MATERIALS SCIENCE SERIES

ANELASTIC RELAXATION IN CRYSTALLINE SOLIDS

A. S. NOWICK and B.S. BERRY

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Thomas J. Watson Research Center
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Yorktown Heights, New York

1972

ACADEMIC PRESS New York and London

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1 7DD

LIBRARY OF CONGRESS CATALOG CARD NUMBER: 70-154378

PRINTED IN THE UNITED STATES OF AMERICA

To Joan and Maureen

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Preface

Anelasticity may be said to have originated as a distinct discipline in 1948 with the publication of Zener's pioneering monograph "Elasticity and Anelasticity of Metals." This book defined anelasticity and demonstrated the usefulness of separating it from the more general viscoelastic behavior treated in the earlier literature. In the years since Zener's book, an enormous growth in the literature of the subject has taken place, as is became apparent that the study of anelastic relaxation could contribute greatly to our understanding of almost the whole gamut of physical and chemical phenomena in crystalline solids. Accordingly, there has been a clear need both for a textbook and an up-to-date monograph on anelasticity in crystals. In writing the present book the authors have hoped, perhaps too ambitiously, to fulfill both requirements.

The first six chapters comprise the formal theory of the subject, the heart of which is covered in Chapters 3–5. It is this theory that provides the glue which holds together the diverse topics to be covered later. In this sense, anelasticity differs from the related topic of "internal friction," which does not possess such a unifying theoretical base. These first six chapters, together with selected readings from the later chapters, can constitute a text for self study, or for a graduate course such as one of the authors (A.S.N.) has given at Columbia University. These early chapters are by no means confined to crystals, and are therefore equally suitable as an introduction to anelasticity in the noncrystalline state, or, with simple changes in notation, to dielectric relaxation. To further aid the usefulness of the book as a text, we have designated with heavy asterisks those sections which may be omitted on a first reading. Also, a rather substantial number of problems are included in the early chapters.

The remainder of the book deals in some detail with the various physical phenomena in crystalline substances which can give rise to anelastic relaxation. In view of the fact that the book is intended for readers with diverse backgrounds, such as mechanical or metallurgical engineers, materials scientists, and solid-state physicists, we have felt it desirable

to present the essential background material on each of the phenomena covered. In the case of the topics dealt with most extensively, namely point defects and dislocations, full chapters (7 and 12) of background material are presented; for other topics, a single section of background material seemed to be sufficient. For each of the physical phenomena, we have attempted to describe the current state of understanding of the anelastic effects in such a way that the reader can then go directly to the literature. With the enormous growth of the literature in the past decade, however, we regret that a complete or exhaustive literature survey could not be attempted, and that often work of some significance had to be omitted. Phenomena which involve ultrasonic attenuation or internal friction but which are not strictly describable as anelastic effects have generally been omitted, although (in Chapters 14 and 17–19) some borderline topics are discussed briefly because of their special importance. Finally, due to the limitations of the book as well as our own backgrounds, we have omitted consideration of organic crystalline compounds.

Acknowledgments

The authors owe a debt of gratitude to many colleagues and students who read individual chapters and contributed valuable criticisms and suggestions, particularly to D. N. Beshers, C. Elbaum, M. C. Franzblau, C. W. Garland, R. Gibala, D. F. Gibbons, W. R. Heller, G. M. Leak, W. P. Mason, D. I. Paul, D. S. Richter, J. T. A. Roberts, E. T. Stephenson, J. C. Swartz, T. J. Turner, and G. P. Williams. We are especially grateful for the support of much of the research leading to this book, as well as partial support of the writing itself, by the U. S. Atomic Energy Commission. We thank Miss Maureen Weaver for her skillful typing of a substantial fraction of the manuscript. Finally, we wish to express our deepest gratitude to our wives for their encouragement and understanding over the several years during which this book was being written.

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Chapter 1 / Characterization of Anelastic Behavior

The first part of this book (Chapters 1–6) deals with the formal theory of anelasticity. We begin with the introduction of a set of postulates which serve to define the subject, and then proceed to develop the consequences of these postulates. The formal theory does not include the physical origins or atomistic mechanisms of anelasticity, but does embrace the interpretation of anelastic behavior as a manifestation of internal relaxation processes.

The function of the present chapter is to lay the groundwork for the formal theory, first, by introducing the postulates, and second, by deducing the characteristic response of an anelastic material to the imposition of certain simple histories of stress or strain. The *response functions* which emerge from this treatment are then taken as the basic manifestations of anelastic behavior. These objectives are, in fact, accomplished in Sections 1.1–1.3, so that the reader interested in the purely theoretical development of the subject may omit the remainder of the chapter.

In Sections 1.4–1.7 we turn to the various dynamical methods that are widely used in studying anelasticity. These methods are of such importance in practice that we have felt it necessary to present them here, even though the material constitutes a digression from the main development of the theory.

1.1 The Meaning of Anelasticity

In order to describe anelasticity in a formal way, it is convenient to consider first an *ideal elastic material*, for which Hooke's law defines the relation between stress σ and strain ε as

$$\sigma = M\varepsilon \tag{1.1-1}$$

or

$$\varepsilon = J\sigma \tag{1.1-2}$$

with

$$M = 1/J \tag{1.1-3}$$

The constant M is called the modulus of elasticity (or often just the *modulus*) while its reciprocal J is called the modulus of compliance (or simply the *compliance*).

