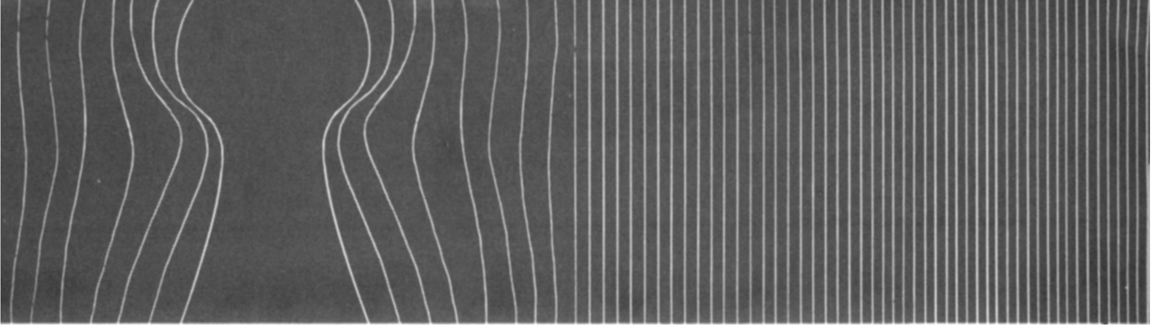
FLUID MECHANICS
AND
SINGULAR PERTURBATIONS



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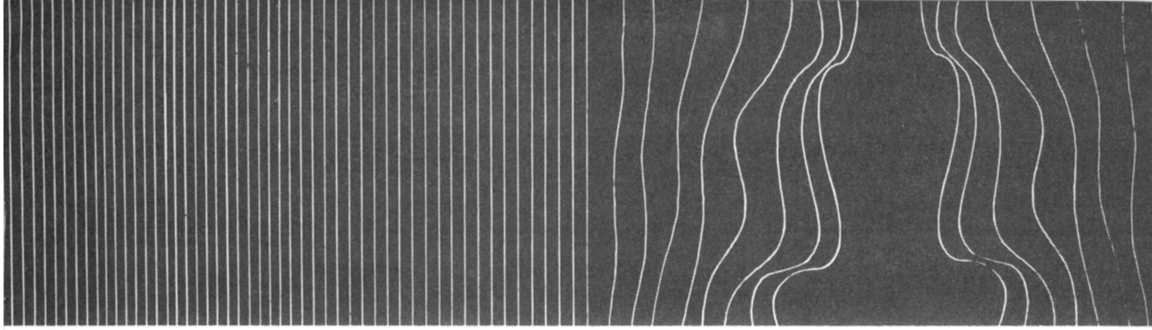

FLUID MECHANICS AND SINGULAR PERTURBATIONS

A Collection of Papers by
SAUL KAPLUN

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FOREWORD

Saul Kaplun was born in Lwow on July 3, 1924, and died in Pasadena, California on February 13, 1964. Shortly after having immigrated to the United States he entered the California Institute of Technology as a freshman in 1942 and, except for wartime service as a radar engineer, spent the remainder of his life there, first as a student and later as a teacher. Upon receiving his Ph.D. in Aeronautical Engineering in 1954 he was made a member of the GALCIT staff. It was my privilege to work very closely with him during the entire span of his short but very fruitful career as a creative scientist.

His only publications are the three papers reprinted in this book. Although it has long been recognized that many deep and highly original ideas are contained in the few pages of these papers, his published work gives a very incomplete picture of his scientific discoveries. Some of his ideas were published indirectly: As a teacher Saul Kaplun excelled in the personal supervision of Ph.D. candidates, and many published Ph.D. theses refer to ideas of Kaplun, with due credit given. An example is the thesis by I-Dee Chang, who worked closely with Kaplun (see Chang (1961) *). The survey article on viscous flow by Lagerstrom (1964) also contains many fundamental ideas due to Saul Kaplun. However, the bulk of Saul Kaplun's research was never published either directly or indirectly. Fortunately, much of it exists in manuscript form. The present book contains edited versions of the most important of those manuscripts.

