GLOBAL EDITION

Finite Element Analysis

Theory and Application with ANSYS

FOURTH EDITION

Saeed Moaveni

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FINITE ELEMENT ANALYSIS

FINITE ELEMENT ANALYSIS

Theory and Application with ANSYS

Fourth Edition

Global Edition

Saeed Moaveni

Minnesota State University, Mankato

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To memories of my mother and father

Contents

Preface

CHANGES IN THE FOURTH EDITION

The fourth edition, consisting of 15 chapters, includes a number of new additions and changes that were incorporated in response to ANSYS revisions and suggestions and requests made by professors, students, and professionals using the third edition of the book. The major changes include:

- Explanation of the changes that were made in the ANSYS's newest release (Chapters 3 and 8)
- Explanation of new element type capabilities (Chapters 3, 4, 6, 8 through 13, and 15)
- A new comprehensive example problem that demonstrates the use of BEAM188 element in modeling beam and frame problems (Chapter 4)
- Modification of twenty example problems to incorporate new ANSYS element types (Chapters 3, 4, 6, 8 through 13, and 15)
- Eight new comprehensive example problems that show in great detail how to use Excel to solve different types of finite element problems (Chapters 2 through 6 and 9 through 12)
- More detail on theory and expanded derivations
- Explanation of new MATLAB revisions in Appendix F

ORGANIZATION

There are many good textbooks already in existence that cover the theory of finite element methods for advanced students. However, none of these books incorporate ANSYS as an integral part of their materials to introduce finite element modeling to undergraduate students and newcomers. In recent years, the use of finite element analysis (FEA) as a design tool has grown rapidly. Easy-to-use, comprehensive packages such as ANSYS, a general-purpose finite element computer program, have become common tools in the hands of design engineers. Unfortunately, many engineers who lack the proper training or understanding of the underlying concepts have been using these tools. This introductory book is written to assist engineering students and practicing engineers new to the field of finite element modeling to gain a clear understanding of the basic concepts. The text offers insight into the theoretical aspects of FEA and also covers some practical aspects of modeling. Great care has been exercised to avoid overwhelming students with theory, yet enough theoretical background is offered to allow individuals to use ANSYS intelligently and effectively. ANSYS is an integral part of this text. In each chapter, the relevant basic theory is discussed first and demonstrated using simple problems with hand calculations. These problems are followed by examples that are solved using ANSYS. Exercises in the text are also presented in this manner. Some exercises require manual calculations, while others, more complex in nature, require the use of ANSYS. The simpler hand-calculation problems will enhance students' understanding of the concepts by encouraging them to go through the necessary steps in a FEA. Design problems are also included at the end of Chapters 3, 4, 6, and 9 through 14.

Various sources of error that can contribute to incorrect results are discussed. A good engineer must always find ways to check the results. While experimental testing of models may be the best way, such testing may be expensive or time consuming. Therefore, whenever possible, throughout this text emphasis is placed on doing a "sanity check" to verify one's FEA. A section at the end of each appropriate chapter is devoted to possible approaches for verifying ANSYS results.

Another unique feature of this book is that the last two chapters are devoted to the introduction of design, material selection, optimization, and parametric programming with ANSYS.

The book is organized into 15 chapters. Chapter 1 reviews basic ideas in finite element analysis. Common formulations, such as direct, potential energy, and weighted residual methods, are discussed. Chapter 2 provides a comprehensive review of matrix algebra. Chapter 3 deals with the analysis of trusses, because trusses offer economical solutions to many engineering structural problems. An overview of the ANSYS program is given in Chapter 3 so that students can begin to use ANSYS right away. Finite element formulation of members under axial loading, beams, and frames are introduced in Chapter 4. Chapter 5 lays the foundation for analysis of one-dimensional problems by introducing one-dimensional linear, quadratic, and cubic elements. Global, local, and natural coordinate systems are also discussed in detail in Chapter 5. An introduction to isoparametric formulation and numerical integration by Gauss–Legendre formulae is also presented in Chapter 5. Chapter 6 considers Galerkin formulation of one-dimensional heat transfer and fluid problems. Two-dimensional linear and higher order elements are introduced in Chapter 7. Gauss–Legendre formulae for two-dimensional integrals are also presented in Chapter 7. In Chapter 8 the essential capabilities and the organization of the ANSYS program are covered. The basic steps in creating and analyzing a model with ANSYS is discussed in detail. Chapter 9 includes the analysis of two-dimensional heat transfer problems with a section devoted to unsteady situations. Chapter 10 provides an analysis of torsion of noncircular shafts and plane stress problems. Dynamic problems are explored in Chapter 11. Review of dynamics and vibrations of mechanical and structural systems are also given in this chapter. In Chapter 12, two-dimensional, ideal fluid-mechanics problems are analyzed. Direct formulation of the piping network problems and underground seepage flow are also discussed. Chapter 13 provides a discussion on three-dimensional elements and formulations. This chapter also presents basic ideas regarding top-down and bottom-up solid modeling methods. The last two chapters of the book are devoted to design and optimization ideas. Design process and material selection are explained in Chapter 14. Design optimization ideas and parametric programming are discussed in Chapter 15. Examples of ANSYS batch files are also given in Chapter 15. Each chapter begins by stating the objectives and concludes by summarizing what the reader should have gained from studying that chapter.

The examples that are solved using ANSYS show in great detail how to use ANSYS to model and analyze a variety of engineering problems. Chapter 8 is also written such that it can be taught right away if the instructor sees the need to start with ANSYS.

A brief review of appropriate fundamental principles in solid mechanics, heat transfer, dynamics, and fluid mechanics is also provided throughout the book. Additionally, when appropriate, students are warned about becoming too quick to generate finite element models for problems for which there exist simple analytical solutions. Mechanical and thermophysical properties of some common materials used in engineering are given in Appendices A and B. Appendices C and D give properties of common area shapes and properties of structural steel shapes, respectively. A comprehensive introduction to MATLAB is given in Appendix F.