For an arbitrary deformation, the stress and strain must be expressed as second-order tensors, and Hooke's law then becomes a set of linear equations expressing each component of the stress tensor in terms of all the components of the strain tensor (or vice versa). For present purposes, this generalization would merely add unnecessary complications, so we shall proceed from the statement of Hooke's law as given by Eqs. (1.1-1) and (1.1-2), leaving the more elaborate treatment to Chapter 6. We note however that the present approach implies a simple mode of deformation, such as pure shear, uniaxial deformation, or hydrostatic deformation. For such cases, the appropriate modulus M will be the shear modulus, Young's modulus, or the bulk modulus, respectively.

There are three conditions defining ideal elastic behavior which are implicit in Eqs. (1.1-1) and (1.1-2). These are: (1) that the strain response to each level of applied stress (or vice versa) has a *unique equilibrium value*; (2) that the equilibrium response is achieved *instantaneously*[†]; (3) that the response is linear (e.g., doubling the stress doubles the strain). It should be noted that a characteristic feature of elasticity, namely the *complete recoverability* of the response upon release of the applied stress or strain, is a corollary of condition (1).

In order to generalize upon ideal elastic behavior, the three conditions listed above may be lifted in various combinations. The possibilities are shown in Table 1-1 together with the name given to the discipline of study in each case. As already mentioned, if all three conditions apply, we have the case of ideal elasticity. If only the restriction of linearity is lifted, "nonlinear elasticity" is the obvious result. If both linearity and complete recoverability are dropped, the material becomes capable of exhibiting "instantaneous plasticity," which is often called "crystal plasticity" because this type of plasticity is best exemplified by the yielding of ductile crystals under high stress. In contrast to the first three entries in Table 1-1, the remaining two entries introduce *time dependence* into the response, by lifting the condition of instantaneity. If this is the only condition lifted, we produce the type of behavior known as anelasticity, the subject of this book. The more general behavior obtained by addition-

[†] Because of the finite velocity of sound, the response is actually instantaneous only in an infinitesimally small sample. The essential point is that Eqs. (1.1-1) and (1.1-2) do not contain time as a variable.

TABLE 1-1

DIFFERENT TYPES OF MECHANICAL BEHAVIOR, CLASSIFIED ACCORDING TO THE CONDITIONS OBEYED BY THE STRESS–STRAIN RELATIONSHIP

	Unique equilibrium relationship (complete recoverability)	Instantaneous	Linear
Ideal elasticity	Yes	Yes	Yes
Nonlinear elasticity	Yes	Yes	No
Instantaneous plasticity	No	Yes	No
Anelasticity	Yes	No	Yes
Linear viscoelasticity	No	No	Yes

ally lifting the condition of complete recoverability is known as linear viscoelasticity, which thus includes anelasticity as a special case.

To summarize the definition of anelasticity, we may employ the following three postulates:

1. For every stress there is a unique equilibrium value of strain, and vice versa.
2. The equilibrium response is achieved only after the passage of sufficient time.†
3. The stress–strain relationship is linear.

The first and third postulates are, of course, merely a repetition of those for ideal elasticity. It should again be recalled that complete recoverability is a corollary of postulate (1), only now the recovery will, in general, be time dependent.

It should be noted that lifting the condition of instantaneity does not imply that *all* of the response of an anelastic material must develop in a time dependent manner, since to do so would produce the special (and unrealistic) case of a material without any component of elastic behavior. Rather, anelasticity implies that, in addition to an instantaneous (elastic) response, there also exists a time dependent nonelastic response.‡ Another point of clarification is concerned with the meaning of the term "linear"

† There is no restriction on the time scale in these considerations, i.e., the achievement of equilibrium may require anything from microseconds (or less) to extremely long periods of time.

‡ Thus, while the term "anelasticity" carries the connotation of being "without elasticity," this meaning applies in a literal sense only to a fraction (and often, a very small fraction) of the total response.

in postulate (3). In the present chapter linearity will be taken to mean that if the stress is doubled at each instant of time, the corresponding strain will also be doubled at every instant (and vice versa). On the other hand, this interpretation does not constitute the full meaning of linearity, as we shall see in Chapter 2. The full meaning of linearity is covered by the following statement: "If a given stress history $\sigma_1(t)$ produces the strain $\varepsilon_1(t)$, and if a stress $\sigma_2(t)$ gives rise to $\varepsilon_2(t)$, then the stress $\sigma_1(t) + \sigma_2(t)$ will give rise to the strain $\varepsilon_1(t) + \varepsilon_2(t)$."

The postulate of linearity is actually incorporated into the definition of anelasticity as a matter of practicality, since the theory becomes extremely difficult otherwise. Fortunately, except for some special circumstances, the observed behavior of materials at low stress levels usually meets the requirement of linearity. The basis for the other two postulates lies in thermodynamic and kinetic concepts which will be discussed briefly now and developed in detail in Chapter 5.

