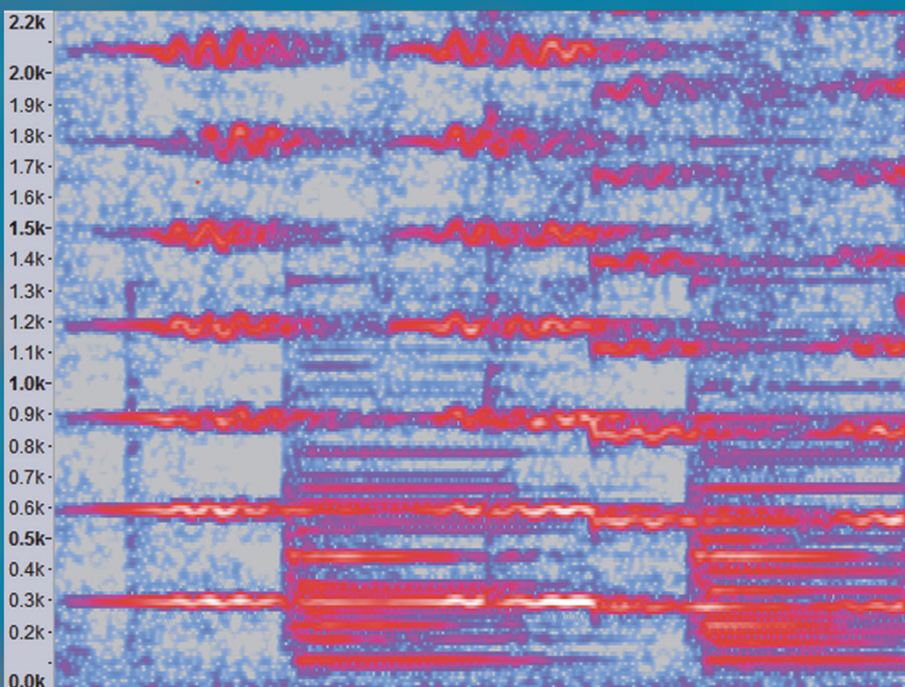
MATHEMATICS AND MUSIC

Composition, Perception, and Performance

SECOND EDITION



James S. Walker

Gary W. Don

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Printed on acid-free paper

International Standard Book Number-13: 978-1-138-58494-5 (Hardback)

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To Angela

Contents

Preface

> *No school would eliminate the study of language, mathematics, or history from its curriculum, yet the study of music, which encompasses so many aspects of these fields and can even contribute to a better understanding of them, is entirely ignored.*
>
> —Daniel Barenboim (2008)

This is the 2nd edition of our book on mathematics and music. As with the 1st edition, our purpose is to explore the connections between mathematics and music. This may seem to be a curious task. Aren't mathematics and music from separate worlds, mathematics from the world of science and music from the world of art? While mathematics does belong to the world of science, one of the goals of science is to understand everything that we experience, and music is no doubt an essential part of human experience. Mathematics has been described as the science of patterns, and we shall see that there are many patterns in music that can be described with mathematics. Mathematics has also been described as the language of the universe, and music itself has been described in such a poetic way. In fact, connections between these two subjects go back thousands of years. For example, the classical Greek mathematician, Pythagoras, contributed the essential ideas for how we quantify changes of pitch for musical tones (musical intervals). The connections between mathematics and music have grown enormously since those ancient days. We will try to explore as many of these connections as possible, in a way that presents both the mathematics and the music to as wide an audience as possible.

What's New in the 2nd Edition

We have added a lot of new material to this edition. The most important additions are the following:

- A large number of color illustrations. The previous edition had only 16 color figures displayed in a special insert. This 2nd edition expands these to over 100 color figures, all displayed within the text itself for ease of reference.
- Two sections have been added on musical composition. Readers are given the opportunity to try composing short pieces of music. These musical composition sections have been very popular with our students, even those who are not musicians.
- A section has been added on analyzing personal performance. We hope this section is useful for readers who sing or play a musical instrument. We describe how you can record your performance, *without the need for professional recording equipment*, and do useful analysis of your musical technique.
- Two sections have been added on the free, but powerful, rhythm generator XronoMorph. Our students have enjoyed using XronoMorph to generate their own original rhythm sequences.
- We have moved the book web page (see below) to a new location, which should be more easily accessible to readers.

Some other changes will be referred to in the following summary of chapters.

Summary of Chapters

Here is a brief description of the main topics covered in the book. For more details, please consult the Table of Contents.

Chapter 1 describes the scientific approach to musical pitch, first worked out by Helmholtz in the 19^{th} century. Helmholtz's theory, which relates pitch to frequency, provides a foundation for understanding different musical scales. One very distinctive aspect of our treatment of this material is our use of ***spectrograms***. A spectrogram is a graphical portrait of the tones within a musical passage, plotting these tones in terms of their frequencies and the time during which they are sounding. We believe that spectrograms are an important tool for understanding and appreciating music, and that they are not difficult to interpret correctly. So we introduce them before we describe the mathematics used to create them; we postpone that discussion to Chapter 4. Although some might object to using a mathematical technique before describing the details underlying it, we believe that the spectrogram examples described here are so compelling, and so dramatically illustrate this material, that we simply had to include them. In any case, they should provide a strong motivation for learning the mathematics of spectrograms described in Chapter 4. Chapter 2 provides a brief introduction to musical notation. It describes just enough notation so that all readers, even those who are not musicians, should be able to read the brief score excerpts that we include in the book. There are a number of such score excerpts in Chapter 3, where we provide some background on basic music theory. This basic music theory is surprisingly mathematical. We emphasize the different musical transformations—scale shiftings, transpositions, inversions—that composers have employed for centuries. These transformations do have a clear mathematical interpretation. The chromatic clock introduced in Chapter 1 plays an important role in understanding this basic music theory. The chapter concludes with a section on composing your own music.

As described in the last paragraph, in Chapter 4 we discuss the mathematics of ***spectrograms***. In addition to the mathematics, we also provide some interesting musical illustrations, such as the phenomenon known as ***beating*** and its relation to musical consonance and dissonance. In Chapter 5 we demonstrate how spectrograms provide revealing insights into musical structure. These insights would be difficult if not impossible to obtain through listening alone, because listening involves mostly short-term memory, while spectrograms can display an analysis of several minutes of music. Furthermore, when videos of spectrograms are traced out as the music is played they allow us to see ahead what tones are to be played, thereby enhancing our anticipation of the music's development. Spectrograms also allow us to detect, and more deeply appreciate, subtle aspects of musical sound quality such as vibrato, dynamic emphasis, and percussion. All of these insights would be difficult, if not impossible, to gain if one only analyzed scores. Spectrograms provide a powerful tool for analyzing the music that we hear, rather than the notes prescribed for musicians to play. Having another tool for analyzing music, in addition to musical scores, is very valuable. One way that spectrograms and scores work together is that spectrograms reveal the overtone structure of the notes played from a musical score. This overtone structure is very important for understanding musical intervals, which are the building blocks of melody and harmony. The chapter concludes with a section on using spectrograms to analyze personal performance. We have written this personal performance section in such a way that it can be used in classes that contain both musicians and non-musicians.

We have described some of the many valuable contributions that spectrograms make to the study and appreciation of music. Our students generally consider the material on spectrograms in Chapters 4 and 5 to be the highlight of the book. Following these chapters, we incorporate rhythm into our study of the mathematical aspects of music. In Chapter 6 we describe how pitch and rhythm share many of the same mathematical features. Most books on music, both in music theory and in mathematical treatments, focus exclusively on pitch and harmony. We believe our treatment of rhythm provides our book with a more complete description of music. For this second edition, we have added four new sections to this chapter. A new section on Bruch's *Kol Nidrei* provides a case study in rhythm and its connection with harmony. Two other new sections discuss two important rhythm types: perfectly balanced rhythms and well-formed rhythms. These sections also illustrate the use of the free

XronoMorph software for generating rhythms. The chapter concludes with another new section, a continuation of the musical composition material that concluded Chapter 3.

The six chapters just described form the core material of the book. The two chapters that follow them describe more advanced mathematical aspects of music. Throughout the book, we make use of geometrical diagrams to aid us in understanding the basic logic of pitch organization and harmony. Chapter 7 explores this connection of geometry with music theory more deeply. The chapter concludes with a new section that introduces some of the most recent geometrical music theory. Chapter 8 describes some of the ways that computers can be used for synthesizing music. Electronically synthesized music is widely used, and we have tried to explain how it works without getting overwhelmed by technicalities. We have added a new section that describes how the widely used pitch processor, Auto-Tune, works. The chapter concludes with another new section that surveys various resources for digital sound synthesis.

Web site

To aid in the study of this book, there is an accompanying web site:

https://jameswalkermathmusic.net/

There are links at the book's web site for videos of many of the spectrograms we discuss in the book. You can also download the musical scores we examine in the book, playable with the free music software MuseScore. We have supplied an online bibliography with many links to free downloadable articles on math and music. Finally, there are links to other web sites related to math and music, including all the ones mentioned in the book.

Prerequisites

To read this book, one needs to have a good background in high school mathematics. We will not assume, however, the ability to read music. The book aims to teach some mathematics, so there are exercises at the end of each section. It also aims to teach how the mathematics relates to music, so many of the exercises involve musical examples. At the end we hope the reader will have a greater mastery of some fundamental mathematics, and a deeper appreciation of music. An appreciation of music made deeper because it is informed by both its mathematical and aesthetic structure.

Music Software

The world of recorded music has been enormously changed in the last three decades or so with the introduction of computer technology. In this book, we use computers to aid in applying mathematics to the analysis of music, and also to the creation of new music. Mostly, we use three *free* software programs. These three free programs are

1. Audacity. An audio editor. We have used it for creating and playing spectrograms.

2. MuseScore. A musical scoring program. We have used it to create brief passages of musical scores, which you can play on MuseScore when studying these passages in the text.

3. XronoMorph. A rhythm generator. In Chapter 6, we discuss how to use it to generate three types of rhythms: perfectly balanced rhythms, well-formed rhythms, and Euclidean rhythms.

The book can be studied without working with these programs, although we encourage you to try them. We provide some tutorials on using these programs in Appendix E.

Order of Chapters

Chapters are mostly organized sequentially. Each chapter uses, to a degree, material from preceding chapters. Chapter 8 is an exception, as it can be read immediately following Chapter 4. Although chapters proceed sequentially, there is some flexibility in how they can be covered in a classroom setting. For example, in our Mathematics and Music course at UW-Eau Claire, we have successfully taught the material using the following sequence:

Chapter 1, Chapter 4, Chapter 5, Chapter 3, Chapter 6, Chapter 8, Chapter 7.

Since typically at least half of the class can play and read music, Chapter 2 is given as optional reading at the start of the class for those students who need to learn basic music notation. Having students work in groups on material, such as Chapter 3 with its emphasis on music theory, can be very helpful for those students who have a great interest in understanding music but lack performance ability. We have found, however, that even students who are not musicians can master the elementary material in Chapter 2 on their own, and then read the music theory in Chapter 3 with understanding.

Acknowledgments

It is a pleasure to acknowledge as many people as I can, who have helped me with this project. Gary Don, professor of music at UWEC, has been a constant supportive colleague from the world of music. Simply listing him as musical consultant for this book does not really do justice to the enjoyable interactions and collaborations that we have been engaged in for nearly two decades. My Mathematics Department Chair at UWEC, Alex Smith, has done everything in his power to help me teach my Mathematics and Music course for more than a decade. Without his hard work on my behalf, this book would simply not exist. Steve Krantz, Professor of Mathematics at Washington University-St. Louis, has given me a lot of encouragement and help in publishing my papers on this subject. The work of Bill Sethares on connections between music, mathematics, and engineering, have provided many insights that helped me in writing this book. I would also like to thank my Executive Editor, Bob Ross, for all of his help.

The scholarly support programs at UWEC—the Center for Excellence in Teaching and Learning and the Office of Research and Sponsored Programs—provided me with grants supporting the research and writing activities needed for producing both editions of this book. One extremely important grant was for funding my sabbatical leave at Macalester College in the academic year 2011–12. While I was at Macalester, I was able to teach a course in Mathematics and Music. I particularly want to thank Karen Saxe, Chair of the Department of Mathematics, Statistics, and Computer Science, and Mark Mazullo, Chair of the Department of Music, at Macalester for arranging my position as Visiting Professor in those departments.

For both editions of this book, my students have given me a lot of help. I would especially like to thank Claire Arneson, Emily Gullerud, Kevin Hulke, Lara Conrad, Hannah Stoelze, Michael Jacobs, Stewart Wallace, Jeanne Knauf, Andrew Jannsen, Gary Baier, Andrew Detra, Kaitlyn Johnstone, Joshua Fuchs, Abigail Doering, Thomas Kokemoor, Carmen Whitehead, Andrew Hanson, Xiaowen Cheng, Jarod Hart, Karyn Muir, Brent McKain, Yeng Chang, and Marisa Berseth.

Finally, a heartfelt thanks to my wife, Angela Huang. I am very grateful for her patient support, and I dedicate this book to her.

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Gary W. Don received his doctorate in music theory from the University of Washington in 1991. He teaches theory and aural skills, 20th-century techniques, counterpoint, and form and analysis as a professor in the department of Music & Theatre Arts at the University of Wisconsin-Eau Claire. The topics of his published articles include Goethe's influence on music theorists of the 19th and 20th centuries, overtone structures in the music of Debussy, and theory pedagogy.

