under Right's as well, so it should perhaps be worth more to him than his middle one? Since we found that the middle edge was worth half a move to Right, the pair of red edges collectively would then be worth at least a whole move to him, counteracting Left's single edge. So maybe Right has the advantage here?

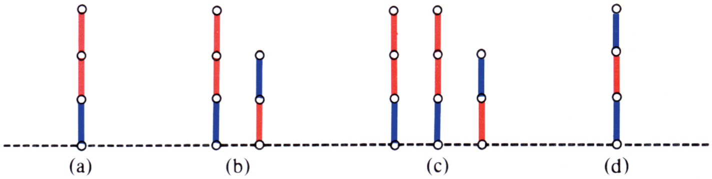


Figure 8. Are Right's Edges worth more than Left's?

This naive opinion is dispelled as soon as play starts, for Left's only move wins the game as soon as he makes it, showing that Fig. 8(a) gives a positive advantage to Left. But when we adjoin half a move for Right as in Fig. 8(b), Right can win by playing first, by removing the topmost edge, or playing second, by removing the highest red edge remaining. So Fig. 8(a), though a positive advantage for Left, is worth even less to him than half a move. Is it perhaps, being three edges high, worth just one-third of a move? No! We leave the reader to show that two copies of Fig. 8(a) exactly balance half a move for Right, by showing that the second player to move wins Fig. 8(c), so that Fig. 8(a) is in fact a quarter move's advantage for Left.

And how much is Fig. 8(d) worth?

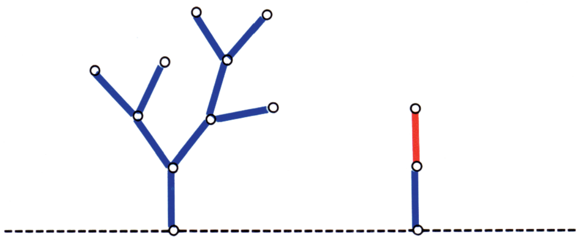


Figure 9. A Hackenbush Position worth $9\frac{1}{2}$.

Figure 9 shows a Hackenbush position of value $9\frac{1}{2}$, since the tree has value 9, and the rest value $\frac{1}{2}$. What are the moves here? Right has a unique red edge, and so a unique move, to a position of value $9+1=10$, but Left can move either at the top of the tree, leaving $8\frac{1}{2}$, *or* by removing the $\frac{1}{2}$ completely, which is a better move, since it leaves value 9. Since Left's best move is to value 9, and Right's to 10, we express this by writing

$$\{9|10\} = 9\tfrac{1}{2} \qquad \text{(“9 slash 10 equals } 9\tfrac{1}{2}\text{”)}$$

In a similar way, we have the more general equation

$$\{n|n+1\} = n + \tfrac{1}{2},$$

of which the simplest case is

$$\{0|1\} = \tfrac{1}{2},$$

with which we began. We also have the simpler equation

$$\{n|\ \} = n + 1$$

for each $n = 0, 1, 2, \ldots$, for if Left has just $n+1$ free moves, he can move so as to leave just n free moves, while Right cannot move at all. The very simplest equation of this type is

$$\{\ |\ \} = 0$$

which expresses the fact that if neither player has a legal move the game has zero value.