Finally, a Web site at *http://www.pearsonglobaleditions.com/moaveni* will be maintained for the following purposes: (1) to share any changes in the upcoming versions of ANSYS; (2) to share additional information on upcoming text revisions; (3) to provide additional homework problems and design problems; and (4) although I have done my best to eliminate errors and mistakes, as is with most books, some errors may still exist. I will post the corrections that are brought to my attention at the site. The Web site will be accessible to all instructors and students.

Thank you for considering this book and I hope you enjoy the fourth edition.

Saeed Moaveni

Acknowledgments

I would like to express my sincere gratitude to ANSYS, Inc. for providing the photographs that appear on page 27 of this book. Descriptions for these photographs are given in Chapter 1 with the images. I would also like to thank ANSYS, Inc. for giving me permission to adapt material from various ANSYS documents, related to capabilities and the organization of ANSYS. The essential capabilities and organizations of ANSYS are covered in Chapters 3, 8, 13, and 15.

As I have mentioned in the Preface, there are many good published books in finite element analysis. When writing this book, several of these books were consulted. They are cited at the end of each appropriate chapter. The reader can benefit from referring to these books and articles.

I am also thankful to all reviewers who offered general and specific comments.

GLOBAL EDITION

Pearson would like to thank and acknowledge the following people for their work on the Global Edition:

CONTRIBUTOR

Jagadeesha T, National Institute of Technology—Calicut

REVIEWERS

Swarup Bag, Indian Institute of Technology—Guwahati
Avinash Parashar, Indian Institute of Technology—Roorkee
Mokhtar B Awang, Universiti Teknologi Petronas—Malaysia
Yash Parikh, Symbiosis Institute of Technology—Pune

FINITE ELEMENT ANALYSIS

CHAPTER 1

Introduction

The finite element method is a numerical procedure that can be used to obtain solutions to a large class of engineering problems involving stress analysis, heat transfer, electromagnetism, and fluid flow. This book was written to help you gain a clear understanding of the fundamental concepts of finite element modeling. Having a clear understanding of the basic concepts will enable you to use a general-purpose finite element software, such as ANSYS, effectively. ANSYS is an integral part of this text. In each chapter, the relevant basic theory behind each respective concept is discussed first. This discussion is followed by examples that are solved using ANSYS. Throughout this text, emphasis is placed on methods by which you may verify your findings from finite element analysis (FEA). In addition, at the end of particular chapters, a section is devoted to the approaches you should consider to verify results generated by using ANSYS.

Some of the exercises provided in this text require manual calculations. The purpose of these exercises is to enhance your understanding of the concepts by encouraging you to go through the necessary steps of FEA. This book can also serve as a reference text for readers who may already be design engineers who are beginning to get involved in finite element modeling and need to know the underlying concepts of FEA.

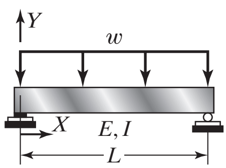
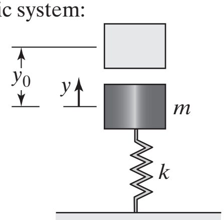
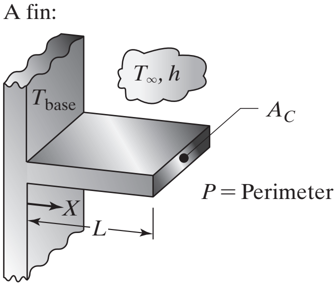
The objective of this chapter is to introduce you to basic concepts in finite element formulation, including direct formulation, the minimum potential energy theorem, and the weighted residual methods. The main topics of Chapter 1 include the following:

1.1 Engineering Problems
1.2 Numerical Methods
1.3 A Brief History of the Finite Element Method and ANSYS
1.4 Basic Steps in the Finite Element Method
1.5 Direct Formulation
1.6 Minimum Total Potential Energy Formulation
1.7 Weighted Residual Formulations
1.8 Verification of Results
1.9 Understanding the Problem

1.1 ENGINEERING PROBLEMS

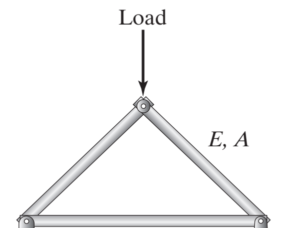
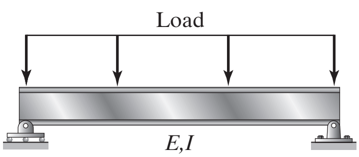
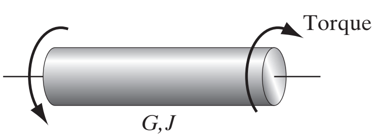
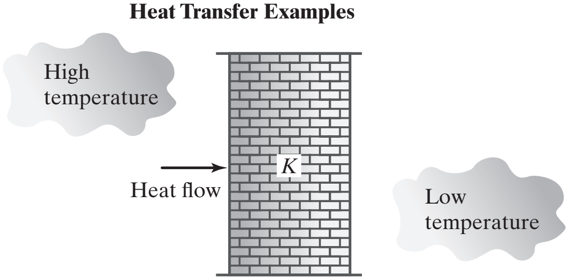
In general, engineering problems are mathematical models of physical situations. Mathematical models of many engineering problems are differential equations with a set of corresponding boundary and/or initial conditions. The differential equations are derived by applying the fundamental laws and principles of nature to a system or a control volume. These governing equations represent balance of mass, force, or energy. When possible, the exact solution of these equations renders detailed behavior of a system under a given set of conditions, as shown by some examples in Table 1.1. The analytical solutions are composed of two parts: (1) a homogenous part and (2) a particular part. In any given engineering problem, there are two sets of design parameters that influence the way in which a system behaves. First, there are those parameters that