Chapter 1

Pitch, Frequency, and Musical Scales

> *...in nature itself, a single note sets up a harmony of its own; and this harmonic series has been the (unconscious) basis of Western European harmony, and the tonal system.*
>
> —Deryk Cooke (1959)

The scientific study of music was put on a firm footing with the seminal work of Hermann von Helmholtz in the middle of the 19th century. His masterpiece, Helmholtz (1954), is still worth studying today. In this chapter we describe Helmholtz's ideas and show how they provide a rationale for musical scales.

1.1 Pitch and Frequency

There is a close connection between pitches in musical tones and the mathematical concept of *frequency*. In the 19th century, Helmholtz did an experiment with tuning forks. He attached a pen to one of the tines of a tuning fork and drew the fork across a piece of paper while it was sounding a specific pitch. The vibration of the pen traced out a simple waveform. We will refer to it as a ***pure tone waveform***. See Figure 1.1.

The most fundamental aspect of a pure tone waveform is that it repeats itself periodically. In physics, the distance from one peak of the wave to the next is called its ***wavelength***. Another term for wavelength is ***cycle***. We have marked one cycle for the pure tone waveform in Figure 1.1. The number of cycles in the pure tone waveform that occur in 1 second is called its ***frequency***. We have this formula for frequency:

$$\text{frequency} = \frac{\text{number of cycles}}{\text{second}}.$$

The unit of cycles/sec for measuring frequency is also called Hz, which is short for Hertz (another German physicist who did fundamental work in the study of frequency). For example, if the cycle shown in Figure 1.1 has a time duration of 0.025 seconds, then the frequency of the pure tone waveform is $1/0.025 = 40$ Hz.

Nowadays, with digital technology, we can record a representation of the sound wave from a tuning fork as a further demonstration of Helmholtz's idea. In Figure 1.2 we show the plot of such a digital waveform recorded from a tuning fork. This tuning fork was designed to match the pitch for the key of middle C on a piano.[1] As described in the caption of Figure 1.2, the frequency for the pure tone waveform produced by the tuning fork is 262 Hz. The maximum y-values of the waveform are all approximately 5000 (and the minimum values are all approximately -5000). This number, 5000, is called the ***amplitude*** of the pure tone. The larger a pure tone's amplitude, the louder the volume of its sound.

To conclude our analysis of pure tones from tuning forks, we look at an extremely important method for displaying the single, constant frequency of the pure tone over time. This method of

[1] See Figure 1.14 on p. 15.

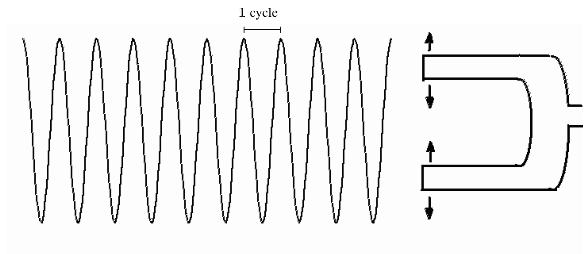

Figure 1.1. Illustration of the famous experiment of Helmholtz. A pen is attached to a tine of tuning fork. As the tuning fork is struck and drawn across a piece of paper at a uniform speed, the pen traces out a pure tone waveform. Distance marked between two peaks of a waveform is called a ***cycle***.

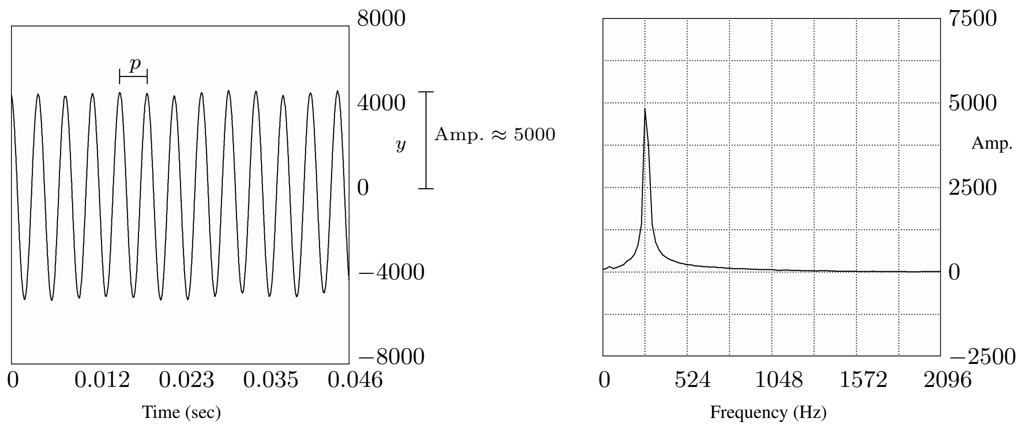

Figure 1.2. Left: The waveform from a recording of a tuning fork with one cycle marked, $p = 0.00382$ seconds. Frequency is about 1 cycle/0.00382 sec $=$ 262 Hz, so the note from the tuning fork is middle C (c.f. Table 1.13, p. 13). Right: Amplitude for the waveform. The height of the spike at frequency 262 Hz is approximately 5000. In Chapter 4 we describe how this amplitude plot was obtained.

display is called a ***spectrogram***. In Figure 1.3 we show a spectrogram of a recording of the sound from the tuning fork discussed above. We will describe how this spectrogram was produced in Chapter 4. We are using it now because it provides such an easily interpretable and compelling picture of the frequency content of this pure tone. The thin horizontal band in the spectrogram is centered on the single frequency of 262 Hz that we found for this tuning fork's pure tone.

We have now shown that there is a definite connection between frequency and pitch. For a pure tone from a tuning fork, there will be a single precise frequency for its waveform. Music, of course, is played on musical instruments rather than tuning forks. So we now turn to the question of frequency and pitch for musical instruments.

1.1.1 Instrumental Tones

There is a huge variety of musical instruments, including human voices, violins, pianos, clarinets, and many others. The tones produced by these instruments are far more complex, and musically interesting, than the tones produced by tuning forks. See the left sides of Figures 1.4 and 1.5 for examples of portions of waveforms from a flute and piano playing the same note. These waveforms have a fundamental cycle, at least approximately. We show this approximate fundamental cycle on the left side of Figure 1.4 for the flute tone, where it is easier to see in the graph. This fundamental cycle determines the frequency of approximately 329 Hz for the note being played. We can see, however, that these waveforms are not cycling in nearly so uniform a manner as the tuning fork waveform

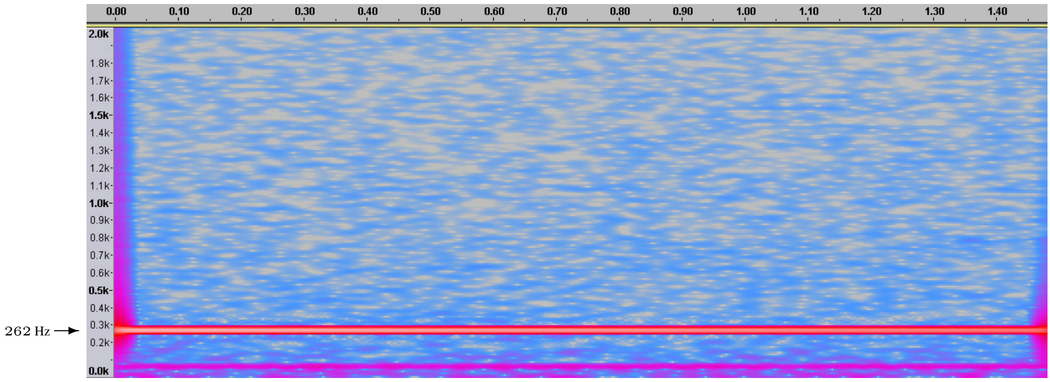

Figure 1.3. Spectrogram of a recording of tone from a tuning fork. Horizontal axis is the time axis, labeled in units of seconds along top. The vertical axis is the frequency axis, labeled in units of kHz (1 kHz = 1000 Hz) along the left side. The thin horizontal band is centered on frequency 262 Hz, the frequency for the tuning fork tone (c.f. Figure 1.2). In color spectrograms: yellow-white is loudest, red is medium loud, purple is faint, blue is barely audible or inaudible. Within those color ranges, brighter is always louder. To watch a video of this spectrogram tracing out in time, as the sound from the tuning fork is played, please visit the book's web site and click on the link ***Videos***.

shown in Figure 1.2. In fact, as we shall examine more closely in Chapter 4, they are *combinations* of several pure tone waveforms of differing frequency and loudness. For both of these instruments, these combined pure tone waveforms have frequencies that are positive integer multiples of 329 Hz:

$$329 \text{ Hz}, \quad 2 \cdot 329 \text{ Hz}, \quad 3 \cdot 329 \text{ Hz}, \quad 4 \cdot 329 \text{ Hz}, \quad 5 \cdot 329 \text{ Hz}, \quad 6 \cdot 329 \text{ Hz}, \ \ldots \tag{1.1}$$

These frequencies are called the ***harmonics*** for these instrumental tones.

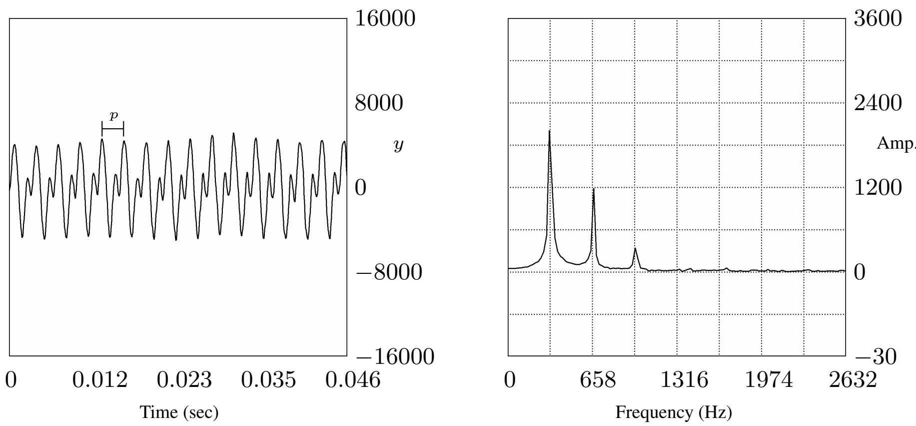

Figure 1.4. Left: Waveform for a recording of a flute playing a single note. Time p for an approximate cycle is $p = 0.00304$ seconds. Fundamental frequency is approximately 329 Hz. Right: Amplitudes of harmonics for the waveform. Harmonics of 329 Hz, $2 \cdot 329$ Hz, and $3 \cdot 329$ Hz, are clearly marked by spikes. Heights of spikes correspond with amplitudes of each pure tone within the complete tone. In Chapter 4 we describe how this plot of amplitudes for harmonics was obtained.

For the flute note, the first two harmonics of 329 Hz and $2 \cdot 329 = 658$ Hz are the loudest, the third harmonic $3 \cdot 329 = 987$ Hz is fainter, and the higher multiples of 329 Hz are fainter still. See the right side of Figure 1.4. In the graph shown there, the heights of the peaks at 329 Hz, 658 Hz, and 987 Hz, correspond to how loud those pitches would be *if those pitches were heard separately.* They are not heard separately, however. It is their combination that produces the complex sound of the flute's tone. We will discuss this point in more detail at the end of this section.

Similarly, the note played by the piano is a combination of harmonics. For the piano, however, the graph on the right of Figure 1.5 shows that the amplitudes of the harmonics are much more equally distributed in size. An interesting feature of this graph is that the magnitude for the harmonic $2 \cdot 329$ Hz is actually larger than the magnitude for 329 Hz. Nevertheless, we refer to 329 Hz as the ***fundamental*** for this piano note since it corresponds to the pitch that the note is sounding at (which is the same pitch as the flute note). We shall emphasize this point again later, as it is a subtle one: ***The fundamental harmonic for a note is the frequency that determines the note's pitch, and that may or may not be the loudest harmonic produced when the note is played.***

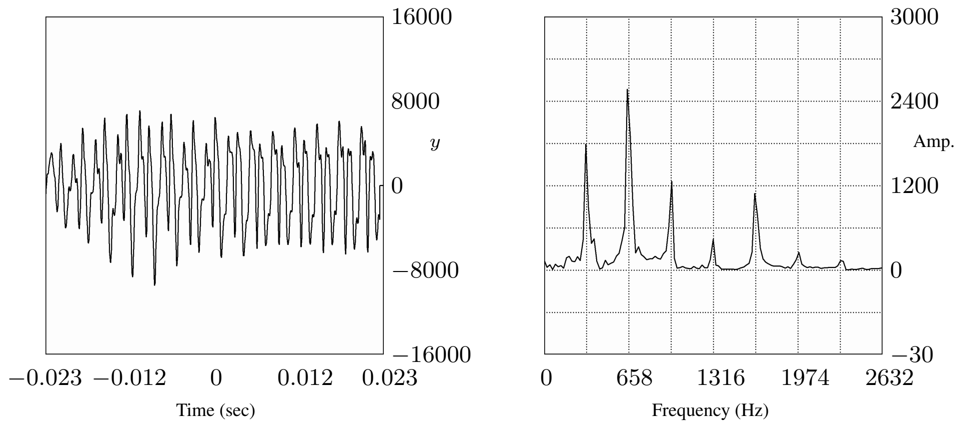

Figure 1.5. Left: Waveform for a recording of a piano playing a single note. Right: Amplitudes of harmonics within the piano note, harmonics marked by spikes at multiples of 329 Hz. The first spike is at 329 Hz, second spike at $2 \cdot 329$ Hz, third spike at $3 \cdot 329$ Hz, up to seventh spike at $7 \cdot 329$ Hz.