It was difficult to decide on the method to be used in editing. Saul Kaplun was a perfectionist; one reason for his reluctance to publish was that, once having arrived at a result, he usually wanted to obtain a still deeper understanding of the problem on which he was working. However, the manuscripts used as a basis for this book were only drafts of articles, or even just informal writeups for use by him and a few co-workers. Thus it is obvious that the manuscripts are incomplete, sometimes repe-

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*See Bibliography at the end of the book.

titious, often informal or obscure, and certainly not in polished form. Still, it was decided to change them as little as possible. The editing consisted in sorting out and selecting manuscripts from a vast amount of material, in making the notation as systematic as possible, in checking for misprints in formulas, and finally in providing introductions and Editors' Notes which attempt to clarify various points. Thus the articles in this book are presented essentially the way Saul Kaplun wrote them, although not in the way he himself would have published them. The reader is asked to excuse the unavoidable imperfections in the main text as well as in the Editors' Notes and to judge the material in this book by its positive merits rather than by its obscurities and mistakes. Obviously, many things would have been corrected by Saul Kaplun if he himself had reworked the drafts. The editors do not claim to have checked all the material in great detail nor to have understood all difficult points of the manuscripts. Additional work would have improved the editing but would have delayed the publication of the material still further.

Thus, in spite of the great amount of editorial work done, most of this book is to be regarded as raw material rather than a finished product. It is my personal conviction that Saul Kaplun's manuscripts, even in their unfinished form, contain a wealth of ideas which should be made available to the scientific community as soon as possible. I believe that extracting these ideas from the material presented here and developing them further will be a scientifically rewarding, although admittedly often difficult, undertaking.

The omissions in this book are due to many causes. Many manuscripts, for example the numerical studies of the separation problem, are too incomplete and fragmentary. Some are lost, such as the studies of the shock-wave boundary-layer interaction which Saul Kaplun made more than ten years ago and which contained some highly interesting examples of singular perturbation problems involving two small parameters. Others are special solutions (for example, of the Oseen equations) which often have been subsequently rediscovered and published by other authors. Although Saul Kaplun found several ingenious solutions to special problems, he did not attach much importance to the solutions in themselves, only to their use in illustrating some fundamental idea. His thesis for the degree of Aeronautical Engineer, "Dimensional Analysis of the Inflation Process of Parachute Canopies" (1951) is available in the GALCIT Library. This paper is not included here; its subject matter was thought to be too special and too far removed from the material in this book. Shortly before his death Saul Kaplun had been invited to give a major address on the separation problem at the Eleventh International Congress of Applied Mechanics in Munich, August 30 to September 5, 1964. A lecture entitled "Boundary Layers and Separation" was given posthumously in his name and has subsequently been published in the Proceedings of the Congress. However, this lecture was based entirely on Chapter I of Part II of the present book and is hence not reprinted here.

The material in this book has been grouped into two parts. Part I, which contains reprints of the three published papers, deals with asymptotic solutions of the Navier-Stokes equations, especially at low Reynolds numbers, and with fundamental ideas in the theory of singular perturbations. Although the problem of flow at low Reynolds numbers provided the incentive for introducing various mathematical ideas and constantly serves as main illustrating example, the mathematical content actually applies to a very wide class of problems in various domains of mathematical physics. Most of the work presented here was done before 1956. Part II, of which nothing has previously been published (except the above-mentioned Munich lecture), is a detailed study of one aspect of the problem of fluid separation, namely the solution of the boundary-layer equations in the vicinity of a point of vanishing skin friction.

References given in the text are all taken from Saul Kaplun's manuscripts. The editors' comments give as references almost exclusively those articles and books which Saul Kaplun had used in his work or, conversely, publications where ideas of his are exposed or utilized; for instance, his work on the separation problem was done independently of the papers on the subject which appeared during the past decade. The editors have therefore not compared Saul Kaplun's results with this literature. Such a comparison would not only have delayed the publication of the present book considerably but is beyond the scope of a book which essentially aims at presenting his own writings.

Footnotes appearing in the text are Saul Kaplun's own notes. The Editors' Notes are placed after each chapter and referred to by a number in square brackets.