TABLE 1.1 Examples of governing differential equations, boundary conditions, initial conditions, and exact solutions for some engineering problems

Problem Type	Governing Equation, Boundary Conditions, or Initial Conditions	Solution
A beam: 	$EI\frac{d^2Y}{dX^2} = \frac{wX(L - X)}{2}$ Boundary conditions: at $X = 0$, $Y = 0$ and at $X = L$, $Y = 0$	Deflection of the beam Y as the function of distance X: $Y = \frac{w}{24EI}(-X^4 + 2LX^3 - L^3X)$
An elastic system: 	$\frac{d^2y}{dt^2} + \omega_n^2 y = 0$ where $\omega_n^2 = \frac{k}{m}$ Initial conditions: at time $t = 0$, $y = y_0$ and at time $t = 0$, $\frac{dy}{dt} = 0$	The position of the mass y as the function of time: $y(t) = y_0 \cos \omega_n t$
A fin: 	$\frac{d^2T}{dX^2} - \frac{hp}{kA_c}(T - T_\infty) = 0$ Boundary conditions: at $X = 0$, $T = T_{base}$ as $L \rightarrow \infty$, $T = T_\infty$	Temperature distribution along the fin as the function of X: $T = T_\infty + (T_{base} - T_\infty)e^{-\sqrt{\frac{hp}{kA_c}}X}$

provide information regarding the *natural behavior* of a given system. These parameters include material and geometric properties such as modulus of elasticity, thermal conductivity, viscosity, and area, and second moment of area. Table 1.2 summarizes the physical properties that define the natural characteristics of various problems.

TABLE 1.2 Physical properties characterizing various engineering systems

Problem Type	Examples of Parameters That Characterize a System
Solid Mechanics Examples	
 A truss	Modulus of elasticity, E; member length, L; cross-sectional area, A
 An elastic plate	Modulus of elasticity, E; length, L; cross-sectional area, A
 A beam	Modulus of elasticity, E; member length, L; second moment of area, I
 A shaft	Modulus of rigidity, G; member length, L; polar moment of inertia of the area, J
Heat Transfer Examples	
 A wall	Thermal conductivity, K; thickness, L; area, A

continued

TABLE 1.2 *Continued*

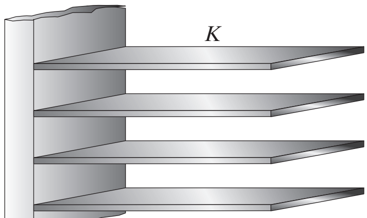
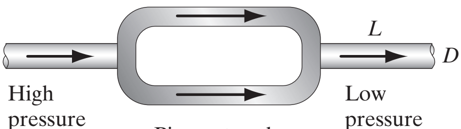
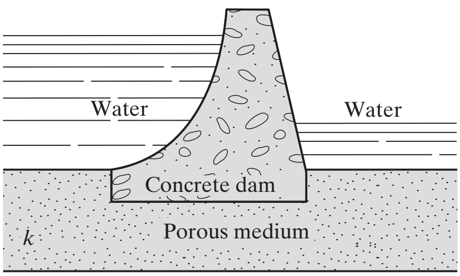
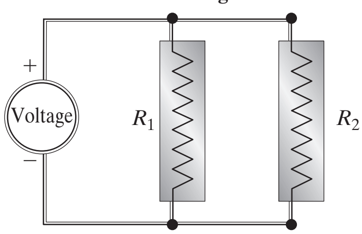
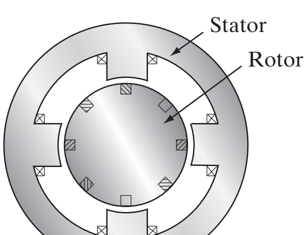
Problem Type	Examples of Parameters That Characterize a System
 Fins	Thermal conductivity, K; perimeter, P; cross-sectional area, A
Fluid Flow Examples  Pipe networks	Fluid viscosity, μ; pipe roughness, e; pipe diameter, D; pipe length, L
 A concrete dam	Soil permeability, k
Electrical and Magnetism Problems  Electrical network	Resistance, R
 Magnetic field of an electric motor	Permeability, μ

TABLE 1.3 Parameters causing disturbances in various engineering systems

Problem Type	Examples of Parameters that Produce Disturbances in a System
Solid Mechanics	External forces and moments; support excitation
Heat Transfer	Temperature difference; heat input
Fluid Flow and Pipe Networks	Pressure difference; rate of flow
Electrical Network	Voltage difference

On the other hand, there are parameters that produce *disturbances* in a system. These types of parameters are summarized in Table 1.3. Examples of these parameters include external forces, moments, temperature difference across a medium, and pressure difference in fluid flow.

The system characteristics as shown in Table 1.2 dictate the natural behavior of a system, and they always appear in the *homogenous part of the solution* of a governing differential equation. In contrast, the parameters that cause the disturbances appear in the *particular solution*. It is important to understand the role of these parameters in finite element modeling in terms of their respective appearances in stiffness or conductance matrices and load or forcing matrices. The system characteristics will always show up in the stiffness matrix, conductance matrix, or resistance matrix, whereas the disturbance parameters will always appear in the load matrix. We will explain the concepts of stiffness, conductance, and load matrices in Section 1.5.