The graphs of amplitudes of harmonics of flute and piano tones shown on the right of Figures 1.4 and 1.5 were obtained by computer processing of recordings of these tones. We shall explain in Chapter 4 how this processing is done.

We summarize this discussion with a definition of these harmonics in instrumental tones.

Definition 1.1.1. *For an instrumental tone that contains frequencies of the form:*

$$\nu_o, \quad 2\nu_o, \quad 3\nu_o, \quad 4\nu_o, \quad 5\nu_o, \quad 6\nu_o, \ \ldots$$

The smallest frequency, ν_o, is called the ***fundamental****. The other frequencies are called* ***overtones****. All of the frequencies are called* ***harmonics****. The* ***first harmonic*** *is ν_o, the* ***second harmonic*** *is $2\nu_o$, the* ***third harmonic*** *is $3\nu_o$, and so on.*

Remark 1.1.2. The physical explanation for why tones from musical instruments contain multiple harmonics is beyond the scope of this book. See Walker (1988, Chap. 3) for a discussion of stringed instruments. For other instruments, consult Olson (1967) and Benade (1990).

1.1.2 Pure Tones Combining to Create an Instrumental Tone

We have described how instrumental tones are combinations of pure tones. We shall now demonstrate this for a single trumpet tone. In Figure 1.6, we show a spectrogram of a recording of a trumpet playing the note of middle C, with fundamental $\nu_o = 262$ Hz, and of the individual pure tones that combine to create this instrumental tone. As a static picture, this spectrogram shows single bright bands corresponding to the individual harmonics of the trumpet tone, and how they combine to produce the complete tone. However, to fully appreciate this spectrogram, it is absolutely necessary to watch a video of it. Please visit the book web page listed in the Preface, and click on the link, Videos. As the spectrogram traces out, you will hear the individual harmonics sounding just like individual tuning forks, as they should because they correspond to the pure tones making up the trumpet tone. At the end of the spectrogram, you will hear all of these pure tones playing together, creating the complete trumpet sound.

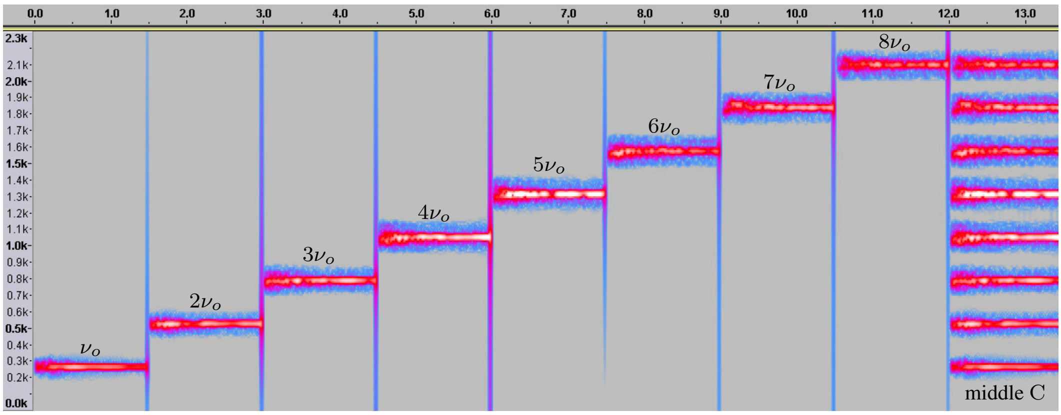

Figure 1.6. Spectrogram illustrating combination of harmonics in a trumpet tone for a single note. From 0.00 to 1.5 seconds: the tone's fundamental of $\nu_o = 262$ Hz displayed as bright band. From 1.5 to 3.0 seconds: the tone's second harmonic of $2\nu_o = 524$ Hz displayed as another bright band. From 3.0 to 4.5 seconds, the tone's third harmonic of $3\nu_o = 786$ Hz displayed as third bright band. Each of the first 8 harmonics are displayed in ascending order from left to right, finishing at 12.0 seconds. Sounds from these harmonics are indistinguishable from tuning fork tones. (Thin vertical bars at the start and end of individual harmonics — heard as clicking noises in the playback — are artifacts of the process of clipping these harmonics out from the original trumpet note recording.) Variations in brightness in these harmonic bands correspond to variations in loudness of sound from the harmonics: brighter parts of bands correspond to louder parts of sound from the harmonics. From 12.0 seconds onward, these 8 harmonics combine to create one tone, the tone of a trumpet.

The reader may have noticed that the separate eight pure tones displayed in this last example sound like a portion of an ascending scale. In the next section, we make this idea precise by discussing the connection between harmonics of instrumental tones and the notes used on musical scales.

Exercises

1.1.1. A pure tone has duration $p = 0.02$ seconds for one cycle. What is its frequency?

1.1.2. A pure tone has duration $p = 0.004$ seconds for one cycle. What is its frequency?

1.1.3. For the graph of amplitudes of harmonics shown on the left of Figure 1.7, estimate the frequencies of the harmonics and find the fundamental frequency.

1.1.4. On the right of Figure 1.7 there is a graph of the magnitudes of the harmonics from a male pronouncing the vowel ō (long o). Estimate the frequencies of the harmonics and find the fundamental frequency.

1.1.5. On the left of Figure 1.8 there is a graph of the magnitudes of the harmonics from a female pronouncing the vowel ā (long a). Estimate the frequencies of the harmonics and find the fundamental frequency.

1.1.6. The right of Figure 1.8 contains a graph of amplitudes of harmonics from a male pronouncing the vowel ā (long a). Estimate frequencies for these harmonics and find the fundamental frequency. What difference do you observe compared to the previous exercise with the female speaker? How do you explain this difference?

1.1.7. Why does the tone from the flute (with the magnitude of its harmonics graphed in Figure 1.4), sound more pure than the tone from the piano (with the magnitude of its harmonics graphed in Figure 1.5)? On the other hand, why does the tone from the piano sound more rich (more complex) than the tone from the flute?

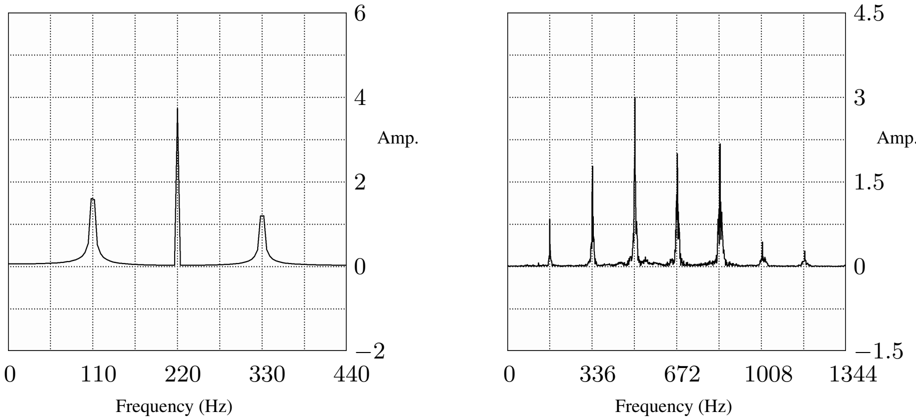

Figure 1.7. Left: Graph of amplitudes of harmonics for Exercise 1.1.3. Right: Graph of amplitudes of harmonics for Exercise 1.1.4, male pronouncing vowel ō (long o).

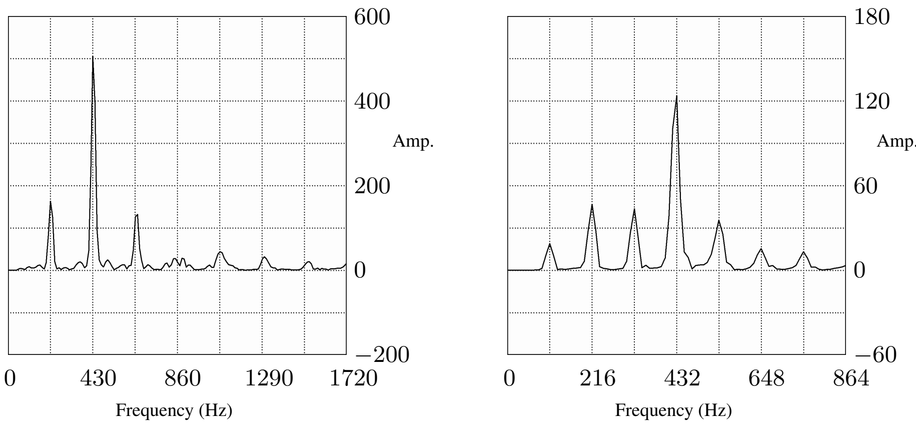

Figure 1.8. Left: Graph of amplitudes of harmonics for Exercise 1.1.5, female pronouncing vowel ā (long a). Right: Graph of amplitudes of harmonics for Exercise 1.1.6, male pronouncing same vowel.

1.1.8. On the left of Figure 1.9 there is a graph of the amplitudes of the harmonics from a male pronouncing the vowel ē (long e). Estimate the frequencies of the harmonics and find the fundamental frequency.

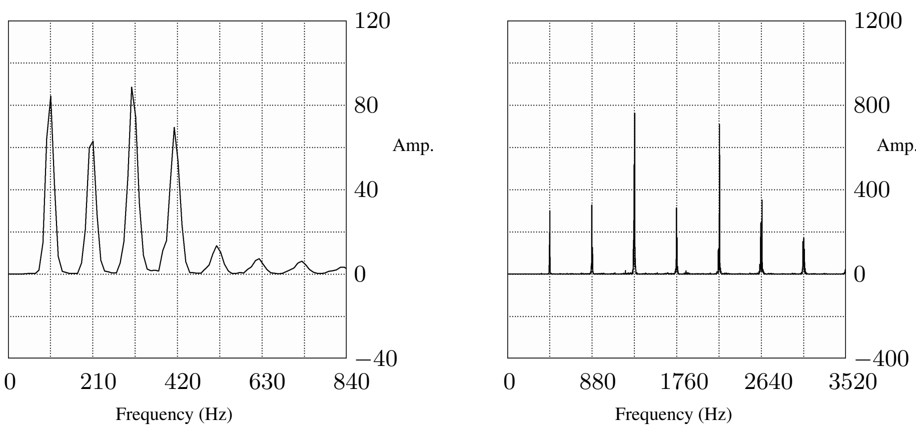

Figure 1.9. Left: Graph of amplitudes of harmonics for Exercise 1.1.8, male pronouncing vowel ē (long e). Right: Graph of amplitudes of harmonics for Exercise 1.1.9, trumpet playing one note.

1.1.9. On the right of Figure 1.9 there is a graph of the amplitudes of the harmonics from a trumpet playing one note. Estimate the frequencies of the harmonics and find the fundamental frequency. Using Table 1.13 on page 13, determine which note is being played.

1.2 Overtones, Pitch Equivalence, and Musical Scales

We have seen that the tones from musical instruments contain harmonics that are all multiples of a fundamental frequency. This fact provides a basis for an equivalence of two tones, when the frequency for one tone is twice the frequency of the other tone. For example, suppose one tone has fundamental 110 Hz, and a second tone has fundamental 220 Hz. Then the harmonics of these tones are (in Hz):

1st tone:	110	220	330	440	550	660	770	880	990	1100	1210	…
2nd tone:		220		440		660		880		1100		…

All of the harmonics for the tone with fundamental 220 Hz are also harmonics for the tone with 110 Hz. Clearly, this will happen whenever two tones have fundamentals of ν_o and $2\nu_o$. We will say that the tone with frequency $2\nu_o$ is ***an octave higher in pitch*** than the tone with frequency ν_o. The term octave comes from the use of 8 notes on the scales commonly used in Western music. On such scales, the eighth tone has double the fundamental of the first tone.

1.2.1 Pitch Equivalence

When one tone is an octave higher in pitch, then the two tones will be nearly indistinguishable ***when they are sounded together***. They are ***harmonically equivalent***. This harmonic equivalence is usually referred to in music theory as ***octave equivalence***. Harmonic equivalence (or octave equivalence) can be shown by playing two notes that are an octave apart, first separately and then together. In Figure 1.10 we show a spectrogram of the resulting sound. This spectrogram shows the harmonics from the tones traced out over time, as bright horizontal bands. One can see how the second tone, with pitch an octave higher, has all of its harmonics contained within those of the first tone. So

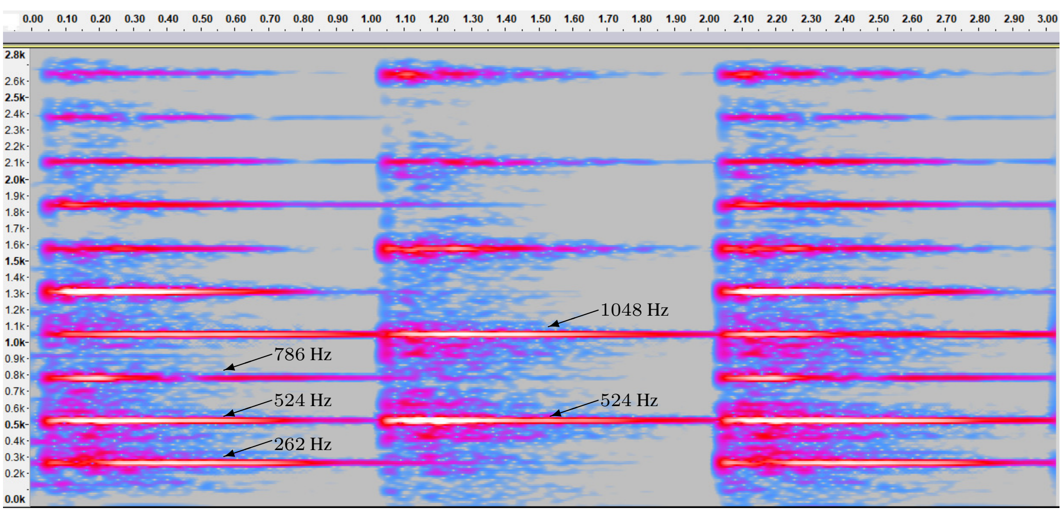

Figure 1.10. Spectrogram of piano tones, illustrating harmonic equivalence (octave equivalence). From 0.00 to 1.00 seconds: piano tone with fundamental frequency 262 Hz. From 1.00 to 2.00 seconds: piano tone with fundamental of twice that frequency. From 2.00 to 3.00 seconds: these two tones sounding together. Tracing out of harmonics from each tone is apparent as horizontal bands, with brightest white bands being the most intense (loudest). The two tones sounding together have an almost identical graph as the first tone.

when the two tones are sounded together, the horizontal bands in the spectrogram appear almost indistinguishable from those of the first tone. The sound of the two tones together sounds much like the first tone, almost as if that first tone was played by striking its key on the piano in a slightly different manner, rather than what was actually done (striking two keys together).