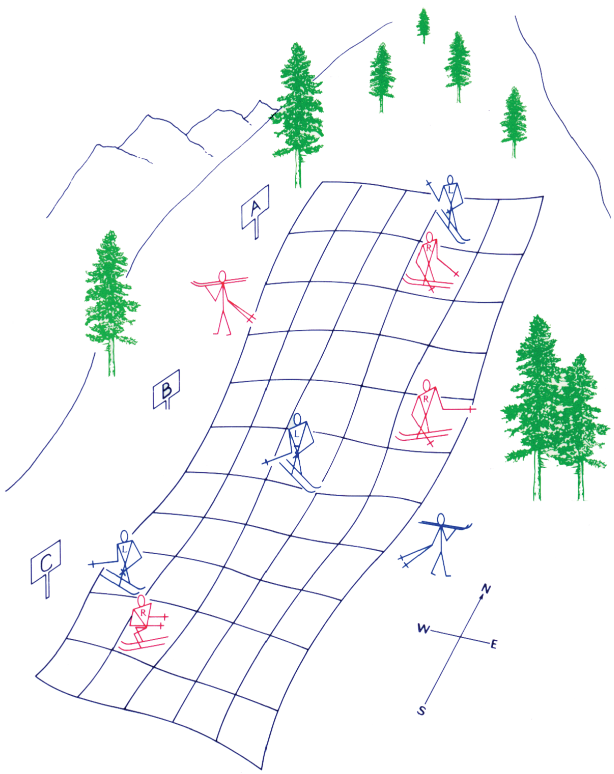


Figure 10. A Game of Ski-Jumps.

Ski-Jumps for Beginners

Figure 10 shows a ski-slope with some skiers in the pay of Left and Right, about to participate in our next game. In a single move, Left may move any skier a square or more Eastwards, or Right any one of his, Westwards, provided there is no other active skier in the way. Such a move may take the skier off the slope; in this case he takes no further part in the game. No two skiers may occupy the same square of the slope. Alternatively a skier on the square immediately *above* one containing a skier of the opposing team, may jump over him onto the square immediately below, provided this is empty. A man jumped over is so humiliated that he will never jump over anyone else—in fact he is demoted from being a *jumper* to an ordinary skier, or *slipper*!

No other kind of move is permitted in this game, so that when all the skiers belonging to one of the players have left the ski-slope, that player cannot move, and a player who cannot move when it is his turn to do so, loses the game. Let's examine some simple positions. Figure 11(a) shows a case when Left's only jumper is already east of Right's, so that no jump is possible. Since Left's man can move 5 times and Right's only 3, the value is $5 - 3 = 2$ spare moves for left.

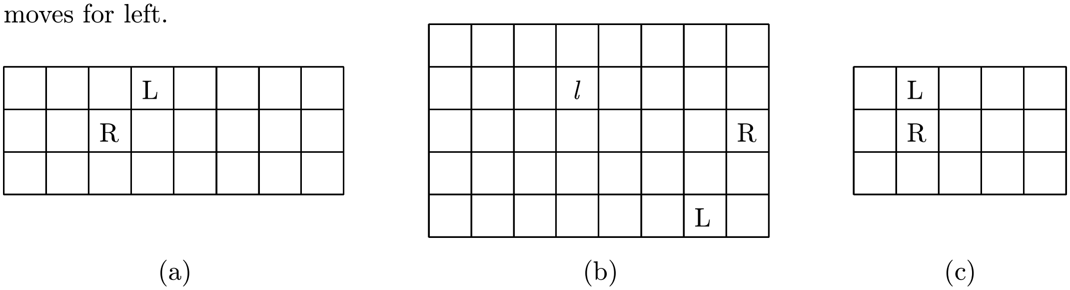


Figure 11. Some Ski-Jumps Positions.

We can similarly evaluate any other position in which no further jumps are possible. Thus in Fig. 11(b) Left has one man on the row above Right's, and another lower down, but still no jump will be possible, for Left's upper man has been demoted to a mere slipper (hence his lower case name, l), while his lower man, being two rows below Right's, is not threatened. Left's two men have collectively $2 + 5$ moves to Right's 8, so the value is

$$2 + 5 - 8 = -1$$

moves to Left, that is, 1 move in favor of Right.

Now let's look at Fig. 11(c), in which Left's man may jump over Right's, if he wishes. If he does so, the value will be $4 - 2 = 2$, which is better than the value $3 - 2 = 1$ he reaches by sliding one place East. If, on the other hand, Right has the move, it will be to a position of value $4 - 1 = 3$. So the position has value

$$\{2|3\} = 2\tfrac{1}{2}$$

moves to Left. More generally, if Left has a single man on the board, with a spaces (and hence $a + 1$ moves) before him, and Right a single man with b spaces before *him*, and one of the two men is now in a position to jump over the other, the value will be

$$a - b + \tfrac{1}{2} \quad \text{or} \quad a - b - \tfrac{1}{2}$$

according as it is Left's or Right's man who has the jump. We can think of an imminent jump as being worth half a move to the player who can make it.

