Stochastic Tools
in Turbulence
APPLIED MATHEMATICS
AND MECHANICS

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STOCHASTIC TOOLS
IN TURBULENCE

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Preface

This book is about the mathematical tools that are available to describe stochastic vector fields and to solve problems relating to them. I am principally interested in turbulence, a phenomenon occurring in fluids, and although the material in this book has applicability beyond the needs of turbulence, much of what is covered arises from these needs. To use a word suggested by R. W. Stewart, the turbulence syndrome includes the following symptoms: The velocity field is such a complicated function of space and time that a statistical description is easier than a detailed description; it is essentially three-dimensional, in the sense that the dynamical mechanism responsible for it (the stretching of vorticity by velocity gradients) can only take place in three dimensions; it is essentially nonlinear and rotational, for the same reasons; a system of partial differential equations exists, relating the instantaneous velocity field to itself at every time and place. Most problems in classical fluid mechanics are reduced to solubility by two-dimensionality, linearity, or irrotationality; in turbulence these familiar and useful techniques must be discarded. Most problems in classical stochastic processes are reduced to solubility by statistical independence, or the assumption of a normal distribution (which is equivalent) or some other stochastic model; because of the governing differential equations, the turbulent velocity field at two space-time points is, in principle, never independent—in fact, the entire dynamical behavior is involved in the departure from statistical independence. The equations, in fact, preclude the assumption of any ad hoc model, although this is often done in the absence of a better idea. The needs of turbulence, then, will best be met by a discussion of stochastic vector fields which emphasizes three-dimensional aspects, and gives short shrift to linear problems and stochastic model building. This book attempts to provide such a treatment. Other books available either emphasize one-dimensional aspects and linear problems, such as are appropriate to communication theory, or emphasize statistical independence, Markov processes, Brownian motion, and other stochastic models relevant to other physical problems and of interest to mathematicians.
The above description does not do justice to several works. Chapters two and six of Monin and Yaglom (1970) have many points of similarity with the present work, differing principally in the selection and weighting of material, and in the mathematical point of view. Yaglom (1962) covers related material, but is limited to stationary processes. Other works, such as Doob (1953) represent a mathematical level that is difficult for the beginning student to grasp. This latter statement is a serious one, which it is necessary to discuss before we can proceed.

This book is intended to satisfy a need somewhere between that of the theoretician and the experimentalist. For the former, nothing will replace a thorough grounding in the various branches of mathematics involved in this subject. From a practical point of view, however, he must be motivated; if one waits to tell him about turbulence until he has studied all the necessary mathematics, one will probably never see him again. It is necessary to provide him first with a background in the structure of the subject, and this will probably be all the coverage that the experimentalist will need.

Many rigorous works are also elegant, a word difficult to define which I have always taken to mean displaying a certain unity of creation and economy of line; it must be very satisfying to produce an elegant piece of work. Most of this book is written from the point of view of generalized functions, and it might have been more elegant to adhere to this point of view throughout. However, I feel that sometimes a point of view and degree of generality may be appropriate for some things, and not for others, and the attempt to present everything from a single point of view may obscure parts of the subject. Accordingly, I have used the ideas of generalized functions whenever they seemed to me to result in simplification. The same is true with regard to very general (unrestrictive) assumptions; as a result, both the point of view and the degree of generality varies from place to place in the book.