It will be recalled first that a thermodynamic substance is one which can assume a continuous succession of unique equilibrium states in response to a series of infinitesimal changes in an external variable. As a consequence, the first postulate of anelasticity, involving the unique equilibrium stress–strain relationship and the corollary of complete recoverability, is satisfied by all materials that qualify as thermodynamic solids. It should be noted that plastic and viscoelastic materials do not qualify as thermodynamic solids. The second postulate of anelasticity means that, in response to a change in the applied mechanical forces, time is required for the equilibration of an anelastic material. In general, the self-adjustment of a thermodynamic system with time toward a new equilibrium state in response to a change in an external variable is termed *relaxation*. Specifically, where the external variable is mechanical (a stress or a strain), the phenomenon is known as *anelastic relaxation* (or mechanical relaxation).[†] If stress is regarded as the independent variable,

[†] Similarly, dielectric or magnetic relaxation may occur under the influence of an electric or magnetic field, respectively. The formal theory of all these phenomena is similar and in large measure interconvertible by appropriate changes in terminology. For dielectric relaxation, electric field strength replaces stress, electric displacement replaces strain, and the dielectric constant replaces the elastic compliance. For magnetic relaxation, the corresponding variables are the magnetic field strength, intensity of magnetization, and magnetic susceptibility, respectively. The only differences in the formal theory of all of these phenomena arise from the tensor nature of stress and strain as compared with the vector nature of the corresponding variables in dielectric and magnetic relaxation. Dielectric relaxation has been the subject of a recent book by Daniel (1967).

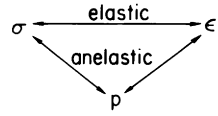

FIG. 1-1. Illustration of the direct and indirect coupling between stress and strain in an anelastic solid, the indirect coupling taking place via an internal variable p.

anelastic relaxation manifests itself as the time-dependent equilibration of the conjugate strain variable (or vice versa). It is important to recognize, however, that this external manifestation of relaxation merely parallels and reflects the adjustment of *internal variables* to new equilibrium values. For simplicity, consider the case where there is just one internal variable p, which is influenced by the stress and contributes to the strain. As shown in Fig. 1-1, the stress is now not only directly linked to strain through the purely elastic coupling, but is also linked indirectly through the internal variable p. For each value of applied stress, we may denote the equilibrium value of p by $\bar{p}$. Thus, as the internal variable relaxes toward equilibrium ($p \rightarrow \bar{p}$), ε concurrently changes toward a corresponding equilibrium value. In order to obtain anelastic behavior, the relaxation $p \rightarrow \bar{p}$ must proceed at a finite rate rather than instantaneously.[†] Such behavior will always occur when a change in p involves a transport process. For example, if p were a parameter characterizing the state of order in an alloy, a change in p would require atomic migration.

In summary, then, anelastic relaxation is inherently a thermodynamic phenomenon which arises from a coupling between stress and strain via certain internal variables which can change to new equilibrium values only through kinetic processes such as diffusion. The external manifestation of this internal relaxation behavior is the time dependent stress–strain behavior given in our previous formal definition of anelasticity. In returning to the development of the formal theory, we shall at first make use of the postulates only, and thereby eliminate any reference to internal variables. After we have gone as far as possible with this approach, internal variables and the thermodynamic basis for relaxation will be reintroduced in Chapter 5, to provide the basis on which specific molecular mechanisms of relaxation can later be discussed.

1.2 Quasi-Static Response Functions

An experiment in which either an applied stress or strain is held constant for any desired period of time is termed *quasi-static*. Under

[†] For, if the change $p \rightarrow \bar{p}$ did not take time, no operational distinction would exist from the case of pure elasticity.

such conditions, anelastic materials exhibit the phenomena of creep, the elastic aftereffect, and stress relaxation. These phenomena, and the response functions used to describe them, are discussed in turn below.

A. Creep

In the creep experiment a stress σ_0 is applied abruptly to the sample at $t = 0$, and held constant while the strain ε is observed as a function of time. The experimental conditions may therefore be expressed by

$$\sigma = \begin{cases} 0, & t < 0 \\ \sigma_0, & t \geq 0 \end{cases}$$

From the requirement of linearity, it is clear that $\varepsilon(t)/\sigma_0$ is independent of σ_0. Accordingly, the response function called the *creep function* $J(t)$ and defined by

$$J(t) \equiv \varepsilon(t)/\sigma_0, \qquad t \geq 0 \tag{1.2-1}$$

characterizes the properties of the solid for the particular mode of deformation and temperature of the experiment. Equation (1.2-1) may be regarded as a generalization of (1.1-2) since for the ideal elastic case $J(t)$ becomes just the constant J. The initial value of $J(t)$ is called the *unrelaxed compliance* J_U since it is a measure of the deformation that occurs when no time is allowed for relaxation to take place. Thus,

$$J(0) \equiv J_U \tag{1.2-2}$$

The left-hand side of Fig. 1-2 contrasts the creep response of an ideal solid with that of the anelastic solid and the more general linear viscoelastic solid. The contrast between curves (b) and (c) is of special interest. In curve (c), following a transient period, the strain increases linearly with time, representing steady-state viscous creep. On the other hand, in (b) the strain approaches a definite final or equilibrium value after a sufficient amount of time. This behavior is in accordance with the second postulate of anelasticity. The equilibrium value of $J(t)$ attained in the anelastic case will be called the *relaxed compliance* J_R. Thus

$$J(\infty) \equiv J_R \tag{1.2-3}$$

Finally, the quantity δJ, called the *relaxation of the compliance* J, is defined as