1.2 NUMERICAL METHODS

There are many practical engineering problems for which we cannot obtain exact solutions. This inability to obtain an exact solution may be attributed to either the complex nature of governing differential equations or the difficulties that arise from dealing with the boundary and initial conditions. To deal with such problems, we resort to numerical approximations. In contrast to analytical solutions, which show the exact behavior of a system at any point within the system, numerical solutions approximate exact solutions only at discrete points, called nodes. The first step of any numerical procedure is discretization. This process divides the medium of interest into a number of small subregions (elements) and nodes. There are two common classes of numerical methods: (1) *finite difference methods* and (2) *finite element methods*. With finite difference methods, the differential equation is written for each node, and the derivatives are replaced by *difference equations*. This approach results in a set of simultaneous linear equations. Although finite difference methods are easy to understand and employ in simple problems, they become difficult to apply to problems with complex geometries or complex boundary conditions. This situation is also true for problems with nonisotropic material properties.

In contrast, the finite element method uses *integral formulations* rather than difference equations to create a system of algebraic equations. Moreover, a continuous function is assumed to represent the approximate solution for each element. The complete solution is then generated by connecting or assembling the individual solutions, allowing for continuity at the interelemental boundaries.

1.3 A BRIEF HISTORY* OF THE FINITE ELEMENT METHOD AND ANSYS

The finite element method is a numerical procedure that can be applied to obtain solutions to a variety of problems in engineering. Steady, transient, linear, or nonlinear problems in stress analysis, heat transfer, fluid flow, and electromagnetism problems may be analyzed with finite element methods. The origin of the modern finite element method may be traced back to the early 1900s when some investigators approximated and modeled elastic continua using discrete equivalent elastic bars. However, Courant (1943) has been credited with being the first person to develop the finite element method. In a paper published in the early 1940s, Courant used piecewise polynomial interpolation over triangular subregions to investigate torsion problems.

The next significant step in the utilization of finite element methods was taken by Boeing in the 1950s when Boeing, followed by others, used triangular stress elements to model airplane wings. Yet, it was not until 1960 that Clough made the term *finite element* popular. During the 1960s, investigators began to apply the finite element method to other areas of engineering, such as heat transfer and seepage flow problems. Zienkiewicz and Cheung (1967) wrote the first book entirely devoted to the finite element method in 1967. In 1971, ANSYS was released for the first time.

ANSYS is a comprehensive general-purpose finite element computer program that contains more than 100,000 lines of code. ANSYS is capable of performing static, dynamic, heat transfer, fluid flow, and electromagnetism analyses. ANSYS has been a leading FEA program for over 40 years. The current version of ANSYS has a completely new look, with multiple windows incorporating a graphical user interface (GUI), pull-down menus, dialog boxes, and a tool bar. Today, you will find ANSYS in use in many engineering fields, including aerospace, automotive, electronics, and nuclear. In order to use ANSYS or any other "canned" FEA computer program intelligently, it is imperative that one first fully understands the underlying basic concepts and limitations of the finite element methods.

ANSYS is a very powerful and impressive engineering tool that may be used to solve a variety of problems (see Table 1.4). However, a user without a basic understanding of the finite element methods will find himself or herself in the same predicament as a computer technician with access to many impressive instruments and tools, but who cannot fix a computer because he or she does not understand the inner workings of a computer!

1.4 BASIC STEPS IN THE FINITE ELEMENT METHOD

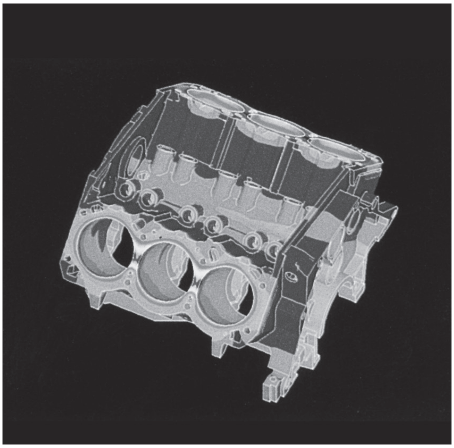
The basic steps involved in any finite element analysis consist of the following:

Preprocessing Phase

1. Create and discretize the solution domain into finite elements; that is, subdivide the problem into nodes and elements.

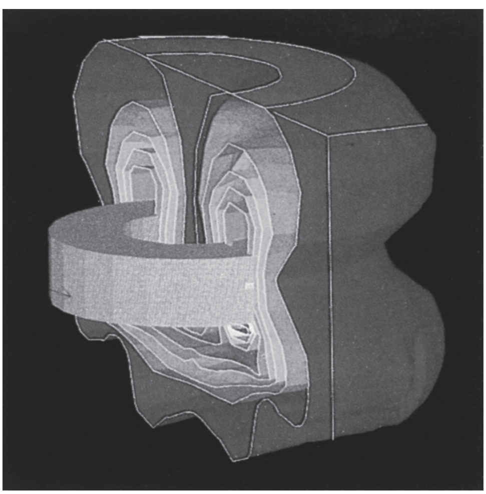
*See Cook et al. (1989) for more detail.

TABLE 1.4 Examples of the capabilities of ANSYS*

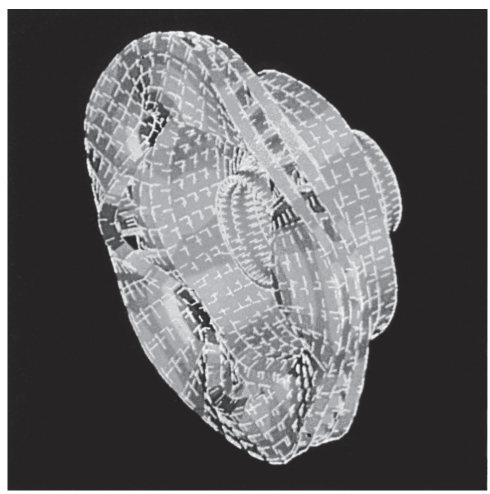


A V6 engine used in front-wheel-drive automobiles analyses were conducted by Analysis & Design Appl. Co. Ltd. (ADAPCO) on behalf of a major U.S. automobile manufacturer to improve product performance. Contours of thermal stress in the engine block are shown in the figure above.