Figure 12 shows all the positions on a 3×5 board in which there are just two men, of which Left's might possibly jump Right's either at his first move or later.

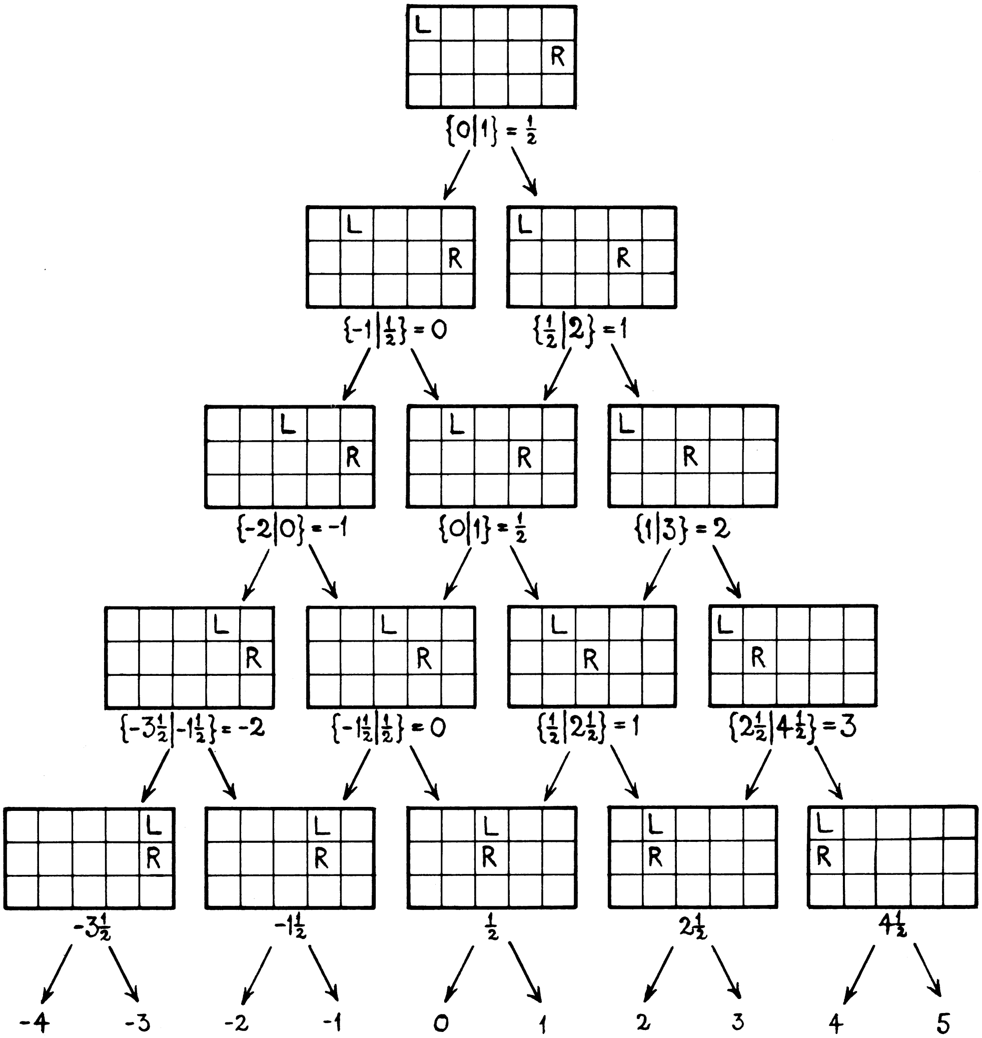


Figure 12. Ski-Jumps Positions on a 3×5 Board.