$$\delta J \equiv J_R - J_U \tag{1.2-4}$$

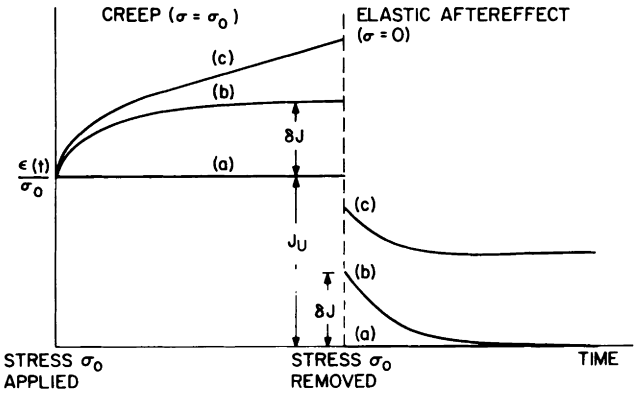

FIG. 1-2. Creep and elastic aftereffect for (a) ideal elastic solid, (b) anelastic solid, and (c) linear viscoelastic solid.

The creep behavior of a material which displays anelasticity, therefore, is such that under unit applied stress the strain increases from an instantaneous value J_U to a final equilibrium value J_R. The time-dependent creep process is also referred to in the literature as "strain relaxation" or "retarded elasticity."

B. ELASTIC AFTEREFFECT (OR CREEP RECOVERY)

If, after a creep experiment has been run for a given time t_1 (not necessarily long enough for attainment of equilibrium), the stress σ_0 is abruptly released, the instantaneous elastic spring-back is in general followed by a time dependent decay of strain. This effect is called the *elastic aftereffect* (in German, *elastische Nachwirkung*) or "creep recovery." In view of the requirement of linearity, the time dependent strain $\varepsilon(t)$ after release of the stress must again be proportional to σ_0. Thus, for the stress history

$$\sigma = \begin{cases} 0, & t < -t_1 \\ \sigma_0, & -t_1 \leq t < 0 \\ 0, & t \geq 0 \end{cases}$$

as illustrated in Fig. 1-3, we define an *aftereffect function* $N_{t_1}(t)$ as

$$N_{t_1}(t) \equiv \varepsilon(t)/\sigma_0, \qquad t \geq 0 \tag{1.2-5}$$

where the subscript t_1 is needed since, in general, the form of the function will depend on the length of time t_1 for which the stress was applied.

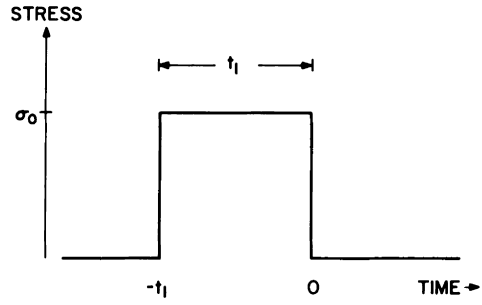

FIG. 1-3. Loading cycle for an elastic aftereffect experiment.

The right-hand side of Fig. 1-2 shows the aftereffect functions for the three cases discussed previously. In the case of the ideal elastic solid there is an immediate return to zero strain and consequently no aftereffect, while for the case of anelasticity, curve (b), total recovery is completed only through the time-dependent aftereffect. By contrast, the linear viscoelastic solid shows only partial recovery; the contribution to the strain due to the steady-state viscous creep is not recoverable.

In view of our interest in relaxation processes and not in linear viscoelasticity in this book, we shall henceforth restrict our attention primarily to anelastic behavior.

C. STRESS RELAXATION

In a stress relaxation experiment a *constant strain* ε_0 is imposed on the specimen at $t = 0$ and maintained for $t \geq 0$ while the stress σ is observed as a function of time. Thus: $\varepsilon = 0$, $t < 0$; $\varepsilon = \varepsilon_0$, $t \geq 0$. By the linearity requirement $\sigma(t)$ will be proportional to ε_0. It is, therefore, convenient to define the *stress relaxation function* $M(t)$ as $\sigma(t)/\varepsilon_0$ (for $t \geq 0$). By analogy to the case of creep, we define the unrelaxed modulus M_U as the ratio $\sigma(0)$ to ε_0, so that

$$M(0) \equiv M_U \tag{1.2-6}$$

From the definition of anelasticity, $M(t)$ must eventually approach a definite equilibrium value, defined as the *relaxed modulus* M_R, i.e.,

$$M(\infty) \equiv M_R \tag{1.2-7}$$

The stress relaxation function for an anelastic material is illustrated in Fig. 1-4.

From the existence of a unique equilibrium relation between stress and strain, it follows that the relaxed modulus is the reciprocal of the relaxed compliance defined by Eq. (1.2-3), i.e.,

$$M_R = 1/J_R \tag{1.2-8}$$

The unrelaxed modulus and compliance are also reciprocals of each other

$$M_U = 1/J_U \tag{1.2-9}$$

This last result follows from the fact that on a short time scale, the material behaves as if it were ideally elastic and therefore Eqs. (1.1-1)–(1.1-3) apply with constants J_U and M_U. Since $J_R > J_U$, M_R must be less than M_U, as shown in Fig. 1-4.