Since the term *octave equivalence* is standard in music theory we shall employ it from now on. It should be remembered, however, that octave equivalence refers exclusively to notes played in harmony. When notes an octave apart are played separately, they are easily distinguishable by their differences in pitch. It is only when they are played together in harmony that they become equivalent. On the other hand, even with notes played separately in melody, there is often an underlying harmonic scheme for which octave equivalence does play a role in analyzing and appreciating the music. For this reason, musicians train themselves to hear the equivalency of separate notes an octave apart in pitch. We will discuss the idea of an underlying harmonic scheme for a melody in Chapter 3.

1.2.2 Musical Scales

There are a wide variety of musical scales used throughout the world. The harmonics of instrumental tones can be used to explain some features of the most commonly employed musical scales. We shall describe two types of scales. First, a pentatonic scale, which has 5 distinct notes. Second, an octave scale, which has 8 notes. ***Our aim here is not to trace the historical development of these scales***. Instead, we aim to show how they have a mathematical structure related to the harmonics of their notes. Similar discussion could be given for many different musical scales. We chose these two because of their wide use throughout the world.

Suppose we begin with a specific note, whose tone has a fundamental frequency ν_o. We shall refer to this note as C. The frequencies of the harmonics for C are

$$\nu_o, \quad 2\nu_o, \quad 3\nu_o, \quad 4\nu_o, \quad 5\nu_o, \quad 6\nu_o, \ \ldots \tag{1.2}$$

The 2^{nd} harmonic, $2\nu_o$, is the fundamental for a tone that we have shown is octave-equivalent to the tone for C. Therefore, we shall also call this note C, and it will be the ending note of our scale. All subsequent notes on our scale will lie between these two C notes, *hence they will have fundamentals that lie between ν_o and $2\nu_o$.*

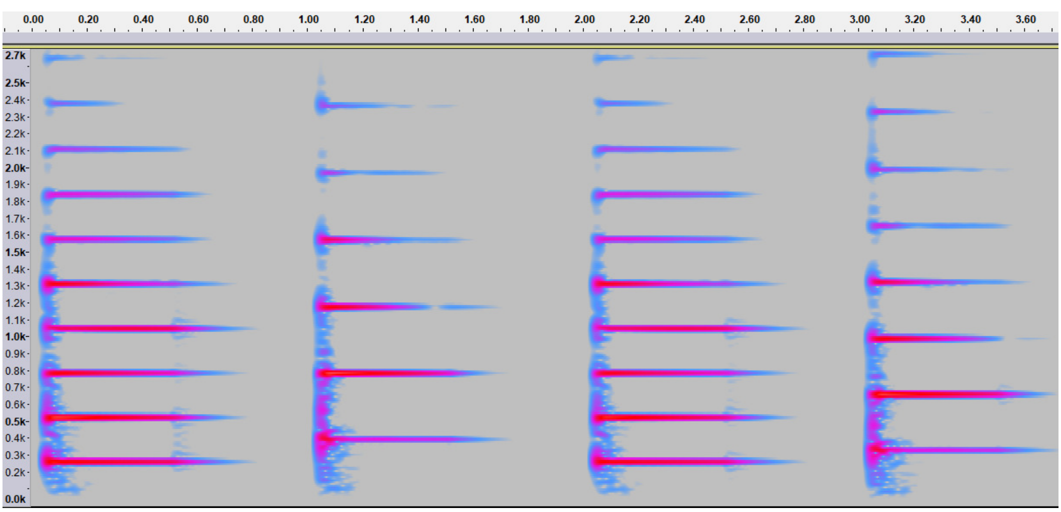

Figure 1.11. Spectrogram of four tones. First tone: Fundamental $\nu_o = 262$ Hz. Second tone: Fundamental $\frac{3}{2}\nu_o$. Third tone: Fundamental ν_o again. Fourth tone: Fundamental $\frac{5}{4}\nu_o$.

The third harmonic, $3\nu_o$, is the fundamental for a higher pitched tone than the ending tone of our scale. If we divide $3\nu_o$ by 2, we obtain an octave-equivalent pitch with fundamental $\frac{3}{2}\nu_o$. That frequency lies halfway between ν_o and $2\nu_o$. We shall use G to designate the note with fundamental frequency $\frac{3}{2}\nu_o$. In Figure 1.11, we show a spectrogram of two tones with fundamentals ν_o and $\frac{3}{2}\nu_o$. The figure shows that if these two tones are played in immediate succession (or even together), then their harmonics are either perfectly matched or well-separated from one another. This phenomenon is called ***acoustic consonance.*** When two distinct harmonics are not well-separated, then the sound waves from these harmonics will clash (interfere). This interference creates a roughness in the sound, which is called ***acoustic dissonance.*** We will explore these notions of acoustic consonance and dissonance more thoroughly in Chapter 4.

Returning to Figure 1.11, we see that the odd-numbered harmonics of the tone for G fit halfway between successive harmonics of the tone for C, and its even-numbered harmonics match with harmonics of the tone for C. In fact, if the note C is lowered in pitch by one octave, then the harmonics for G are a subset of the harmonics of this lower pitched C. To see this, we note that $\frac{1}{2}\nu_o$ is the fundamental for the octave lower C note. Therefore, the harmonics for this C and for G satisfy

$$\begin{array}{rcccccccccccc} \text{octave lower C:} & \frac{1}{2}\nu_o & \nu_o & \frac{3}{2}\nu_o & 2\nu_o & \frac{5}{2}\nu_o & 3\nu_o & \frac{7}{2}\nu_o & 4\nu_o & \frac{9}{2}\nu_o & 5\nu_o & \dots \\ \text{G:} & & & \frac{3}{2}\nu_o & & & 3\nu_o & & & \frac{9}{2}\nu_o & & \dots \end{array}$$

The octave lower C provides a perfectly acoustically consonant *harmonic foundation* for the pitch G. This use of a lower pitched, bass note as a harmonic foundation for other notes is a very common practice in music.[2]

We now have three notes on our scale:

$$\begin{array}{ccc} \text{C} & \text{G} & \text{C} \\ \nu_o & \frac{3}{2}\nu_o & 2\nu_o \end{array} \tag{1.3}$$

Throughout the world, almost all musical scales contain two notes whose fundamentals are in the ratio 2 to 1, and an additional note whose fundamental lies at the midpoint between them. Later in the book, we shall see that this additional note (with fundamental $\frac{3}{2}\nu_o$) is dividing the octave into a *perfect fifth* (from ν_o to $\frac{3}{2}\nu_o$) and a *perfect fourth* (from $\frac{3}{2}\nu_o$ to $2\nu_o$). See Exercises 3.1.3 and 3.1.16.

Musical scales, however, contain more than three notes. To get another note for our scale, we continue looking at the harmonics in (1.2). The next harmonic would be $4\nu_o$, but this corresponds to a note octave-equivalent to C. So we go to the next harmonic $5\nu_o$. The frequency $5\nu_o$ does not lie between ν_o and $2\nu_o$. To get an octave-equivalent tone that does lie between ν_o and $2\nu_o$, we divide $5\nu_o$ by 4. We shall use E to designate the note with fundamental frequency $\frac{5}{4}\nu_o$. In Figure 1.11, we show a spectrogram of two tones with fundamentals ν_o and $\frac{5}{4}\nu_o$. The figure shows that if these two notes are played in immediate succession (or even together), then they have a high degree of acoustic consonance. Every fourth harmonic for E matches a harmonic for C. This is because $4 \cdot \frac{5}{4}\nu_o = 5\nu_o$. The other harmonics for E also fit nicely in between harmonics for C and G. In fact, if the note C was lowered in pitch by two octaves, then the harmonics for E and G would all match perfectly with subsets of the harmonics of this lower pitched C. This two-octave lowered pitch C then would provide a perfectly acoustically consonant harmonic foundation for the other two notes, E and G.

We now have four notes on our scale:

$$\begin{array}{cccc} \text{C} & \text{E} & \text{G} & \text{C} \\ \nu_o & \frac{5}{4}\nu_o & \frac{3}{2}\nu_o & 2\nu_o \end{array} \tag{1.4}$$

It is interesting to note that $\frac{5}{4}\nu_o = \frac{1}{2}\left(\nu_o + \frac{3}{2}\nu_o\right)$, so the fundamental for E lies at the midpoint between the fundamentals for C and G. A large percentage of musical scales in the world have a note with a fundamental determined by this midpoint relation. Later in the book, we shall see that this note

[2] For example, see Exercises 1.4.10 and 1.4.11.

(with fundamental $\frac{5}{4}\nu_o$) is dividing the perfect fifth (from ν_o to $\frac{3}{2}\nu_o$) into a *major third* (from ν_o to $\frac{5}{4}\nu_o$) and a *minor third* (from $\frac{5}{4}\nu_o$ to $\frac{3}{2}\nu_o$). See Exercise 3.1.4.

To get additional notes on our scale we could continue with the process of looking at harmonics of C. We shall take a different route, however, which we need to do in order to obtain two of the most commonly used scales.

The note G has a fundamental that is $\frac{3}{2}$ times the fundamental for C. If we also multiply the fundamental for G by $\frac{3}{2}$, then we get $\frac{9}{4}\nu_o$. That frequency is greater than $2\nu_o$. To obtain an octave-equivalent tone with fundamental between ν_o and $2\nu_o$, we divide $\frac{9}{4}\nu_o$ by 2, obtaining $\frac{9}{8}\nu_o$. We shall use D to designate the note with fundamental $\frac{9}{8}\nu_o$.

To get the fundamental for D we multiplied the fundamental for G by $\frac{3}{2}$, and applied octave equivalency. For consistency, we would also like E to be related to a note on our scale by multiplying the fundamental of that note by $\frac{3}{2}$ (and employing octave equivalency if needed). We can find that note by dividing the fundamental for E by $\frac{3}{2}$, which produces the frequency $\frac{5}{4}\nu_o \,/\, \frac{3}{2} = \frac{5}{6}\nu_o$. The frequency $\frac{5}{6}\nu_o$ does not lie between ν_o and $2\nu_o$. To get an octave-equivalent tone that does lie between ν_o and $2\nu_o$, we multiply $\frac{5}{6}\nu_o$ by 2, obtaining the frequency $\frac{5}{3}\nu_o$. We shall use A to designate the tone with fundamental frequency $\frac{5}{3}\,\nu_o$. We now have the following scale:

$$\begin{array}{cccccc} \text{C} & \text{D} & \text{E} & \text{G} & \text{A} & \text{C} \\ \nu_o & \frac{9}{8}\nu_o & \frac{5}{4}\nu_o & \frac{3}{2}\nu_o & \frac{5}{3}\nu_o & 2\nu_o \end{array} \tag{1.5}$$

This is a *pentatonic major scale.* Many cultures throughout the world use a pentatonic major scale. Chinese folk music is a prime example. Most Western music, however, is based on an octave scale.

Octave scale

We will now derive one type of octave scale by adding more notes to the pentatonic major scale. To obtain the fundamental for G we multiplied ν_o by $\frac{3}{2}$. If we look for a note whose fundamental, when multiplied by $\frac{3}{2}$, is equal to ν_o (the fundamental for C), we get a fundamental of $\frac{2}{3}\nu_o$. The frequency $\frac{2}{3}\nu_o$ does not lie between ν_o and $2\nu_o$. To get an octave-equivalent tone, we multiply by 2, obtaining frequency $\frac{4}{3}\nu_o$. We shall use F to designate the tone with fundamental frequency $\frac{4}{3}\nu_o$.

To complete our octave scale, we multiply the fundamental $\frac{5}{4}\nu_o$ for note E by $\frac{3}{2}$, obtaining the frequency $\frac{15}{8}\nu_o$. We use B to designate the tone with fundamental frequency $\frac{15}{8}\nu_o$.

Our octave scale is now complete. Here are its notes and frequencies:

$$\begin{array}{cccccccc} \text{C} & \text{D} & \text{E} & \text{F} & \text{G} & \text{A} & \text{B} & \text{C} \\ \nu_o & \frac{9}{8}\nu_o & \frac{5}{4}\nu_o & \frac{4}{3}\nu_o & \frac{3}{2}\nu_o & \frac{5}{3}\nu_o & \frac{15}{8}\nu_o & 2\nu_o \end{array} \tag{1.6}$$

This scale is called a ***just major scale***. In Table 1.12 we show a standard tuning system for this type of scale. It is also called an ***eight-tone just scale***. The term "just" refers to the fact that the harmonics for various combinations of notes—such as C, E, and G—have perfect acoustic consonance. Several other combinations of notes also have perfect acoustic consonance; we describe some of them in the exercises.