Thanks are due to the many persons who aided in the publication of this book. L. N. Howard, co-editor of Part I, helped greatly in organizing and clarifying Saul Kaplun's intuitive, rather than formal and rigorous, approach to the theory of singular perturbations. Comments made by George W. Bluman during the final stages of editing Part I proved to be of great value. Ching-Shi Liu, co-editor of Part II, did the major part of checking and organizing of Saul Kaplun's manuscripts on the separation problem. This task which was often difficult and time-consuming, not only because of Saul Kaplun's highly original approach to the subject, but also because of the large amount of concrete analysis and detailed computations involved. Stephen W. Childress and I-Dee Chang rendered substantial help in the editing, both having worked closely with Saul Kaplun. I-Dee Chang had actually collaborated with Saul Kaplun in writing sections of the notes on separation. Milton Van Dyke helped in various essential ways. Mrs. Joy Smelser did the major part of the difficult technical typing of the edited manuscript; the remainder of the typing as well as all other secretarial work involved was done by Mrs. Vivian Davies. Verlag Birkhäuser and the Editorial Board of the Journal of Mathematics and Mechanics kindly gave permission for reprinting the articles appearing here as Chapter I and Chapters II and III, respectively. Thanks are due to Academic Press for realizing the scientific value of Saul Kaplun's work and for guaranteeing a speedy publication of the edited manuscripts. Saul Kaplun's research was originally sponsored by the Office of Naval Research and, since 1958, by the Air Force Office of Scientific Research. The U. S. Air Force also sponsored the preparation of this book (Grant AF-AFOSR-338-65). Finally, tribute is due to the late Clark B. Millikan who, as Director of GALCIT, showed great confidence in Saul Kaplun's research and who actively encouraged the posthumous publication of his work.

November, 1966,
GALCIT

P. A. LAGERSTROM

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PART I

THEORY OF SINGULAR PERTURBATIONS WITH APPLICATIONS TO THE ASYMPTOTIC THEORY OF THE NAVIER-STOKES EQUATIONS

Edited by

P. A. Lagerstrom and L. N. Howard

INTRODUCTION TO PART I.

Saul Kaplun developed his ideas about singular perturbations in the course of studying asymptotic solutions of the Navier-Stokes equations, in particular solutions for viscous flow at low Reynolds numbers. As a result, almost all his discussion of asymptotic theory and technique is given in connection with problems of viscous flow. This Introduction will attempt a survey of his basic ideas. A historical account of how he developed these ideas will be given here. It is hoped that such an account will be of help for understanding the logic of his ideas although, for reasons stated, concepts from Fluid Dynamics will be used. However, Kaplun's theory of singular perturbations could, in principle, be presented without any reference to viscous flow. Therefore some simple mathematical models will be given later on in this Introduction and it will be indicated briefly how Kaplun's ideas can be illustrated with the aid of these equations.

Development of an Asymptotic Theory for Flow at Low Reynolds Numbers. In this historical account we shall have to assume that a reader is familiar with the basic theory of viscous fluids. As reference for this theory the reader may consult standard textbooks. We refer also to Van Dyke (1964) which deals specifically with perturbation problems and applies Kaplun's technique to a large variety of problems and to Lagerstrom (1964) which, in the discussion of viscous flow, makes extensive use of ideas due to Kaplun. The notation used in this Introduction is explained in the articles reprinted here as Chapters II and III.

The problem of discussing Prandtl's boundary-layer theory (for flow at large Reynolds numbers) in the light of the theory of asymptotic expansions had occupied various research workers at GALCIT since before 1950. Kaplun's first contribution to this research was his doctor's thesis on the role of the coordinate system in boundary-layer theory; the published version of this is reprinted here as Chapter I. Very little comment is needed on this chapter (in fact, Editors' Notes were dispensed with). It is a self-contained paper which solves the given problem indicated by the title in an ingenious way by going directly to the heart of the matter. In essence the problem posed is completely solved, although obviously several generalizations of the theory suggest themselves.