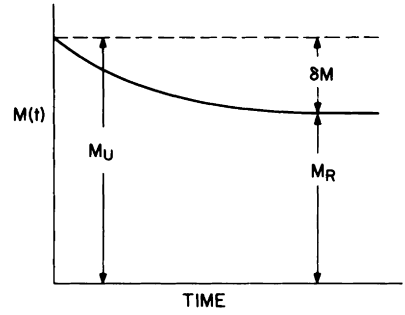

FIG. 1-4. Stress relaxation of an anelastic solid.

It is also useful to define the quantity δM, called *the relaxation of the modulus* M, by

$$\delta M \equiv M_U - M_R \tag{1.2-10}$$

This quantity is also marked on Fig. 1-4. It should be noted that whereas M_U and M_R are, respectively, reciprocals of J_U and J_R, δM is not the reciprocal of δJ; in fact, $\delta M = \delta J/J_U J_R$.

D. NORMALIZED CREEP AND STRESS RELAXATION FUNCTIONS; RELAXATION STRENGTH

In parts A–C of this section we have introduced three quasi-static response functions, $J(t)$, $N_{t_1}(t)$, and $M(t)$. For some purposes it is desirable to introduce a *normalized creep function* $\psi(t)$ defined, for $t \geq 0$, by

$$J(t) = J_U + \delta J\, \psi(t) \tag{1.2-11}$$

It is clear (e.g., from comparison of this equation with Fig. 1-2) that $\psi(t)$ is defined so as to increase monotonically between the extreme values

$$\psi(0) = 0, \qquad \psi(\infty) = 1 \tag{1.2-12}$$

It is also convenient to define a dimensionless quantity Δ, called the *relaxation strength*, by the equation

$$\Delta \equiv \delta J/J_U = \delta M/M_R \tag{1.2-13}$$

The last equality follows from Eqs. (1.2-8)–(1.2-10). Equation (1.2-11) may also be written

$$J(t) = J_U[1 + \Delta \cdot \psi(t)] \tag{1.2-14}$$

In a similar way, we may define a *normalized stress relaxation function* $\varphi(t)$ by the equations

$$M(t) = M_R + \delta M\, \varphi(t) = M_R[1 + \Delta \cdot \varphi(t)] \tag{1.2-15}$$

The function $\varphi(t)$ is a monotonically *decreasing* function between the extreme values

$$\varphi(0) = 1, \qquad \varphi(\infty) = 0 \tag{1.2-16}$$

The following useful relations follow from the definitions of Δ, δJ, and δM:

$$J_R = J_U(1 + \Delta) \tag{1.2-17}$$

$$M_U = M_R(1 + \Delta) \tag{1.2-18}$$

1.3 The Primary Dynamic Response Functions

The quasi-static experiments described in Section 1.2 are used to obtain information on the behavior of materials over periods of several seconds and longer. For information about the behavior of a material at much shorter times, dynamic experiments are more appropriate. In these experiments a stress (or strain) which is periodic in time is imposed on the system, and the phase lag of the strain behind the stress is determined. The behavior of the system is most conveniently described with the aid of complex notation. Let the stress be written as

$$\sigma = \sigma_0 e^{i\omega t} \tag{1.3-1}$$

where σ_0 is the stress amplitude and ω the circular frequency of vibration ($\omega = 2\pi f$, where f is the vibration frequency). The requirement of linearity of the stress–strain relation assures us that the strain is periodic with the same frequency, and therefore expressible in the form

$$\varepsilon = \varepsilon_0 e^{i(\omega t - \phi)} \tag{1.3-2}$$

where ε_0 is the strain amplitude and ϕ is the angle by which the strain lags behind the stress, often called the *loss angle*. Linearity also means that the ratio ε_0/σ_0 is independent of σ_0. Clearly, for ideal elasticity $\phi = 0$, and the ratio ε/σ gives the elastic compliance of the material J. For the anelastic case, however, ϕ is in general not zero, so that the ratio ε/σ is a complex quantity. Let us call this quantity the *complex compliance* $J^*(\omega)$, noting that in general it must be a function of ω. Thus,

$$J^*(\omega) \equiv \varepsilon/\sigma = |J|(\omega)e^{-i\phi(\omega)} \tag{1.3-3}$$

The quantity $|J|(\omega)$, which is the absolute value of J^*, is called the *absolute dynamic compliance*, and is given by

$$|J|(\omega) = \varepsilon_0/\sigma_0 \tag{1.3-4}$$

In terms of this description, two real dynamic response functions of a material have been defined, namely, $|J|(\omega)$ and $\phi(\omega)$. It is also convenient to introduce two other real response functions which are very closely related to $|J|(\omega)$ and $\phi(\omega)$. We first write Eq. (1.3-2) in the alternative form

$$\varepsilon = (\varepsilon_1 - i\varepsilon_2)e^{i\omega t} \tag{1.3-5}$$

where ε_1 is the amplitude of the component of ε in phase with the stress and ε_2 the amplitude of the component which is 90° out of phase. Dividing through by σ, we obtain

$$J^*(\omega) = J_1(\omega) - iJ_2(\omega) \tag{1.3-6}$$

where $J_1(\omega) \equiv \varepsilon_1/\sigma_0$ is the real part of $J^*(\omega)$ (sometimes called the "storage compliance") and $J_2(\omega) \equiv \varepsilon_2/\sigma_0$ is the imaginary part of $J^*(\omega)$ (sometimes called the "loss compliance"). The conventional vector diagram for the complex quantity J^* given in Fig. 1-5 shows the relationships between the various response functions discussed. From this diagram it is clear that the functions $|J|$ and ϕ are related to J_1 and J_2