The frequencies shown in (1.6) have an interesting pattern of ratios. Here we show these ratios, written as multiplying factors of the fundamentals for successive notes:

$$\begin{array}{c} \nu_o \xrightarrow{\cdot\frac{9}{8}} \frac{9}{8}\nu_o \xrightarrow{\cdot\frac{10}{9}} \frac{5}{4}\nu_o \xrightarrow{\cdot\frac{16}{15}} \frac{4}{3}\nu_o \xrightarrow{\cdot\frac{9}{8}} \frac{3}{2}\nu_o \xrightarrow{\cdot\frac{10}{9}} \frac{5}{3}\nu_o \xrightarrow{\cdot\frac{9}{8}} \frac{15}{8}\nu_o \xrightarrow{\cdot\frac{16}{15}} 2\nu_o. \\ \text{C} \quad \text{D} \quad \text{E} \quad \text{F} \quad \text{G} \quad \text{A} \quad \text{B} \quad \text{C} \end{array} \tag{1.7}$$

Our brains interpret equal frequency ratios as equivalent changes in pitch. For example, going from C to D is heard as an equivalent change in pitch as going from F to G. In both cases, the frequency ratios of the pitch changes are $\frac{9}{8}$. Similarly, going from E to F is heard as an equivalent change in pitch as going from B to C. In both cases, the frequency ratios of the pitch changes are $\frac{16}{15}$. We will explore this point further later in the chapter, as it is a crucial fact about pitch change.

The ratios in (1.7) are not all the same, however. The fractions $\frac{9}{8}$ and $\frac{10}{9}$ differ by about 1% in magnitude. Consequently there is a slight variation in the pitch change in going from C to D, compared to going from D to E. Likewise there is a slight variation in pitch change in going from F to G, versus going from G to A. These variations in pitch change are the cause of different major scales, using our just tuning, being inconsistent with each other. For example, if one builds a major scale starting with the note G having frequency $\frac{3}{2}\nu_o$, then the second note in that scale is $\frac{9}{8} \cdot \frac{3}{2}\nu_o = \frac{27}{16}\nu_o$. This note is very close in pitch to A in the scale shown in (1.6), but not exactly the same. For instance, if $\nu_o = 256$ Hz, the fundamental for C in the fourth octave in Table 1.12, then $\frac{27}{16}\nu_o = 432$ Hz. The frequency for A in the fourth octave in Table 1.12 is 427 Hz. So the two notes will have a noticeable difference in pitch, the tone for A at 427 Hz will sound slightly lower in pitch (slightly flatter) than the tone for A at 432 Hz. These two major scales will sound out of tune if played together. Different musical instruments are often tuned to major scales with different starting notes, so this inconsistency is a real problem with just tuning.

There is one further problem with the 8-tone just scale that we described above: It is incomplete. For example, the note D has fundamental $\frac{9}{8}$ times the fundamental ν_o for the starting note C. For reasons of symmetry, or completeness, we look for a note whose fundamental times $\frac{9}{8}$ will equal a fundamental for C. In order for this fundamental to lie between ν_o and $2\nu_o$, we divide $\frac{9}{8}$ into $2\nu_o$ to get $\frac{16}{9}\nu_o$. The note with fundamental $\frac{16}{9}\nu_o$ lies between A and B. Depending on the musical context, it is either called $A^\sharp$ (A-sharp) or $B^\flat$ (B-flat). We will not pursue this question further, since introducing new notes into the just scale only increases the inconsistencies in frequency ratios that we have already observed.

By using a different scale, an equal-tempered scale, these inconsistencies in frequency ratios simply disappear. We discuss the 12-tone equal-tempered scale in the next section.

Exercises

1.2.1. Show that if the note G (with fundamental $\frac{3}{2}\nu_o$) is lowered in pitch by two octaves, then the harmonics for D (with fundamental $\frac{9}{8}\nu_o$) all match perfectly with a subset of the harmonics for this lower pitched G.

1.2.2. In the text it was stated that *if the note* C *(with fundamental* ν_o*) is lowered in pitch by two octaves, then the harmonics for* G *(with fundamental* $\frac{3}{2}\nu_o$*) and the harmonics for* E *(with fundamental* $\frac{5}{4}\nu_o$*) all match perfectly with a subset of the harmonics for this lower pitched* C. Verify that this statement is correct.

1.2.3. Show that if the note F (with fundamental $\frac{4}{3}\nu_o$) is lowered in pitch by two octaves, then the harmonics for A (with fundamental $\frac{5}{3}\nu_o$) and the harmonics for C (with fundamental $2\nu_o$) all match perfectly with subsets of the harmonics for this lower pitched F.

Table 1.12 PITCH AND FREQUENCY FOR EIGHT-TONE JUST SCALE*

Frequency Ratio	Octave 1	Octave 2	Octave 3	Octave 4	Octave 5	Octave 6	Octave 7
1	C 32	C 64	C 128	C 256	C 512	C 1024	C 2048
9/8	D 36	D 72	D 144	D 288	D 576	D 1152	D 2304
5/4	E 40	E 80	E 160	E 320	E 640	E 1280	E 2560
4/3	F 43	F 85	F 171	F 341	F 683	F 1365	F 2731
3/2	G 48	G 96	G 192	G 384	G 768	G 1536	G 3072
5/3	A 53	A 107	A 213	A 427	A 853	A 1707	A 3413
15/8	B 60	B 120	B 240	B 480	B 960	B 1920	B 3840
2	C 64	C 128	C 256	C 512	C 1024	C 2048	C 4096

*Fundamentals for C notes are exact. If needed, other fundamentals are rounded to nearest Hz.

1.2.4. Show that if the note G (with fundamental $\frac{3}{2}\nu_o$) was lowered in pitch by two octaves, then the harmonics for B (with fundamental $\frac{15}{8}\nu_o$) and the harmonics for D (with fundamental $\frac{9}{8}\nu_o$) would all match perfectly with subsets of the harmonics for this lower pitched G.

1.2.5. Explain why the sixth harmonic $6\nu_o$ for the note C does not introduce any new note into the just scale we have described.

1.2.6. Suppose the seventh harmonic $7\nu_o$ for the note C is used to introduce a new note into the just scale. Where would this note lie in relation to the scale in (1.6)? What if the eleventh harmonic is used to introduce a new note; where would this note lie on the scale in (1.6)? Why did we skip over the ninth and tenth harmonics?

1.3 The 12-Tone Equal-Tempered Scale

In this section we describe a different musical scale that eliminates the inconsistencies in frequency ratios in the just scale, and has additional notes that make it a relatively complete scale. This musical scale is called the 12-tone equal-tempered scale.

To explain how the 12-tone equal-tempered scale is defined, we need to closely examine the inconsistencies in frequency ratios in the eight-tone just scale. These frequency ratios are shown in (1.7). We notice that these frequency ratios split into two categories, large and small (major and minor):

$$\text{Major:}\quad \tfrac{9}{8} = 1.125, \quad \tfrac{10}{9} = 1.111. \qquad \text{Minor:}\quad \tfrac{16}{15} = 1.0667.$$

The major frequency ratios are used five times in (1.7), while the minor frequency ratio is used twice.

We also observe that the square of the minor frequency ratio $\frac{16}{15}$ is nearly equal to both of the major frequency ratios:

$$\begin{aligned}\left(\frac{16}{15}\right)^2 &= 1.13778\\ &\approx 1.125 \;= \frac{9}{8}\\ &\approx 1.11111 \;= \frac{10}{9}\,.\end{aligned}$$

Suppose for a moment that these approximations are close enough to be regarded as equalities. Then, denoting the minor frequency ratio $\frac{16}{15}$ by $\mathbf{r}$, we will suppose that $\frac{9}{8} = \mathbf{r}^2$ and $\frac{10}{9} = \mathbf{r}^2$. This fiction allows us to rewrite (1.7) as follows:

$$\begin{array}{cccccccc}\nu_o \xrightarrow{\cdot\mathbf{r}^2} & \mathbf{r}^2\,\nu_o \xrightarrow{\cdot\mathbf{r}^2} & \mathbf{r}^4\,\nu_o \xrightarrow{\cdot\mathbf{r}} & \mathbf{r}^5\,\nu_o \xrightarrow{\cdot\mathbf{r}^2} & \mathbf{r}^7\,\nu_o \xrightarrow{\cdot\mathbf{r}^2} & \mathbf{r}^9\,\nu_o \xrightarrow{\cdot\mathbf{r}^2} & \mathbf{r}^{11}\,\nu_o \xrightarrow{\cdot\mathbf{r}} & \mathbf{r}^{12}\nu_o = 2\nu_o.\\ \text{C} & \text{D} & \text{E} & \text{F} & \text{G} & \text{A} & \text{B} & \text{C}\end{array} \tag{1.8}$$

From this calculation we have derived the equation $\mathbf{r}^{12}\nu_o = 2\nu_o$. Dividing out ν_o, we obtain $\mathbf{r}^{12} = 2$. Therefore, $\mathbf{r}$ is the twelfth root of 2:

$$\mathbf{r} = 2^{1/12}. \tag{1.9}$$

Using $\mathbf{r} = 2^{1/12}$ for each minor frequency ratio, and $\mathbf{r}^2 = 2^{2/12}$ for each major frequency ratio, completely removes the inconsistency that we described above for the eight-tone just scale.

Using this frequency ratio $\mathbf{r}$, we can also add more notes to our major scale in (1.8). For example, if we divide the highest frequency $2\nu_o = \mathbf{r}^{12}\nu_o$ by $\mathbf{r}^2$, then we get $\mathbf{r}^{10}\nu_o$. The note with fundamental $\mathbf{r}^{10}\nu_o$ lies equally between—in terms of frequency ratios—the fundamentals $\mathbf{r}^9\,\nu_o$ for note A and $\mathbf{r}^{11}\,\nu_o$ for note B. This new note can be written as either A$^\sharp$ or B$^\flat$. The symbol $\sharp$ (a sharp) indicates that the note A has been raised in pitch by the frequency ratio $\mathbf{r}$, while the symbol $\flat$ (a flat) indicates that the note B has been lowered in pitch by the frequency ratio $\mathbf{r}$. What this leads to is that multiplying ν_o successively by $\mathbf{r}$ produces a new scale, the ***12-tone equal-tempered scale***. To save verbiage, we shall also refer to it as the ***chromatic scale***.[3] In Table 1.13 we show a standard tuning system for this

[3] There are other chromatic scales besides the 12-tone equal-tempered scale, but we will not discuss them in this book.

type of scale. For convenience, we have used only sharps in this table. In each octave, the frequencies are the same for the five pairs: C♯, D♭, and D♯, E♭, and F♯, G♭, and G♯, A♭, and A♯, B♭.

Table 1.13 PITCH AND FREQUENCY FOR 12-TONE EQUAL-TEMPERED SCALE*

Frequency Ratio	Octave 1	Octave 2	Octave 3	Octave 4	Octave 5	Octave 6	Octave 7
1	C 33	C 65	C 131	C 262	C 523	C 1047	C 2093
$2^{1/12}$	C♯ 35	C♯ 69	C♯ 139	C♯ 277	C♯ 554	C♯ 1109	C♯ 2218
$2^{2/12}$	D 37	D 73	D 147	D 294	D 587	D 1175	D 2349
$2^{3/12}$	D♯ 39	D♯ 78	D♯ 156	D♯ 311	D♯ 622	D♯ 1245	D♯ 2489
$2^{4/12}$	E 41	E 82	E 165	E 330	E 659	E 1319	E 2637
$2^{5/12}$	F 44	F 87	F 175	F 349	F 699	F 1397	F 2794
$2^{6/12}$	F♯ 46	F♯ 93	F♯ 185	F♯ 370	F♯ 740	F♯ 1475	F♯ 2960
$2^{7/12}$	G 49	G 98	G 196	G 392	G 784	G 1568	G 3136
$2^{8/12}$	G♯ 52	G♯ 104	G♯ 208	G♯ 415	G♯ 831	G♯ 1661	G♯ 3322
$2^{9/12}$	A 55	A 110	A 220	A 440	A 880	A 1760	A 3520
$2^{10/12}$	A♯ 58	A♯ 117	A♯ 233	A♯ 466	A♯ 932	A♯ 1865	A♯ 3729
$2^{11/12}$	B 62	B 124	B 247	B 494	B 988	B 1976	B 3951
2	C_2 65	C_3 131	C_4 262	C_5 523	C_6 1047	C_7 2093	C_8 4186

*Frequencies for fundamentals are rounded to nearest Hz, except for the A notes which are exact.

The tuning system for the chromatic scale shown in Table 1.13 is now established as an International Standard. It is used, for example, in tuning many orchestral instruments, including pianos and harps, and also guitars.[4] In the next section, we will describe how the chromatic scale provides the basis for the most commonly used major scales in Western music. In contrast to the just scale, major scales beginning with different notes are perfectly consistent when the 12-tone equal-tempered system is used.