Large deflection capabilities of ANSYS were utilized by engineers at Today's Kids, a toy manufacturer, to confirm failure locations on the company's play slide, shown in the figure above, when the slide is subjected to overload. This nonlinear analysis capability is required to detect these stresses because of the product's structural behavior.



Electromagnetic capabilities of ANSYS, which include the use of both vector and scalar potentials interfaced through a specialized element, as well as a three-dimensional graphics representation of far-field decay through infinite boundary elements, are depicted in this analysis of a bath plate, shown in the figure above. Isocontours are used to depict the intensity of the H-field.



Structural Analysis Engineering Corporation used ANSYS to determine the natural frequency of a rotor in a disk-brake assembly. In this analysis, 50 modes of vibration, which are considered to contribute to brake squeal, were found to exist in the light-truck brake rotor.

*Photographs courtesy of ANSYS, Inc., Canonsburg, PA.

2. Assume a shape function to represent the physical behavior of an element; that is, a continuous function is assumed to represent the approximate behavior (solution) of an element.
3. Develop equations for an element.
4. Assemble the elements to present the entire problem. Construct the global stiffness matrix.
5. Apply boundary conditions, initial conditions, and loading.

Solution Phase

6. Solve a set of linear or nonlinear algebraic equations simultaneously to obtain nodal results, such as displacement values at different nodes or temperature values at different nodes in a heat transfer problem.

Postprocessing Phase

7. Obtain other important information. At this point, you may be interested in values of principal stresses, heat fluxes, and so on.

In general, there are several approaches to formulating finite element problems: (1) *direct formulation,* (2) *the minimum total potential energy formulation,* and (3) *weighted residual formulations*. Again, it is important to note that the basic steps involved in any finite element analysis, regardless of how we generate the finite element model, will be the same as those listed above.

1.5 DIRECT FORMULATION

The following problem illustrates the steps and the procedure involved in direct formulation.

EXAMPLE 1.1

Consider a bar with a variable cross section supporting a load P, as shown in Figure 1.1. The bar is fixed at one end and carries the load P at the other end. Let us designate the width of the bar at the top by w_1, at the bottom by w_2, its thickness by t, and its length by L. The bar's modulus of elasticity will be denoted by E. We are interested in approximating how much the bar will deflect at various points along its length when it is subjected to the load P. We will neglect the weight of the bar in the following analysis, assuming that the applied load is considerably larger than the weight of the bar:

Preprocessing Phase

1. *Discretize the solution domain into finite elements.*
   We begin by subdividing the problem into nodes and elements. In order to highlight the basic steps in a finite element analysis, we will keep this problem simple and thus represent it by a model that has five nodes and four elements, as shown in Figure 1.2. However, note that we can increase the accuracy of our results by generating a model with additional nodes and elements. This task is left as an exercise for you to complete. (See Problem 1 at the end of this chapter.) The given bar

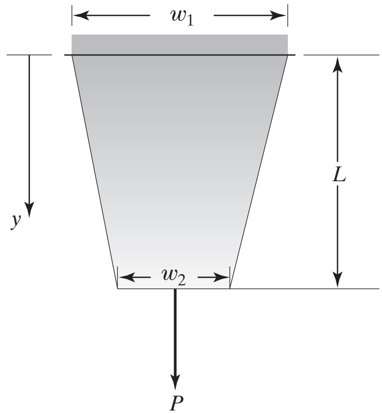


FIGURE 1.1 A bar under axial loading.

is modeled using four individual segments (elements), with each segment having a uniform cross section. The cross-sectional area of each element is represented by an average area of the cross sections at the nodes that make up (define) the element. This model is shown in Figure 1.2.

2. *Assume a solution that approximates the behavior of an element.*
In order to study the behavior of a typical element, let's consider the deflection of a solid member with a uniform cross section A that has a length ℓ when subjected to a force F, as shown in Figure 1.3.

The average stress σ in the member is given by

$$\sigma = \frac{F}{A} \tag{1.1}$$

The average normal strain ε of the member is defined as the change in length $\Delta\ell$ per unit original length ℓ of the member:

$$\varepsilon = \frac{\Delta\ell}{\ell} \tag{1.2}$$

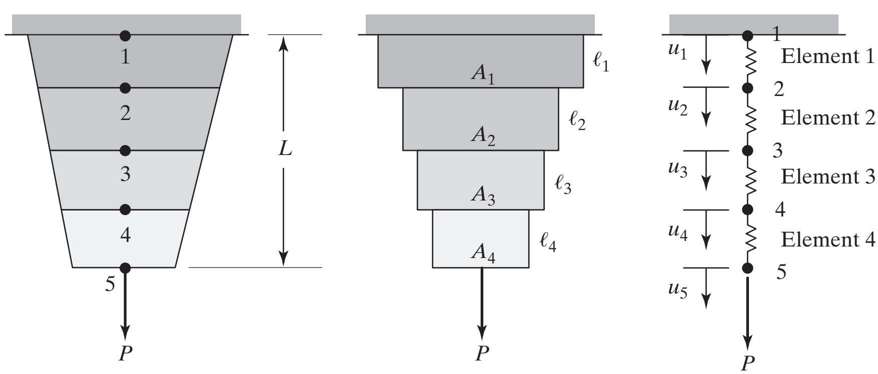


FIGURE 1.2 Subdividing the bar into elements and nodes.

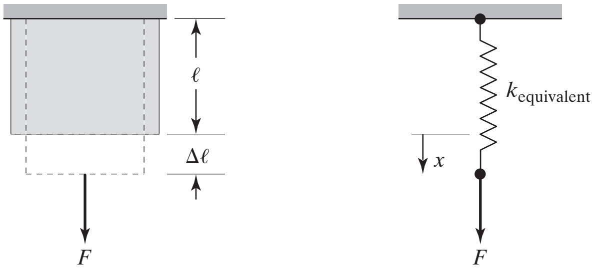


FIGURE 1.3 A solid member of uniform cross section subjected to a force F.