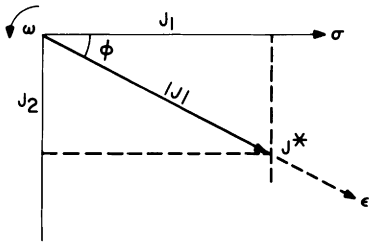

FIG. 1-5. Vector diagram in the complex plane showing the phase relationships between stress, strain, and the complex compliance. The entire diagram rotates about the origin with angular frequency ω.

by the simple equations

$$|J|^2 = J_1^2 + J_2^2 \tag{1.3-7}$$

$$\tan\phi = J_2/J_1 \tag{1.3-8}$$

In a similar way, we could have regarded the periodic strain as given, and the stress as leading the strain by angle ϕ. We then define the *complex modulus* $M^*(\omega)$ by

$$M^*(\omega) \equiv \sigma/\varepsilon = |M|(\omega)e^{i\phi(\omega)} \tag{1.3-9}$$

The absolute value $|M|(\omega)$ is called the *absolute dynamic modulus.* Comparison of (1.3-9) with (1.3-3) shows that the complex compliance is simply the reciprocal of the complex modulus and that $|M|(\omega)$ and $|J|(\omega)$ are also reciprocals. Thus,

$$M^*(\omega) = [J^*(\omega)]^{-1} \quad \text{and} \quad |M|(\omega) = [|J|(\omega)]^{-1} \tag{1.3-10}$$

It is also useful to write $M^*(\omega)$ in the alternate form

$$M^*(\omega) = M_1(\omega) + iM_2(\omega) \tag{1.3-11}$$

where $M_1(\omega)$ and $M_2(\omega)$ are, respectively, the real and imaginary parts of $M^*(\omega)$. A vector diagram similar to Fig. 1-5, or direct comparison of Eqs. (1.3-9) and (1.3-11), gives

$$|M|^2 = M_1^2 + M_2^2 \tag{1.3-12}$$

$$\tan\phi = M_2/M_1 \tag{1.3-13}$$

Comparing these results with Eqs. (1.3-7) and (1.3-8) shows that $M_2/M_1 = J_2/J_1$; also

$$J_1 = M_1/|M|^2 = [M_1(1 + \tan^2\phi)]^{-1} \tag{1.3-14}$$

and

$$J_2 = M_2/|M|^2 \quad \text{or} \quad M_2 = J_2/|J|^2 \tag{1.3-15}$$

Thus, while the complex quantities $J^*(\omega)$ and $M^*(\omega)$ are reciprocals of one another, J_1 is not the reciprocal of M_1, nor is J_2 the reciprocal of M_2.

A very useful approximation results under the assumption that $\phi^2 \ll 1$. Since $\tan\phi = \phi(1 + \frac{1}{3}\phi^2 + \cdots)$ we find that under this assumption $\tan\phi \doteq \phi$. (The symbol $\doteq$ will be used here to signify equality to within terms of the order ϕ^2.) For $\phi = 0.2$, for example, the use of ϕ in place of $\tan\phi$ amounts to an error of about 1%. In view of the fact that $\phi = 0.2$ represents a much higher value of ϕ than is commonly encountered in the anelasticity of crystals, the above approximation can generally be used without significant error. To within this approximation $M_1 \doteq |M|$, $J_1 \doteq |J|$, and from (1.3-14),

$$M_1 \doteq J_1^{-1} \tag{1.3-14a}$$

i.e., the quantities M_1 and J_1 are reciprocals of each other to within terms of order ϕ^2.

At sufficiently low frequencies, the strain will be proportional to the stress, with the relaxed compliance as proportionality constant, so that

$$J^*(0) = 1/M^*(0) = J_R \tag{1.3-16}$$

Conversely, at very high frequencies,

$$J^*(\infty) = 1/M^*(\infty) = J_U \tag{1.3-17}$$

The analogy with Eqs. (1.2-2), (1.2-3), (1.2-6), and (1.2-7) should be noted. Equations (1.3-16) and (1.3-17) state that the complex compliance at the two extremes of frequency is real, and equal to the relaxed and unrelaxed compliance, respectively. It follows that

$$J_2(0) = J_2(\infty) = 0 \tag{1.3-18}$$

and similarly for $M_2(\omega)$.

The significance of the quantities J_1 and J_2 as "storage compliance" and "loss compliance," respectively, is illustrated by calculating the energy stored and the energy dissipated in a cycle of vibration. The energy per unit volume at any phase in the cycle is $\int \sigma\, d\varepsilon$, taken between the start of the cycle up to the point of interest. Thus, the energy ΔW dissipated in a full cycle, per unit of volume, is easily shown (see Problem 1-3) to be

$$\Delta W = \oint \sigma\, d\varepsilon = \pi J_2 \sigma_0^2 \tag{1.3-19}$$

On the other hand, the maximum stored energy W per unit volume is given by

$$W = \int_{\omega t=0}^{\pi/2} \sigma \, d\varepsilon = \tfrac{1}{2} J_1 \sigma_0^2 \tag{1.3-20}$$

The ratio of the energy dissipated to the maximum stored energy (when expressed as a percentage) is often called the "specific damping capacity." From Eqs. (1.3-19), (1.3-20), and (1.3-8), it is clear that this ratio is related to the loss angle ϕ by

$$\Delta W/W = 2\pi(J_2/J_1) = 2\pi \tan \phi \tag{1.3-21}$$

In a similar way, taking $\varepsilon = \varepsilon_0 \cos \omega t$ and $\sigma = M^* \varepsilon$ with M^* given by Eq. (1.3-11) we can show that

$$\Delta W = \pi M_2 \varepsilon_0^2 \tag{1.3-22}$$

$$W = \tfrac{1}{2} M_1 \varepsilon_0^2 \tag{1.3-23}$$

and again arrive at the result $\Delta W/W = 2\pi \tan \phi$.