The first column in Table 1.13 is included to emphasize the fact that the notes in each octave are all generated by the same frequency ratios, relative to the initial note C. Using $\mathbf{r} = 2^{1/12}$, and ν_o for the frequency of the C note that begins an octave, the frequencies of each note in that octave have this pattern:

$$\begin{array}{ccccccc} \nu_o \cdot \mathbf{r}^0 & \nu_o \cdot \mathbf{r}^1 & \nu_o \cdot \mathbf{r}^2 & \nu_o \cdot \mathbf{r}^3 & \nu_o \cdot \mathbf{r}^4 & \nu_o \cdot \mathbf{r}^5 & \\ \mathrm{C} & \mathrm{C}^\sharp & \mathrm{D} & \mathrm{D}^\sharp & \mathrm{E} & \mathrm{F} & \\ \nu_o \cdot \mathbf{r}^6 & \nu_o \cdot \mathbf{r}^7 & \nu_o \cdot \mathbf{r}^8 & \nu_o \cdot \mathbf{r}^9 & \nu_o \cdot \mathbf{r}^{10} & \nu_o \cdot \mathbf{r}^{11} & \nu_o \cdot \mathbf{r}^{12} \\ \mathrm{F}^\sharp & \mathrm{G} & \mathrm{G}^\sharp & \mathrm{A} & \mathrm{A}^\sharp & \mathrm{B} & \mathrm{C} \end{array} \tag{1.10}$$

The final note C, with fundamental $\nu_o \cdot \mathbf{r}^{12} = \nu_o \cdot 2$, is an octave above the initial note C.

Example 1.3.1. Tuning note for the chromatic scale. The fundamentals shown for the notes in Table 1.13 are based on A in the 4^{th} octave as the reference note. We will denote this note by A_4. The fundamental of 440 Hz for A_4 is exact, as are the fundamentals for all of the other A notes. The frequencies in the table for the rest of the notes, however, are only approximations of their exact values. For instance, the exact value for the fundamental of C_4, C in the 4^{th} octave, is not 262 Hz (as shown in Table 1.13). Its exact value can be found from the fact that the fundamental for A_4 is $\nu_o \cdot \mathbf{r}^9$,

[4] An interesting exception is that stringed instruments in an orchestra are typically tuned with their fundamentals in ratios of 3 : 2. See Section 1.4.4 on page 20 for further discussion.

where ν_o is the fundamental for C_4. We calculate as follows:

$$\begin{aligned}\nu_o &= (\nu_o \cdot \mathbf{r}^9) \cdot \mathbf{r}^{-9} \\ &= (440) \cdot \frac{1}{\mathbf{r}^9} \\ &= \frac{440}{2^{9/12}} \\ &= 261.625565300599.\end{aligned}$$

Therefore, the fundamental for C_4 is 261.625565300599 Hz. The value of 262 Hz shown in Table 1.13 is an approximation of this more exact value.

Example 1.3.2. Finding fundamentals of notes using C as reference. As shown in the previous example, the note C_4 has fundamental

$$261.625565300599 \text{ Hz} \approx 262 \text{ Hz}.$$

Therefore $\mathrm{C}_4^\sharp$ has fundamental

$$\begin{aligned}261.625565300599 \cdot \mathbf{r}^1 &= 261.625565300599 \cdot 2^{1/12} \\ &= 277.182630976873 \\ &\approx 277 \text{ Hz}\end{aligned}$$

as shown in the table. Moreover, G_4 has fundamental

$$\begin{aligned}261.625565300599 \cdot \mathbf{r}^7 &= 261.625565300599 \cdot 2^{7/12} \\ &= 391.99543598175 \\ &\approx 392 \text{ Hz}\end{aligned}$$

as shown in the table.

The chromatic scale provides the basis for most of the scales used throughout the Western world. We look at these scales in the next section.

Exercises

1.3.1. On the right of Figure 1.4, page 3, there is a graph of the amplitudes for harmonics of a flute tone. Using Table 1.13, estimate what note is being played.

1.3.2. On the right of Figure 1.5, page 4, there is a graph of the amplitudes for harmonics of a piano tone. Using Table 1.13, estimate what note is being played. Why is this the same note as for the flute in the previous exercise?

1.3.3. On the right of Figure 1.9, page 6, there is a graph of the amplitudes for harmonics of a trumpet tone. Using Table 1.13, estimate what note is being played.

1.3.4. On the left of Figure 1.9, page 6, there is a graph of the amplitudes for harmonics for a male speaking the vowel ē (long e). Setting aside the fact that this male is not singing, find the notes in Table 1.12 and in Table 1.13 that are the best estimates of the note for the tone of this person's vowel.

1.3.5. On the left of Figure 1.8, page 6, there is a graph of the amplitudes for harmonics for a female speaking the vowel ā (long a). Setting aside the fact that this female is not singing, find the notes in Table 1.12 and in Table 1.13 that are the best estimates of the note for the tone of this person's vowel.

1.3.6. On the right of Figure 1.8, page 6, there is a graph of the amplitudes for harmonics for a male speaking the vowel ā (long a). Setting aside the fact that this male is not singing, find the notes in Table 1.12 and in Table 1.13 that are the best estimates of the tone of this person's vowel.

1.4 Musical Scales within the Chromatic Scale

In this section we will describe the standard Western musical scales. These are most easily explained using a piano keyboard. Almost all pianos are tuned to play the pitches described in Table 1.13. The layout for a piano keyboard is shown in Figure 1.14. The notes on this piano keyboard span a pitch range of 7 octaves,[5] a huge pitch range which contains the pitch ranges of just about every instrument used in music.

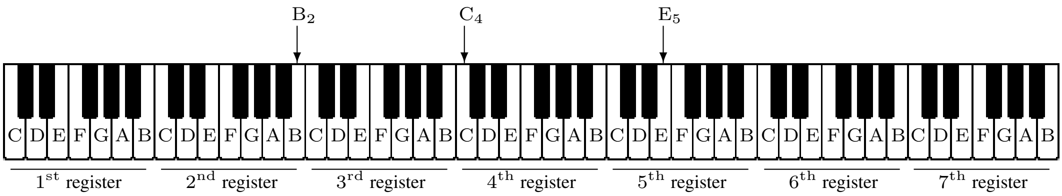

Figure 1.14. Piano keyboard spanning 7 octave registers. The 1st octave register contains notes with lowest pitch. Each successive register contains notes with higher pitch than the preceding one, until notes with the highest pitch in the 7th register. In each register, white keys are labeled by notes from C-major scale. If it is important to state what register a specific note belongs to, then a subscript is used. For example, the note C in the 4th register is C_4, also known as middle C. Note B in the second register is B_2, and so forth.

1.4.1 The C-Major Scale

The most straightforward musical scale is the C-major scale, since it consists entirely of white notes on the keyboard. It corresponds to 8 adjacent white keys, an *octave,* on the keyboard shown in Figure 1.14. In Figure 1.15 we show one section of the keyboard with these 8 adjacent white keys. These white keys are labeled C, D, E, F, G, A, B, C. The last C is octave-equivalent to the first C, just an octave higher in pitch.

These 8 white keys on the piano will play a C-major scale:

$$\mathrm{C} \quad \mathrm{D} \quad \mathrm{E} \quad \mathrm{F} \quad \mathrm{G} \quad \mathrm{A} \quad \mathrm{B} \quad \mathrm{C}. \tag{1.11}$$

If you play this scale, you will hear the familiar sound of "Do-Re-Mi-Fa-So-La-Ti-Do" that school children learn to sing in music classes.

The black keys are for notes whose pitches lie above or below the pitches of the notes on adjacent white keys. For example, $C^\sharp$ has a pitch above C but below D. On the other hand, $D^\flat$ has a pitch below D but above C. On the piano, which is tuned to the chromatic scale, the two notes $C^\sharp$ and $D^\flat$ *have the same pitch.* They are called ***enharmonic***. We have indicated this by writing both $C^\sharp$ and $D^\flat$ on the first black key on the left side of Figure 1.15. Similar remarks apply to the rest of the black keys. If we were to strike both white and black keys in order from left to right, beginning with the note C on the left, then we would play this scale:

$$\mathrm{C} \quad \mathrm{C}^\sharp \quad \mathrm{D} \quad \mathrm{D}^\sharp \quad \mathrm{E} \quad \mathrm{F} \quad \mathrm{F}^\sharp \quad \mathrm{G} \quad \mathrm{G}^\sharp \quad \mathrm{A} \quad \mathrm{A}^\sharp \quad \mathrm{B} \quad \mathrm{C} \tag{1.12}$$

which is the chromatic scale discussed in the previous section. A C-major scale is called a *diatonic scale*. A chromatic scale and a C-major scale are shown in Figure 1.17, along with two other scales that we shall discuss soon.

In Figure 1.15 there are black keys in between some white keys but not others. For example, the key for D is 2 keys over to the right from the key for C. Likewise, the key for E is 2 keys over from

[5] A typical piano keyboard has four additional keys, three to the left and one to the right of the keys shown in Figure 1.14. To explain the basic principles more clearly, we have omitted these keys, which are rarely played.

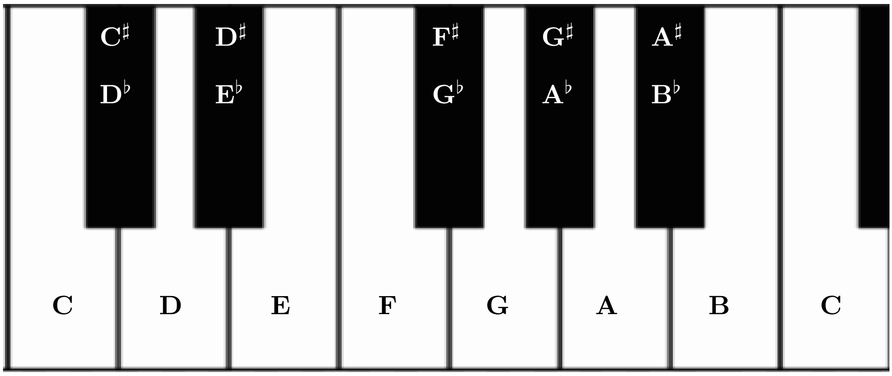

Figure 1.15. Keys on a piano keyboard spanning one octave, starting at C. White keys are C-major scale. All black and white keys are chromatic scale.

the key for D. However, the key for F is just 1 key over from the key for E. We have this pattern of how many keys over to the right each white key is to the next on the piano:

$$\mathrm{C} \xrightarrow{+2\text{ keys}} \mathrm{D} \xrightarrow{+2\text{ keys}} \mathrm{E} \xrightarrow{+1\text{ key}} \mathrm{F} \xrightarrow{+2\text{ keys}} \mathrm{G} \xrightarrow{+2\text{ keys}} \mathrm{A} \xrightarrow{+2\text{ keys}} \mathrm{B} \xrightarrow{+1\text{ key}} \mathrm{C}.$$

Because a move of $+2$ keys occurs much more often than a move of $+1$ key, musical terminology refers to a move of $+2$ keys as a ***whole step*** and a move of $+1$ key as a ***half step.*** A whole step equals two half steps.[6] Using this terminology we can write instead

$$\mathrm{C} \xrightarrow{2\text{ half steps}} \mathrm{D} \xrightarrow{2\text{ half steps}} \mathrm{E} \xrightarrow{1\text{ half step}} \mathrm{F} \xrightarrow{2\text{ half steps}} \mathrm{G} \xrightarrow{2\text{ half steps}} \mathrm{A} \xrightarrow{2\text{ half steps}} \mathrm{B} \xrightarrow{1\text{ half step}} \mathrm{C}.$$

This pattern of half steps exactly matches with the powers of the frequency factor $\mathbf{r}$ used in (1.8) on page 12:

$$\begin{array}{cccccccc} \nu_o \xrightarrow{\cdot\mathbf{r}^2} & \mathbf{r}^2\,\nu_o \xrightarrow{\cdot\mathbf{r}^2} & \mathbf{r}^4\,\nu_o \xrightarrow{\cdot\mathbf{r}} & \mathbf{r}^5\,\nu_o \xrightarrow{\cdot\mathbf{r}^2} & \mathbf{r}^7\,\nu_o \xrightarrow{\cdot\mathbf{r}^2} & \mathbf{r}^9\,\nu_o \xrightarrow{\cdot\mathbf{r}^2} & \mathbf{r}^{11}\,\nu_o \xrightarrow{\cdot\mathbf{r}} & \mathbf{r}^{12}\nu_o = 2\nu_o. \\ \mathrm{C} & \mathrm{D} & \mathrm{E} & \mathrm{F} & \mathrm{G} & \mathrm{A} & \mathrm{B} & \mathrm{C} \end{array}$$

Consequently, we can use this pattern of key distances or half steps:

$$2 \quad 2 \quad 1 \quad 2 \quad 2 \quad 2 \quad 1 \tag{1.13}$$

to create major scales with any starting note.

1.4.2 Other Major Scales

To create another major scale, we use the pattern of half steps in (1.13), starting with a different note than C. Based on our previous discussion of harmony, we will choose the fifth note in a given scale to begin a new scale. For the C-major scale, the fifth note is G. So we shall start with the note G and use the half step numbers in (1.13) as key distances on the piano keyboard. This will produce the G-major scale. To perform our work, we will start with the first key marked G on the left side of the piano keyboard section shown in Figure 1.16. Referring to this figure, and using our pattern of half steps, we get the following results:

$$\mathrm{G} \xrightarrow{2\text{ half steps}} \mathrm{A} \xrightarrow{2\text{ half steps}} \mathrm{B} \xrightarrow{1\text{ half step}} \mathrm{C} \xrightarrow{2\text{ half steps}} \mathrm{D} \xrightarrow{2\text{ half steps}} \mathrm{E} \xrightarrow{2\text{ half steps}} \mathrm{F}^\sharp \xrightarrow{1\text{ half step}} \mathrm{G}.$$

[6] Another terminology is *semitone* for half step, and *whole tone* for whole step. We shall use the terms half step and whole step, since they might be familiar already to many readers.