Over the elastic region, the stress and strain are related by Hooke's law, according to the equation

$$\sigma = E\varepsilon \tag{1.3}$$

where E is the modulus of elasticity of the material. Combining Eqs. (1.1), (1.2), and (1.3) and simplifying, we have

$$F = \left(\frac{AE}{\ell}\right)\Delta\ell \tag{1.4}$$

Note that Eq. (1.4) is similar to the equation for a linear spring, $F = kx$. Therefore, a centrally loaded member of uniform cross section may be modeled as a spring with an equivalent stiffness of

$$k_{eq} = \frac{AE}{\ell} \tag{1.5}$$

Turning our attention to Example 1.1, we note once again that the bar's cross section varies in the y-direction. As a first approximation, we model the bar as a series of centrally loaded members with different cross sections, as shown in Figure 1.2. Thus, the bar is represented by a model consisting of four elastic springs (elements) in series, and the elastic behavior of an element with nodes i and $i + 1$ is modeled by an equivalent linear spring according to the equation

$$f = k_{eq}(u_{i+1} - u_i) = \frac{A_{avg}E}{\ell}(u_{i+1} - u_i) = \frac{(A_{i+1} + A_i)E}{2\ell}(u_{i+1} - u_i) \tag{1.6}$$

where u_{i+1} and u_i are the deflections at nodes $i + 1$ and i, and the equivalent element stiffness is given by

$$k_{eq} = \frac{(A_{i+1} + A_i)E}{2\ell} \tag{1.7}$$

A_i and A_{i+1} are the cross-sectional areas of the member at nodes i and $i + 1$ respectively, and ℓ is the length of the element. Employing the above model, let us consider the forces acting on each node. The free-body diagram of nodes, which shows the forces acting on nodes 1 through 5 of this model, is depicted in Figure 1.4.

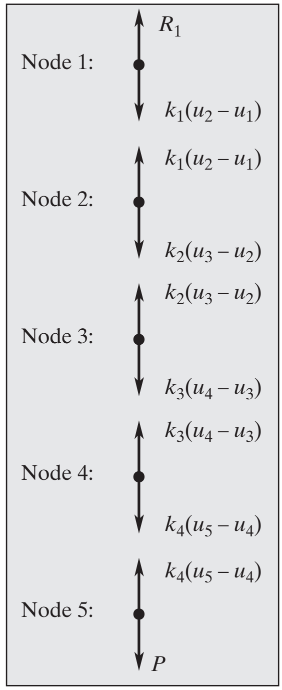


FIGURE 1.4 Free body diagram of the nodes in Example 1.1.

Static equilibrium requires that the sum of the forces acting on each node be zero. This requirement creates the following five equations:

$$
\begin{aligned}
&\text{node 1:} \quad R_1 - k_1(u_2 - u_1) = 0 \\
&\text{node 2:} \quad k_1(u_2 - u_1) - k_2(u_3 - u_2) = 0 \\
&\text{node 3:} \quad k_2(u_3 - u_2) - k_3(u_4 - u_3) = 0 \\
&\text{node 4:} \quad k_3(u_4 - u_3) - k_4(u_5 - u_4) = 0 \\
&\text{node 5:} \quad k_4(u_5 - u_4) - P = 0
\end{aligned} \tag{1.8}
$$

Rearranging the equilibrium equations given by Eq. (1.8) by separating the reaction force R_1 and the applied external force P from the internal forces, we have

$$
\begin{array}{llllllllll}
k_1u_1 & -k_1u_2 & & & & & & & & = -R_1 \\
-k_1u_1 & +k_1u_2 & +k_2u_2 & -k_2u_3 & & & & & & = 0 \\
 & & -k_2u_2 & +k_2u_3 & +k_3u_3 & -k_3u_4 & & & & = 0 \\
 & & & & -k_3u_3 & +k_3u_4 & +k_4u_4 & -k_4u_5 & & = 0 \\
 & & & & & & -k_4u_4 & +k_4u_5 & & = P
\end{array} \tag{1.9}
$$

Presenting the equilibrium equations of Eq. (1.9) in a matrix form, we have

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} -R_1 \\ 0 \\ 0 \\ 0 \\ P \end{Bmatrix} \tag{1.10}$$

It is also important to distinguish between the reaction forces and the applied loads in the load matrix. To do so, the matrix relation of Eq. (1.10) is written as

$$\begin{Bmatrix} -R_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P \end{Bmatrix} \tag{1.11}$$

We can readily show that under additional nodal loads and other fixed boundary conditions, the relationship given by Eq. (1.11) can be put into the general form

$$\{\mathbf{R}\} = [\mathbf{K}]\{\mathbf{u}\} - \{\mathbf{F}\} \tag{1.12}$$

which stands for

$$\{\textbf{reaction matrix}\} = [\textbf{stiffness matrix}]\{\textbf{displacement matrix}\} - \{\textbf{load matrix}\}$$

Note the difference between applied load matrix $\{\mathbf{F}\}$ and the reaction force matrix $\{\mathbf{R}\}$.

Turning our attention to Example 1.1 again, we find that because the bar is fixed at the top, the displacement of node 1 is zero. Hence, there are only four unknown nodal displacement values, u_2, u_3, u_4, and u_5. The reaction force at node 1, R_1, is also unknown—all together, there are five unknowns. Because there are five equilibrium equations, as given by Eq. (1.11), we should be able to solve for all of the unknowns. However, it is important to note that even though the number of unknowns match the number of equations, the system of equations contains two different types of unknowns—displacement and reaction force. In order to eliminate the need to consider the unknown reaction force simultaneously and focus first on unknown displacements, we make use of the known boundary condition and replace the first row of Eq. (1.10) with a row that reads $u_1 = 0$. The application of the boundary condition $u_1 = 0$ eliminates the need to consider the unknown reaction force in our system of equations and creates a set of equations with the displacements being the only unknowns.