Because of the fact that ϕ (or $\tan \phi$) gives a measure of the fractional energy loss per cycle due to anelastic behavior, the quantity ϕ is commonly known as the *internal friction* of the material.

In summary, the above description of the dynamic behavior of anelastic solids leaves us with a choice of several pairs of dynamic response functions that may be used to designate the properties of the material. These are $|J|$ and ϕ, J_1 and J_2, $|M|$ and ϕ, or M_1 and M_2, each of which is a function of the frequency ω. Each of the four pairs of response functions may be converted to another pair by means of equations given above. Relationships between the members of each pair, e.g., between $J_1(\omega)$ and $J_2(\omega)$, have not appeared as a result of these considerations. To obtain such relations requires a further development of the theory involving the Boltzmann superposition principle, which will be discussed in Chapter 2.

1.4 Additional Dynamic Response Functions

The dynamic response functions considered in Section 1.3 can only be measured directly in an experiment carried out at frequencies well below any resonances of the mechanical system used. Such an experiment, which will be referred to as a "subresonant experiment," is very simple to carry

out, in principle.† It is merely necessary to set a specimen into forced vibration at a frequency ω and to measure the amplitudes of stress and strain and their relative phases to obtain $\phi(\omega)$ and $|J|(\omega)$ [or $J_1(\omega)$ and $J_2(\omega)$]. These dynamic functions are theoretically the most useful ones, and in fact are all that are needed for the development of the formal theory of anelasticity in Chapters 2–6. In practice, however, the measurement of the phase angle $\phi(\omega)$ is difficult when it is very small, which is usually the case for crystalline materials. Accordingly, subresonant methods are not generally used. Instead, anelastic materials are usually tested at frequencies where the inertia of the system is appreciable. These methods are conveniently divided into two types: (a) methods employing resonant systems vibrating at a natural frequency, either in forced vibration or in free decay, and (b) wave propagation methods. These two types of methods will be discussed from an experimental point of view in Chapter 20. In the remainder of the present chapter our objective will be to study the theory of these methods in order to obtain the relationships between the response functions derived from them and the primary response functions of Section 1.3. Resonant systems will be dealt with in Section 1.5 and Appendices A and F, while wave propagation methods are discussed in Section 1.6.

1.5 Resonant Systems with Large External Inertia

In general, a resonant system must have two elements: the "elastic" element (which in fact may be anelastic), and the inertia. The situation is simplified considerably when a rigid inertia member which is large compared to the inertia of the specimen is added. Such a system has only one degree of freedom, since the motion of the system can be described completely in terms of a single coordinate. An example is a wire sample gripped at the top, and having a large weight hanging freely at the bottom; this system can be set either into longitudinal or into torsional oscillation. The latter case represents the well-known "torsion pendulum," in which the strain at any point can be expressed in terms

† A mechanical system shows resonance-type behavior if it contains inertial components as well as elastic ones, as, e.g., in the case of a mass on an ideal spring. Near resonance, the inertial and elastic forces are of comparable magnitude. However, at suitably low subresonant frequencies, the inertial force is negligible with respect to the elastic force. For an anelastic material, the subresonant response is therefore determined only by $J^*(\omega)$.

of a single displacement parameter, usually chosen to be the angular twist of the inertia member. We shall designate the parameter which describes the displacement of the inertia member by a coordinate x, and the "force" acting on the specimen by F_s. The quantity F_s is the force variable conjugate to x (e.g., the torque acting on the specimen in the case of the torsion pendulum). The quantities F_s and x are, respectively, proportional to the stress and the strain at some chosen point in the sample. If we choose the point of maximum stress σ_{max} and maximum strain ε_{max} we may write

$$F_s = C_1\sigma_{max}, \qquad x = C_2\varepsilon_{max} \tag{1.5-1}$$

where the proportionality constants C_1 and C_2 involve the shape and dimensions of the sample (see Problem 1-4).

The model that may be used to represent the resonant system with one degree of freedom is that of a mass on a spring as shown in Fig. 1-6.

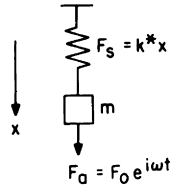

FIG. 1-6. Mechanical model for a resonant system with one degree of freedom, incorporating an anelastic spring.