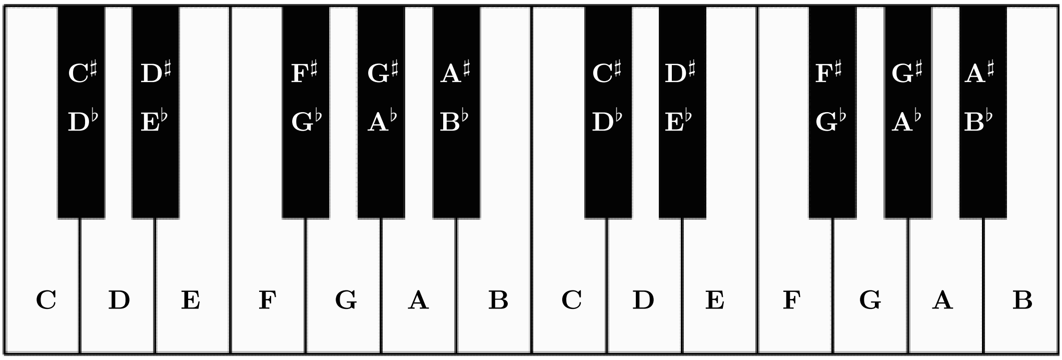

Figure 1.16. Keys on a piano keyboard spanning two registers.

Notice that going two half steps to the right from the key for E gets us to the key marked either $F^\sharp$ or $G^\flat$. We chose $F^\sharp$ as the note for the black key, rather than $G^\flat$. This was done to avoid repeating a note letter. If we had chosen $G^\flat$, then we would have had $G^\flat$ and G as the last two notes for our scale. Clearly, that would lead to some confusion. We shall always follow the standard musical convention of avoiding using the same note letter twice in our scales.

To summarize, we have found that the G-major scale is

$$\text{G} \quad \text{A} \quad \text{B} \quad \text{C} \quad \text{D} \quad \text{E} \quad \text{F}^\sharp \quad \text{G}. \tag{1.14}$$

See Figure 1.17.

Remark 1.4.1. Before we do another example, we should point out the meaning of sharps and flats in terms of half steps. If a note has a sharp on it, then that means the note is raised in pitch by one half step. For example, $C^\sharp$ is one half step above C. Or, $E^\sharp$ is one half step above E. This last example is interesting because the note F is also one half step above E. The notes $E^\sharp$ and F are enharmonic (have the same pitch). If a flat is applied to a note, then the note is lowered one half step. So $D^\flat$ is one half step below D. There are enharmonic cases here as well. For instance, the notes $C^\flat$ and B are enharmonic.

Returning to our discussion of major scales, suppose we start on the fifth note D of the G-major scale. We will produce the scale known as D-major. Using our pattern of half steps, starting with the leftmost key for D in Figure 1.16, we get the following results:

$$\text{D} \xrightarrow{\text{2 half steps}} \text{E} \xrightarrow{\text{2 half steps}} \text{F}^\sharp \xrightarrow{\text{1 half step}} \text{G} \xrightarrow{\text{2 half steps}} \text{A} \xrightarrow{\text{2 half steps}} \text{B} \xrightarrow{\text{2 half steps}} \text{C}^\sharp \xrightarrow{\text{1 half step}} \text{D}.$$

So the D-major scale is

$$\text{D} \quad \text{E} \quad \text{F}^\sharp \quad \text{G} \quad \text{A} \quad \text{B} \quad \text{C}^\sharp \quad \text{D}. \tag{1.15}$$

See Figure 1.17.

Try playing these three scales, C-major, G-major, and D-major, on a real or virtual instrument to hear the similarity in the pitch changes. In each case, you should hear the familiar scale of "Do-Re-Mi-Fa-So-La-Ti-Do," just over different pitch ranges. The frequency ratios for successive pitches are the same in each scale, and our brains detect this equivalence.

1.4.3 Scales and Clock Arithmetic

The method of using the piano keyboard for describing scales is workable, but a bit cumbersome. There is an elegant mathematical way of creating scales using the notion of *clock arithmetic*. We will introduce this method in an informal way in this section.

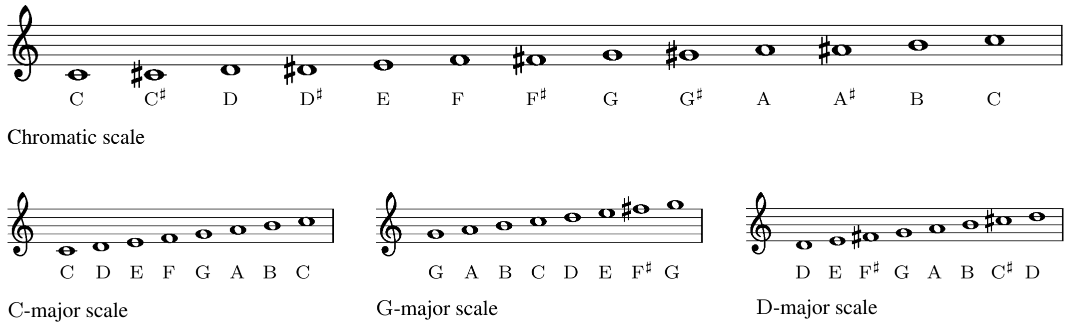

Figure 1.17. Some musical scales. Try playing these scales with a real instrument, or a virtual one (using software such as MUSESCORE). For diatonic scales, shown below the chromatic scale, you will hear essentially the same scale but with different pitch ranges.

Notice that in the piano keyboard shown in Figure 1.14 the sequence of keys in each register just repeats again and again (it is said to be a *periodic* sequence). We can also see that there is a single half step used to go from one key to the next on the chromatic scale formed by the 12 keys in each register. The easiest way to express this periodic repetition is to pair these notes with the 12 hours on a clock face as shown in Figure 1.18. We could imagine physically wrapping the piano keyboard around the clock: Starting with the leftmost key for the note C_1 at hour 0, and pairing each successive key with each successive hour. Each C-key in each register will then pair up with hour 0, and the keyboard will wrap around the clock 7 times. We shall refer to this clock as the ***chromatic clock.***

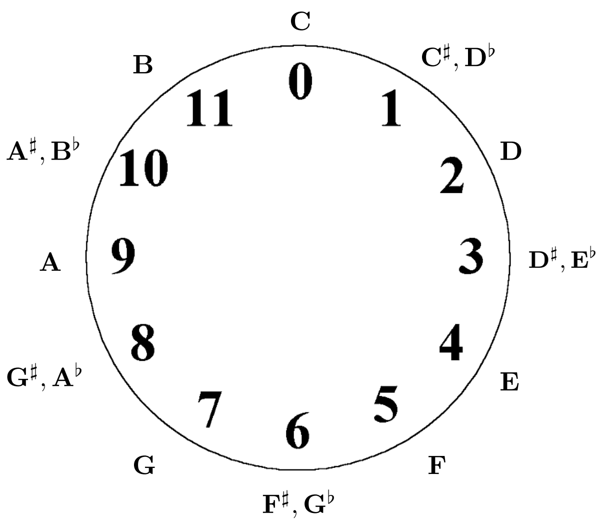

Figure 1.18. Chromatic clock. The chromatic scale arranged on a clock face (starting at hour **0**).

The hour values on the chromatic clock are the powers of the frequency factor **r** used for creating the chromatic scale. The chromatic clock has the top hour start at 0, instead of 12, which facilitates the arithmetic we shall be using. With the chromatic clock, rather than counting half steps using a piano keyboard, we instead count hour distances. So, for instance, going from C to D is a 2-hour change, which corresponds precisely to the 2 half steps we used previously. Or, going from E to F is a 1-hour change, corresponding precisely to the 1 half step we used before. When hour 12 is reached, then the equality $\mathbf{r}^{12} = 2$ corresponds to the octave-equivalence of two C-notes differing by one octave.

Remark 1.4.2. On the chromatic clock we have written only the letter names for notes, disregarding their registers. Since we identified the chromatic scale in each register with the hours on the clock, we can regard each note on the clock as specifying a set of pitches from all of the registers. For example, the note **C** on the clock denotes the set of pitches $\{\mathrm{C}_1, \mathrm{C}_2, \dots, \mathrm{C}_7\}$. So we can write

$$\mathbf{C} = \{\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4, \mathrm{C}_5, \mathrm{C}_6, \mathrm{C}_7\}.$$

Likewise, we have

$$\mathbf{C}^\sharp = \{\mathrm{C}^\sharp_1, \mathrm{C}^\sharp_2, \mathrm{C}^\sharp_3, \mathrm{C}^\sharp_4, \mathrm{C}^\sharp_5, \mathrm{C}^\sharp_6, \mathrm{C}^\sharp_7\}$$

and so on. In the examples that follow we will keep track of hour changes around the clock, and whenever we pass the top hour 0 going clockwise it will mean we have passed into a higher register. We will not usually write register subscripts, however, as they can almost always be inferred from context.

Using the chromatic clock, we can create major scales using the sequence

$$2 \quad 2 \quad 1 \quad 2 \quad 2 \quad 2 \quad 1 \tag{1.16}$$

as hours to add in moving around the hour positions on the clock. We agree, however, that when we circle around the top of the clock (the number 0) we will always reset to 0 by subtracting 12 (just as in telling time with this type of clock). This convention will allow us to correctly identify the note names, which is all we need for this application of chromatic clock arithmetic.

Example 1.4.3. C-major scale. To make the C-major scale, we start with the note C at hour 0, and then apply the sequence in (1.16) as hour moves around the chromatic clock. We perform these calculations:

$$\begin{array}{ccccccccccccccc} 0 & \xrightarrow{+2} & 2 & \xrightarrow{+2} & 4 & \xrightarrow{+1} & 5 & \xrightarrow{+2} & 7 & \xrightarrow{+2} & 9 & \xrightarrow{+2} & 11 & \xrightarrow[-12 \text{ at top}]{+1} & 0 \\ \mathrm{C} & & \mathrm{D} & & \mathrm{E} & & \mathrm{F} & & \mathrm{G} & & \mathrm{A} & & \mathrm{B} & & \mathrm{C}. \end{array} \tag{1.17}$$

Notice that we subtracted 12 when reaching the top of the clock (so hour 0, not 12). Thus we have obtained the C-major scale:

$$\mathrm{C} \quad \mathrm{D} \quad \mathrm{E} \quad \mathrm{F} \quad \mathrm{G} \quad \mathrm{A} \quad \mathrm{B} \quad \mathrm{C}. \tag{1.18}$$

Example 1.4.4. G-major scale. To make the G-major scale, we start at G which has hour 7. We then make the following calculations:

$$\begin{array}{ccccccccccccccc} 7 & \xrightarrow{+2} & 9 & \xrightarrow{+2} & 11 & \xrightarrow[-12 \text{ at top}]{+1} & 0 & \xrightarrow{+2} & 2 & \xrightarrow{+2} & 4 & \xrightarrow{+2} & 6 & \xrightarrow{+1} & 7 \\ \mathrm{G} & & \mathrm{A} & & \mathrm{B} & & \mathrm{C} & & \mathrm{D} & & \mathrm{E} & & \mathrm{F}^\sharp & & \mathrm{G}. \end{array}$$

Thus, we have obtained the G-major scale:

$$\mathrm{G} \quad \mathrm{A} \quad \mathrm{B} \quad \mathrm{C} \quad \mathrm{D} \quad \mathrm{E} \quad \mathrm{F}^\sharp \quad \mathrm{G}. \tag{1.19}$$

Example 1.4.5. D-major scale. We can also use addition on the chromatic clock to produce the D-major scale. The note D has hour 2, and the calculations go as follows:

$$\begin{array}{ccccccccccccccc} 2 & \xrightarrow{+2} & 4 & \xrightarrow{+2} & 6 & \xrightarrow{+1} & 7 & \xrightarrow{+2} & 9 & \xrightarrow{+2} & 11 & \xrightarrow[-12 \text{ at top}]{+2} & 1 & \xrightarrow{+1} & 2 \\ \mathrm{D} & & \mathrm{E} & & \mathrm{F}^\sharp & & \mathrm{G} & & \mathrm{A} & & \mathrm{B} & & \mathrm{C}^\sharp & & \mathrm{D} \end{array}$$

to get the D-major scale:

$$\mathrm{D} \quad \mathrm{E} \quad \mathrm{F}^\sharp \quad \mathrm{G} \quad \mathrm{A} \quad \mathrm{B} \quad \mathrm{C}^\sharp \quad \mathrm{D}. \tag{1.20}$$

Example 1.4.6. A-major scale. Finally, let's go up a fifth from the D-major scale. We will start from its fifth note, A, and create the A-major scale. The hour number for A is 9 and the calculations go as follows:

$$\begin{array}{ccccccccccccccc} 9 & \xrightarrow{+2} & 11 & \xrightarrow[-12 \text{ at top}]{+2} & 1 & \xrightarrow{+1} & 2 & \xrightarrow{+2} & 4 & \xrightarrow{+2} & 6 & \xrightarrow{+2} & 8 & \xrightarrow{+1} & 9 \\ \text{A} & & \text{B} & & \text{C}^\sharp & & \text{D} & & \text{E} & & \text{F}^\sharp & & \text{G}^\sharp & & \text{A} \end{array}$$

to obtain the A-major scale:

$$\text{A} \quad \text{B} \quad \text{C}^\sharp \quad \text{D} \quad \text{E} \quad \text{F}^\sharp \quad \text{G}^\sharp \quad \text{A}. \tag{1.21}$$

Remark 1.4.7. Looking at these successive scales, we see that every time we went up a fifth we found the same notes as on the previous scale, except for the next to the last note, where we obtained a sharped note from the previous scale's note. For example, here are the G-major and D-major scales:

$$\text{G} \quad \text{A} \quad \text{B} \quad \text{C} \quad \text{D} \quad \text{E} \quad \text{F}^\sharp \quad \text{G}$$

$$\text{D} \quad \text{E} \quad \text{F}^\sharp \quad \text{G} \quad \text{A} \quad \text{B} \quad \text{C}^\sharp \quad \text{D}.$$

The D-major scale can be rapidly obtained by just writing down the notes of the G-major scale, starting with D, and simply remembering to sharp the C that comes right before the ending D. This phenomenon is summarized in the famous "Circle of Fifths," which we describe in Chapter 2.