After having completed his thesis, Kaplun started his work on the theory of flow at low Reynolds numbers. The problems of flow at large and small Reynolds numbers, respectively, exhibit a certain similarity. In the former case one may obtain an approximate solution by neglecting the viscous terms in the Navier-Stokes equations. This approximation is, however, not uniformly valid near the boundary of a solid. Prandtl's boundary-layer theory of 1904 was an ingenious attempt to cope with this difficulty and is in fact a classical example of one of the important methods in the theory of singular perturbations. For the case of flow at low Reynolds numbers Stokes proposed in 1850 that the transport terms be neglected in the Navier-Stokes equations. The success as well as the difficulties of this theory are well-known. A partial explanation of the difficulties was found by Oseen around 1910, who pointed out that the Stokes approximation was not uniformly valid at infinity. Oseen introduced the "extended Stokes equations", now known as the Oseen equations. These equations evidently were an improvement over the Stokes equations in some respects; however a rational discussion of their meaning and their validity was missing. Anyway in both the case of large Reynolds numbers and that of small Reynolds numbers a basic fact is the lack of uniform validity of certain approximations. Guided by this similarity Kaplun started to investigate whether a unified theory of singular perturbations could be applied to both cases. The ultimate result of his investigations was not only the first systematic explanation of meaning of the Stokes and Oseen solutions and their place in an asymptotic expansion of the corresponding Navier-Stokes solutions, but also some very significant and deep ideas about singular perturbation problems, powerful and general enough to be applicable to a wide variety of problems.

Previously many authors had applied the idea of limit processes to theory of flow at large Reynolds numbers: an outer limit, which yields the inviscid solution and an inner limit which yields the boundary-layer solution (cf. Chapter I). Similarly (cf. Chapters II and III) as Reynolds number tends to zero one may consider two principal limits.

The outer limit (Oseen limit) may be thought of as letting the characteristic length of the body go to zero while everything else is fixed. The Navier-Stokes equations are unchanged under this limit. The inner limit (Stokes limit) may be conveniently thought of as letting viscosity tend to infinity; applied to the Navier-Stokes equations it yields the Stokes equations. Kaplun introduced the systematic use of these two limits. Under the outer limit the finite body shrinks to a point which cannot disturb the flow. Thus it is plausible that the free stream velocity $\vec{q}^*$ at any fixed point tends to the free-stream value $\vec{i}$. The convergence is, however, not uniform near the body since $\vec{q}^* = 0$ at the surface of the body. Or, to put it slightly differently, a very small body introduced in a uniform stream causes a perturbation which is very small except at the surface of the body. It is this lack of uniformity of the outer limit near the body which suggests the introduction of the inner limit described above. An equivalent description of the inner limit is the following: As the body shrinks to a point, say the origin, one measures the velocity, not at a fixed point, but at a point which approaches the origin at the same rate at which the body shrinks. Using either description one sees that the value $q^* = 0$ at the body does not "get lost" when the inner limit process is applied. Note that Oseen emphasized the lack of uniformity of the Stokes equations at ∞; Kaplun emphasized the lack of uniformity of the outer limit near the body. In fact he insisted that while the outer and inner limits are the two principal limits (see end of Section 1.2 of Chapter IV) the outer limit has priority, it is "more principal". (See end of Section 1.5 of Chapter IV). The physical reason for this is that the nature of the perturbation problem is that a uniform stream ($= \vec{i} =$ outer limit of $\vec{q}^*$) is being perturbed by the presence of a small body. This is physically reasonable, and the mathematical technique for constructing the asymptotic expansions consists in first finding the leading term of the outer solution (in this case $\vec{g}_0 = \vec{i}$) and then matching a solution of the inner (Stokes) equations to the outer approximation $\vec{i}$.

The technique of using inner and outer limits was, of course, not new. The new thing in the reasoning described above consisted essentially in realizing that this technique could be applied to the problem of flow at low Reynolds numbers and in finding suitable inner and outer limits and finally in recognizing the priority of the outer limit. According to the state of the art in 1953-54 the next thing would be to construct the inner and outer expansions, in principle obtainable by repeated applications of the respective limit processes to the exact solutions and in practice by solving certain approximate differential equations with appropriate boundary conditions and matching conditions. In boundary-layer theory (see Chapter I) the simplest form of the matching condition states that the tangential velocity component at the wall given by the outer solution has the same value as that given by the inner (boundary-layer) solution at infinity. A more general form is that the inner limit of the outer limit should agree with the outer limit of the inner limit. A still more general form is that some partial sum of the outer expansion, evaluated for small values of the argument, should agree with some partial sum of the inner expansion, evaluated for large values of the argument.