When the specimen material is perfectly elastic, the force F_s that *the mass exerts on the spring* may be written $F_s = kx$, where k is the spring constant. On the other hand, if we wish to allow for an anelastic specimen such that $\sigma = M^*\varepsilon$ (where M^* is in general a complex quantity), we must, in view of Eq. (1.5-1), allow for a *complex spring constant* k^* in the present model. Accordingly, we have

$$F_s = k^*x = k_1(1 + i\tan\phi)x \tag{1.5-2}$$

The quantity k^* is proportional to M^* where, in fact, the proportionality constant is equal to C_1/C_2, the ratio of the two constants which appear in Eq. (1.5-1). We now consider two ways in which the response of this system may be studied.[†]

[†] It should be remembered in what follows that the terminology "force," "mass," and "displacement" is symbolic and would actually become "torque," "moment of inertia," and "angle of twist" in the case of the torsion pendulum.

A. Forced Vibrations of the System

In a forced-vibration experiment a periodic force F_a is applied to the mass m as shown in Fig. 1-6. We may write, in complex notation,

$$F_a = F_0 e^{i\omega t} \tag{1.5-3}$$

The equation of motion of the system shown in Fig. 1-6 is

$$m\ddot{x} = F_a - F_s \tag{1.5-4}$$

and the steady-state solution may be anticipated to be of the form

$$x = x_0^* e^{i\omega t} = x_0 e^{i(\omega t - \theta)} \tag{1.5-5}$$

where θ is the phase angle by which x lags behind F_a (not to be confused with the loss angle ϕ by which x lags behind F_s).

It is then easy to show that

$$x_0^2 = |\, x_0^* \,|^2 = \frac{(F_0/m)^2}{(\omega_r^2 - \omega^2)^2 + \omega_r^4 \tan^2 \phi} \tag{1.5-6}$$

where

$$\omega_r \equiv (k_1/m)^{1/2} \tag{1.5-7}$$

Equation (1.5-6) gives the dependence of the displacement amplitude x_0 on the frequency ω, showing that x_0 goes through a maximum at $\omega = \omega_r$ and falls to small values for $\omega \gg \omega_r$ and $\omega \ll \omega_r$. This is the behavior known as *resonance*, and ω_r is the *resonant frequency*. In general, since k_1 is a function of frequency, ω_r also depends on ω. The reader should also show (Problem 1-5) that the phase angle θ is given by

$$\tan \theta = \frac{\omega_r^2 \tan \phi}{\omega_r^2 - \omega^2} \tag{1.5-8}$$

Equation (1.5-8) shows that θ changes rapidly in passing through resonance, from the value $\theta = \phi$ for $\omega \ll \omega_r$, to a value of $\theta = \pi$ for $\omega \gg \omega_r$, passing through $\theta = \pi/2$ for $\omega = \omega_r$. The resonance equation can be simplified further if $\phi \ll 1$ (a more stringent assumption than the earlier approximation that $\phi^2 \ll 1$). In this case, the entire resonance peak is localized near $\omega = \omega_r$. We may then substitute: $\omega_r + \omega \simeq 2\omega_r$ or $\omega_r^2 - \omega^2 \simeq (\omega_r - \omega)2\omega_r$; also $\phi(\omega) \simeq \phi(\omega_r)$ and $k_1(\omega) \simeq k_1(\omega_r)$ may

both be considered as constants. Equation (1.5-6) then becomes

$$x_0^2 \simeq \frac{(F_0/m)^2\omega_r^{-2}}{4(\omega - \omega_r)^2 + \omega_r^2[\phi(\omega_r)]^2}. \tag{1.5-9}$$

This dependence on frequency is of the form

$$x_0^2 \propto [(\omega - \omega_r)^2 + \omega_r^2\alpha^2]^{-1}$$

where α is a constant. This equation represents the well-known "Lorentzian" curve plotted in Fig. 1-7. It is easy to show that if ω_1 and ω_2 are the two values of frequency at which x_0^2 falls to half maximum value (or x_0 to $2^{-1/2}$ of its maximum), then

$$(\omega_2 - \omega_1)/\omega_r \equiv \mathcal{Q}^{-1} = \phi \tag{1.5-10}$$

where this definition of $\mathcal{Q}$ corresponds to the one commonly used in the description of resonant electrical circuits. Equation (1.5-10) shows that the loss angle ϕ, which is a measure of the internal friction of the system, is obtainable directly from the width of the resonance peak at half-maximum in a plot of (amplitude)^2 versus frequency (or at $2^{-1/2}$ of maximum when amplitude is plotted versus frequency). In passing, it may be noted that the quantity $\mathcal{Q}$, defined above in terms of the sharpness of the resonance peak, is actually the magnification factor of the resonant system (see Problem 1-6).

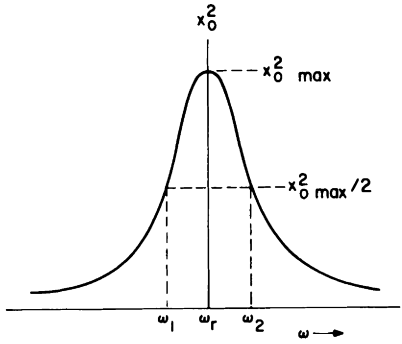

FIG. 1-7. Lorentzian form for the resonance peak in forced vibration.

It should be remembered that Eq. (1.5-10) is valid only for $\phi \ll 1$, which means $(\omega_2 - \omega_1) \ll \omega_r$ (sharp resonance peak). If first we consider the effect of letting ϕ become larger but keeping ϕ and ω_r independent of ω, examination of Eq. (1.5-6) shows that the resonance