Thus, application of the boundary condition leads to the following matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P \end{Bmatrix} \tag{1.13}$$

The solution of the above matrix yields the nodal displacement values. It should be clear from the above explanation and examining Eq. (1.13) that for solid mechanics problems, the application of boundary conditions to the finite element formulations transforms the system of equations as given by Eq. (1.11) to a new general form that is made up of only the stiffness matrix, the displacement matrix, and the load matrix:

$$[\textbf{stiffness matrix}]\{\textbf{displacement matrix}\} = \{\textbf{load matrix}\}$$

After we solve for the nodal displacement values, from the above relationship, we use Eq. (1.12) to solve for the reaction force(s). In the next section, we will develop the general elemental stiffness matrix and discuss the construction of the global stiffness matrix by inspection.

3. *Develop equations for an element.*
Because each of the elements in Example 1.1 has two nodes, and with each node we have associated a displacement, we need to create two equations for each element. These equations must involve nodal displacements and the element's stiffness. Consider the internally transmitted forces f_i and f_{i+1} and the end displacements u_i and u_{i+1} of an element, which are shown in Figure 1.5.

Static equilibrium conditions require that the sum of f_i and f_{i+1} be zero. Note that the sum of f_i and f_{i+1} is zero regardless of which representation of Figure 1.5 is selected. However, for the sake of consistency in the forthcoming derivation, we will use the representation given by Figure 1.5(b), so that f_i and f_{i+1} are given in

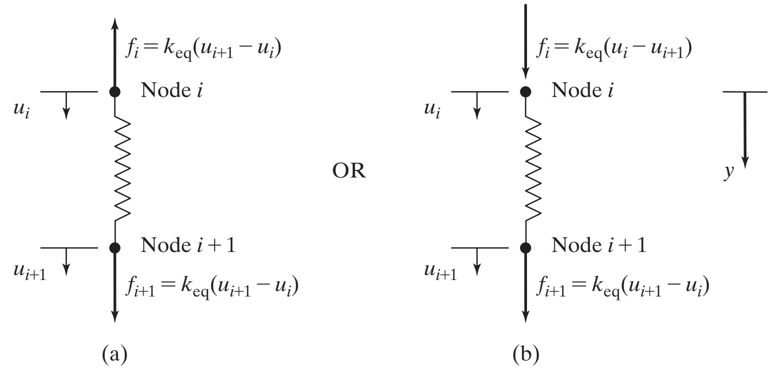


FIGURE 1.5 Internally transmitted forces through an arbitrary element.

the positive y-direction. Thus, we write the transmitted forces at nodes i and $i + 1$ according to the following equations:

$$
\begin{aligned}
f_i &= k_{eq}(u_i - u_{i+1}) \\
f_{i+1} &= k_{eq}(u_{i+1} - u_i)
\end{aligned}
\tag{1.14}
$$

Equation (1.14) can be expressed in a matrix form by

$$
\begin{Bmatrix} f_i \\ f_{i+1} \end{Bmatrix} = \begin{bmatrix} k_{eq} & -k_{eq} \\ -k_{eq} & k_{eq} \end{bmatrix} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}
\tag{1.15}
$$

4. *Assemble the elements to present the entire problem.*
Applying the elemental description given by Eq. (1.15) to all elements and assembling them (putting them together) will lead to the formation of the global stiffness matrix. The stiffness matrix for element (1) is given by

$$
[\mathbf{K}]^{(1)} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}
$$

and its position in the global stiffness matrix is given by

$$
[\mathbf{K}]^{(1G)} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix}
$$

The nodal displacement matrix is shown alongside the position of element 1 in the global stiffness matrix to aid us to observe the contribution of a node to its neighboring elements. Similarly, for elements (2), (3), and (4), we have

$$
[\mathbf{K}]^{(2)} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}
$$

and its position in the global matrix

$$
[\mathbf{K}]^{(2G)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix}
$$

$$
[\mathbf{K}]^{(3)} = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix}
$$

and its position in the global matrix

$$
[\mathbf{K}]^{(3G)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix}
$$

and

$$[\mathbf{K}]^{(4)} = \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix}$$

and its position in the global matrix

$$[\mathbf{K}]^{(4G)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix}$$

The final global stiffness matrix is obtained by assembling, or adding together, each element's position in the global stiffness matrix:

$$[\mathbf{K}]^{(G)} = [\mathbf{K}]^{(1G)} + [\mathbf{K}]^{(2G)} + [\mathbf{K}]^{(3G)} + [\mathbf{K}]^{(4G)}$$

$$[\mathbf{K}]^{(G)} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \tag{1.16}$$

Note that the global stiffness matrix obtained using elemental description, as given by Eq. (1.16), is identical to the global stiffness matrix we obtained earlier from the analysis of the free-body diagrams of the nodes, as given by the left-hand side of Eq. (1.10).

5. *Apply boundary conditions and loads.*
The bar is fixed at the top, which leads to the boundary condition $u_1 = 0$. The external load P is applied at node 5. Applying these conditions results in the following set of linear equations.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P \end{Bmatrix} \tag{1.17}$$

Again, note that the first row of the matrix in Eq. (1.17) must contain a 1 followed by four 0s to read $u_1 = 0$, the given boundary condition. As explained earlier, also note that in solid mechanics problems, the finite element formulation will always lead to the following general form:

$$[\textbf{stiffness matrix}]\{\textbf{displacement matrix}\} = \{\textbf{load matrix}\}$$

Solution Phase

6. *Solve a system of algebraic equations simultaneously.*
In order to obtain numerical values of the nodal displacements, let us assume that $E = 10.4 \times 10^6$ lb/in^2 (aluminum), $w_1 = 2$ in, $w_2 = 1$ in, $t = 0.125$ in, $L = 10$ in, and $P = 1000$ lb. You may consult Table 1.5 while working toward the solution.