Consider now the case of two-dimensional flow. In this case the first two forms of the matching condition fail completely; the last form may be verified a posteriori but hides the nature of matching. To see that the first two forms fail we observe that the inner limit of $\vec{q}^*$ is zero (an analogy with the heat equation makes this quite plausible a priori and it may be made still more plausible a posteriori). Since the outer limit of $\vec{q}^*$ is $\vec{i}$, it is obvious that the inner and outer limit cannot be matched by any process. This led Kaplun to a thorough rethinking of the principles underlying the techniques of matching.

The basic question is: What are the essential conditions for two asymptotic approximations to match? To answer this question Kaplun considered the partially ordered set of equivalence classes (ord classes, ord f) of functions f(Re) such that ord Re $\leq$ ord f $\leq$ ord 1. (For the definition of these terms see the section on Intermediate Limits in Chapter II or Section 1 of Chapter IV). To each such f(Re) there is a corresponding limit process defined; the order classes are also used in describing the domain of validity of an approximation. (cf. Chapter II and Section 1.2 of Chapter IV). Let us consider the two-dimensional problems in polar coordinates and disregard the dependence on θ (which does not lead to non-uniformities). The outer variable is then $\tilde{r} = Ur/\nu$ and the inner variable is $r^* = \tilde{r}/Re$. Let w be a flow quantity, say $\vec{q}^*$, and w_1 and w_2 two approximations. Assume that w_1 is uniformly valid in the order domain D_1, consisting of all f such that ord g_1 < ord f $\leq$ ord 1. This means that as Re tends to zero $w - w_1$ tends to zero uniformly for $f_1(\mathrm{Re}) \leq \tilde{r} < \infty$ where f_1 is any function in the order domain D. In other words, given any δ one may find an $\epsilon_{1\delta}$ such that $|w - w_1| < \delta$ in the domain bounded by the horizontal lines $\mathrm{Re} = \epsilon_{1\delta}$ and $\mathrm{Re} = 0$ and the curve $\tilde{r} = f_1(\mathrm{Re})$ (See Fig. 1).

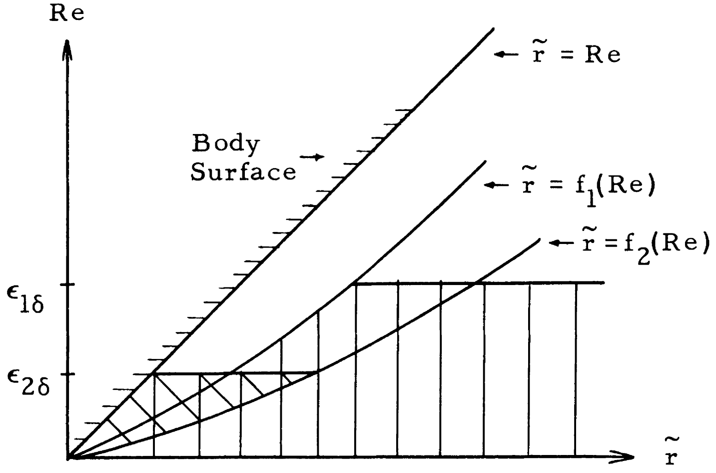

Fig. 1 Overlapping Domains of Validity

Similarly, we may speak of w_2 being a uniformly valid approximation in an order domain D_2 consisting of all $f(\mathrm{Re})$ such that $\mathrm{ord}\ \mathrm{Re} \leq f < \mathrm{ord}\ g_2$. This means that $w - w_2$ approaches zero uniformly for $\mathrm{Re} \leq \tilde{r} \leq f_2(\mathrm{Re})$ where $\mathrm{ord}\ f_2$ is any order class in D_2. Clearly w_1 and w_2 can be matched only if they have domains of validity which overlap, that is which have order classes in common. In the present case this means that $\mathrm{ord}\ g_1 < \mathrm{ord}\ g_2$. The functions f_1 and f_2 may then be chosen such that $f_1 < f_2$ for $\mathrm{Re} > 0$, and there exists a function f_3, lying inbetween f_1 and f_2. A matching condition is then simply $\lim_{f_3}(w_1 - w_2) = 0$.