1.4.4 Relation between Just and Equal-Tempered Tunings

Although we have treated just and equal tempered tunings separately, we did note the close connection between them. Here is a listing of the frequency ratios for a major scale in equal temperament and the ratios that they approximate in the just scale:

$$\begin{aligned} \mathbf{r}^0 &= 1 \text{ (exact)} \\ \mathbf{r}^2 &= 1.122462\cdots \approx 1.125 = \frac{9}{8} \\ \mathbf{r}^4 &= 1.259921\cdots \approx 1.25 = \frac{5}{4} \\ \mathbf{r}^5 &= 1.334839\cdots \approx 1.3333\cdots = \frac{4}{3} \\ \mathbf{r}^7 &= 1.498307\cdots \approx 1.5 = \frac{3}{2} \\ \mathbf{r}^9 &= 1.681792\cdots \approx 1.6666\cdots = \frac{5}{3} \\ \mathbf{r}^{11} &= 1.887748\cdots \approx 1.875 = \frac{15}{8} \\ \mathbf{r}^{12} &= 2 \text{ (exact)} \end{aligned} \tag{1.22}$$

Musicians make use of these close approximations. The composer and physicist, John Powell, has concisely described how they do it:

> The "just" system can only be used to good effect by instruments that don't have fixed notes—like violins, violas, cellos, trombones and, most important, the human voice. On these instruments it is possible for good players to adjust their notes so that the combinations of notes always have lovely, simple relationships. [Powell (2010, 131)]

By "simple relationships," Powell means that the fundamentals for the combined notes have the simple fractional relationships found in just tuning. These simple fractional relationships provide the perfect matches of different harmonics from the combined notes, a property that equal-tempered tuning does not have. Nevertheless, since the vast majority of Western music is written with the equal-tempered system in mind, we shall concentrate on that system from here on.

Exercises

1.4.1. Use the clock arithmetic method to go up a fifth from the A-major scale, shown in (1.21), to find the E-major scale.

1.4.2. Explain why going up a fifth note in a scale is always done by adding 7 hours on the chromatic clock.

1.4.3. Going down a fifth. Here is the chromatic scale written using flats:

$$\text{C} \quad \text{D}^\flat \quad \text{D} \quad \text{E}^\flat \quad \text{E} \quad \text{F} \quad \text{G}^\flat \quad \text{G} \quad \text{A}^\flat \quad \text{A} \quad \text{B}^\flat \quad \text{B} \quad \text{C} \tag{1.23}$$

Now, suppose you have the C-major scale:

C D E F G A B C

and you want to make a new major scale that begins with the fifth note ***down*** from the note C at the right end of this scale. That would be the note F. **(a)** Use the clock arithmetic method to find the F-major scale, starting with this note F. [Note: Make sure you do not use a note letter more than once, resulting in one flatted note.] **(b)** From the F-major scale that you just found, make a new major scale that begins with the fifth note down from the highest pitched note (the second F) in the F-major scale.

1.4.4. Starting with the note $\text{F}^\sharp$, use this sequence of additions on the chromatic clock:

$$+2 \qquad +2 \qquad +3 \qquad +2 \qquad +3$$

to generate a pentatonic major scale. (Notice that its five distinct notes are the five black keys on a piano.)

1.4.5. Harmonic minor scales. A harmonic minor scale is obtained from a given starting note by using the following sequence of additions in our clock arithmetic method:

$$+2 \qquad +1 \qquad +2 \qquad +2 \qquad +1 \qquad +3 \qquad +1.$$

Use these additions to generate a harmonic minor scale that begins with C (the harmonic C-minor scale). (Remember to not repeat note letters; this will determine how you employ flats or sharps.)

1.4.6. Natural minor scales. A natural minor scale is obtained from a major scale by starting from its sixth note. For example, if the scale is C-major, then its sixth note is A. The natural A-minor scale is

A B C D E F G A.

(a) Verify that the natural A-minor scale, given above, corresponds to the following sequence of additions on the chromatic clock:

$$+2 \qquad +1 \qquad +2 \qquad +2 \qquad +1 \qquad +2 \qquad +2.$$

(b) Use these additions to generate a natural minor scale beginning with G (natural G-minor scale).
(c) Explain why this natural G-minor scale has the same notes as the $\text{B}^\flat$-major scale.

Modes for Major Scales

A major scale can be played in various modes. Each mode is obtained by using a different starting note for the scale. For example, if the scale is C-major:

C D E F G A B C

then its ***Dorian mode*** is obtained by starting on its second note:

$$\mathrm{D} \quad \mathrm{E} \quad \mathrm{F} \quad \mathrm{G} \quad \mathrm{A} \quad \mathrm{B} \quad \mathrm{C} \quad \mathrm{D}.$$

Although this scale was obtained from the C-major scale, it is usually designated by listing its starting note, D, and stating that it is Dorian mode. Thus, it is referred to as D-Dorian.

If one starts on the fourth note of the C-major scale, then the notes are in ***Lydian mode***:

$$\mathrm{F} \quad \mathrm{G} \quad \mathrm{A} \quad \mathrm{B} \quad \mathrm{C} \quad \mathrm{D} \quad \mathrm{E} \quad \mathrm{F}$$

This scale is called F-Lydian. If one starts on the fifth note of the C-major scale, then the notes are in ***Mixolydian mode***:

$$\mathrm{G} \quad \mathrm{A} \quad \mathrm{B} \quad \mathrm{C} \quad \mathrm{D} \quad \mathrm{E} \quad \mathrm{F} \quad \mathrm{G}.$$

This scale is called G-Mixolydian.

We will not be dealing extensively with modes in this book. These three modes should suffice to illustrate the idea. Their significance will be clearer once we describe chords and chord progressions in Chapter 3. Here are a few exercises that relate these modes to our clock arithmetic method.

1.4.7. **(a)** Verify that the D-Dorian mode, given above, corresponds to the following sequence of hour additions on the chromatic clock:

$$+2 \quad +1 \quad +2 \quad +2 \quad +2 \quad +1 \quad +2. \tag{1.24}$$

(b) Use this sequence of hour additions to find the scale for the G-Dorian mode. **(c)** Use this sequence of hour additions to find the scale for the C-Dorian mode.

1.4.8. **(a)** Verify that the G-Mixolydian mode, given above, corresponds to the following sequence of hour additions on the chromatic clock:

$$+2 \quad +2 \quad +1 \quad +2 \quad +2 \quad +1 \quad +2. \tag{1.25}$$

(b) Use this sequence of hour additions to find the scale for the E-Mixolydian mode. **(c)** Use this sequence of hour additions to find the scale for the C-Mixolydian mode.

1.4.9. **(a)** Verify that the F-Lydian mode, given above, corresponds to the following sequence of hour additions on the chromatic clock:

$$+2 \quad +2 \quad +2 \quad +1 \quad +2 \quad +2 \quad +1. \tag{1.26}$$

(b) Use this sequence of hour additions to find the scale for the D-Lydian mode. **(c)** Use this sequence of hour additions to find the scale for the C-Lydian mode.

1.4.10. In the song *Think of Me* (music by Andrew Loyld Webber), the first piano chord used consists of the simultaneous notes D_2, A_3, D_4, and $\mathrm{F}_4^\sharp$. Assuming that D_2 has fundamental ν_o, find the powers of $\mathbf{r}$ that multiply ν_o to obtain the fundamentals for these four notes. What simple fractions are closely approximated by these powers of $\mathbf{r}$, and what do these fractions have to do with the acoustic consonance discussed in Section 1.2?

1.4.11. In Beethoven's *Moonlight Sonata,* the following five notes are used in an (arpeggiated) chord: A_1, A_2, A_3, $\mathrm{C}_4^\sharp$, and E_4. Assuming that A_1 has fundamental ν_o, find the powers of $\mathbf{r}$ that multiply ν_o to obtain the fundamentals for these five notes. What simple fractions are closely approximated by these powers of $\mathbf{r}$, and what do these fractions have to do with the acoustic consonance discussed in Section 1.2?

1.4.12. Harmonic major scales. A harmonic major scale is obtained from a given starting note by using the following sequence of additions in our clock arithmetic method:

$$+2 \qquad +2 \qquad +1 \qquad +2 \qquad +1 \qquad +3 \qquad +1.$$

Use these additions to generate a harmonic major scale that begins with C (the harmonic C-major scale). Also use these additions to generate a harmonic major scale that begins with A (the harmonic A-major scale).

1.4.13. Melodic minor scales. A melodic minor scale is obtained from a given starting note by using the following sequence of additions in our clock arithmetic method:

$$+2 \qquad +1 \qquad +2 \qquad +2 \qquad +2 \qquad +2 \qquad +1.$$

Use these additions to find the melodic minor scale that begins with A (the melodic A-minor scale). (The melodic minor scale is used in ascending melodies in minor keys, while descending melodies use the natural minor scale.)

1.5 Logarithms

By working with half steps to represent changes in pitch from one note to the next we have, in effect, done an operation with logarithms. In this section we shall define logarithms in a precise way, and show exactly how logarithms describe the half step changes that we have been working with. We shall also discuss a few other ways they occur in music. Logarithms often occur in mathematics related to human perception, such as the musically important perceptions of pitch (frequency), loudness (sound volume), and rhythm (note durations).

1.5.1 Half Steps and Logarithms

We begin our discussion of logarithms by showing how they relate to the half step changes in pitch that we used for constructing musical scales. In (1.8) we described the C-major scale using these multiplications by powers of $\mathbf{r} = 2^{1/12}$:

$$\begin{array}{cccccccccccccccc} \nu_o & \xrightarrow{\cdot \mathbf{r}^2} & \mathbf{r}^2\,\nu_o & \xrightarrow{\cdot \mathbf{r}^2} & \mathbf{r}^4\,\nu_o & \xrightarrow{\cdot \mathbf{r}} & \mathbf{r}^5\,\nu_o & \xrightarrow{\cdot \mathbf{r}^2} & \mathbf{r}^7\,\nu_o & \xrightarrow{\cdot \mathbf{r}^2} & \mathbf{r}^9\,\nu_o & \xrightarrow{\cdot \mathbf{r}^2} & \mathbf{r}^{11}\,\nu_o & \xrightarrow{\cdot \mathbf{r}} & \mathbf{r}^{12}\nu_o = 2\nu_o. \\ \mathrm{C} & & \mathrm{D} & & \mathrm{E} & & \mathrm{F} & & \mathrm{G} & & \mathrm{A} & & \mathrm{B} & & \mathrm{C} \end{array} \tag{1.27}$$

Later, in (1.17), we described this same scale using half step changes:

$$\begin{array}{cccccccccccccccc} 0 & \xrightarrow{+2} & 2 & \xrightarrow{+2} & 4 & \xrightarrow{+1} & 5 & \xrightarrow{+2} & 7 & \xrightarrow{+2} & 9 & \xrightarrow{+2} & 11 & \xrightarrow[-12 \text{ at top}]{+1} & 0 \\ \mathrm{C} & & \mathrm{D} & & \mathrm{E} & & \mathrm{F} & & \mathrm{G} & & \mathrm{A} & & \mathrm{B} & & \mathrm{C}. \end{array} \tag{1.28}$$

We can see that, except for the change to 0 when 12 is reached, this latter calculation is *using addition of the powers of* $\mathbf{r}$ *rather than multiplication by* $\mathbf{r}$ *raised to those powers.* These powers of $\mathbf{r}$ are logarithms. To be more precise, we have the following definition.

Definition 1.5.1. *The **logarithm**, base* $\mathbf{r}$*, of a positive number* c *is the power of* $\mathbf{r}$ *that is needed to produce* c*. We write* $\log_{\mathbf{r}} c$ *to express this power. In other words,* $\log_{\mathbf{r}} c$ *satisfies:*

$$\mathbf{r}^{\log_{\mathbf{r}} c} = c. \tag{1.29}$$

At first sight, this definition does not appear to relate to our calculations with the C-major scale. However, if we have $c = \mathbf{r}^p$, then the power needed for obtaining c on the base $\mathbf{r}$ is p. Therefore, we have the following identity:

$$\log_{\mathbf{r}} (\mathbf{r}^p) = p. \tag{1.30}$$

Identity (1.30) implies that the hour changes used in (1.28) are the logarithms, base $\mathbf{r}$, of the frequency factors used in (1.27).

The logarithm, base $\mathbf{r}$, of a given positive number is unique. The reason for the uniqueness of the logarithm is shown in Figure 1.19.

We have introduced logarithms by relating them to constructing the C-major scale. However, they also can be used to describe any frequency changes occurring from one pitch to another. Here are some basic illustrations of how this is done.

Example 1.5.2. Frequency change by half steps. **(a)** If one pitch has fundamental 220 Hz, and another pitch has fundamental 440 Hz, then we have

$$\frac{440}{220} = 2.$$

However, since $\mathbf{r} = 2^{1/12}$, we also know that

$$\mathbf{r}^{12} = 2.$$