TABLE 1.5 Properties of the elements in Example 1.1

Element	Nodes	Average Cross-Sectional Area (in^2)	Length (in)	Modulus of Elasticity (lb/in^2)	Element's Stiffness Coefficient (lb/in)
1	1 2	0.234375	2.5	10.4×10^6	975×10^3
2	2 3	0.203125	2.5	10.4×10^6	845×10^3
3	3 4	0.171875	2.5	10.4×10^6	715×10^3
4	4 5	0.140625	2.5	10.4×10^6	585×10^3

The variation of the cross-sectional area of the bar in the y-direction can be expressed by:

$$A(y) = \left(w_1 + \left(\frac{w_2 - w_1}{L}\right)y\right)t = \left(2 + \frac{(1-2)}{10}y\right)(0.125) = 0.25 - 0.0125y \quad (1.18)$$

Using Eq. (1.18), we can compute the cross-sectional areas at each node:

$A_1 = 0.25$ in^2 $\qquad A_2 = 0.25 - 0.0125(2.5) = 0.21875$ in^2

$A_3 = 0.25 - 0.0125(5.0) = 0.1875$ in^2 $\qquad A_4 = 0.25 - 0.0125(7.5) = 0.15625$ in^2

$A_5 = 0.125$ in^2

Next, the equivalent stiffness coefficient for each element is computed from the equations

$$k_{eq} = \frac{(A_{i+1} + A_i)E}{2\ell}$$

$$k_1 = \frac{(0.21875 + 0.25)(10.4 \times 10^6)}{2(2.5)} = 975 \times 10^3 \frac{\text{lb}}{\text{in}}$$

$$k_2 = \frac{(0.1875 + 0.21875)(10.4 \times 10^6)}{2(2.5)} = 845 \times 10^3 \frac{\text{lb}}{\text{in}}$$

$$k_3 = \frac{(0.15625 + 0.1875)(10.4 \times 10^6)}{2(2.5)} = 715 \times 10^3 \frac{\text{lb}}{\text{in}}$$

$$k_4 = \frac{(0.125 + 0.15625)(10.4 \times 10^6)}{2(2.5)} = 585 \times 10^3 \frac{\text{lb}}{\text{in}}$$

and the elemental matrices are

$$[\mathbf{K}]^{(1)} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} = 10^3 \begin{bmatrix} 975 & -975 \\ -975 & 975 \end{bmatrix}$$

$$[\mathbf{K}]^{(2)} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = 10^3 \begin{bmatrix} 845 & -845 \\ -845 & 845 \end{bmatrix}$$

$$[\mathbf{K}]^{(3)} = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} = 10^3 \begin{bmatrix} 715 & -715 \\ -715 & 715 \end{bmatrix}$$

$$[\mathbf{K}]^{(4)} = \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix} = 10^3 \begin{bmatrix} 585 & -585 \\ -585 & 585 \end{bmatrix}$$

Assembling the elemental matrices leads to the generation of the global stiffness matrix:

$$[\mathbf{K}]^{(G)} = 10^3 \begin{bmatrix} 975 & -975 & 0 & 0 & 0 \\ -975 & 975 + 845 & -845 & 0 & 0 \\ 0 & -845 & 845 + 715 & -715 & 0 \\ 0 & 0 & -715 & 715 + 585 & -585 \\ 0 & 0 & 0 & -585 & 585 \end{bmatrix}$$

Applying the boundary condition $u_1 = 0$ and the load $P = 1000$ lb, we get

$$10^3 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -975 & 1820 & -845 & 0 & 0 \\ 0 & -845 & 1560 & -715 & 0 \\ 0 & 0 & -715 & 1300 & -585 \\ 0 & 0 & 0 & -585 & 585 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 10^3 \end{Bmatrix}$$

Because in the second row, the -975 coefficient gets multiplied by $u_1 = 0$, we need only to solve the following 4×4 matrix:

$$10^3 \begin{bmatrix} 1820 & -845 & 0 & 0 \\ -845 & 1560 & -715 & 0 \\ 0 & -715 & 1300 & -585 \\ 0 & 0 & -585 & 585 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 10^3 \end{Bmatrix}$$

The displacement solution is $u_1 = 0, u_2 = 0.001026$ in, $u_3 = 0.002210$ in, $u_4 = 0.003608$ in, and $u_5 = 0.005317$ in.

Postprocessing Phase

7. *Obtain other information.*

For Example 1.1, we may be interested in obtaining other information, such as the average normal stresses in each element. These values can be determined from the equation

$$\sigma = \frac{f}{A_{\text{avg}}} = \frac{k_{\text{eq}}(u_{i+1} - u_i)}{A_{\text{avg}}} = \frac{\dfrac{A_{\text{avg}}E}{\ell}(u_{i+1} - u_i)}{A_{\text{avg}}} = E\left(\frac{u_{i+1} - u_i}{\ell}\right) \quad (1.19)$$

Since the displacements of different nodes are known, Eq. (1.19) could have been obtained directly from the relationship between the stresses and strains,

$$\sigma = E\varepsilon = E\left(\frac{u_{i+1} - u_i}{\ell}\right) \tag{1.20}$$

Employing Eq. (1.20) in Example 1.1, we compute the average normal stress for each element as

$$\sigma^{(1)} = E\left(\frac{u_2 - u_1}{\ell}\right) = \frac{(10.4 \times 10^6)(0.001026 - 0)}{2.5} = 4268\,\frac{\text{lb}}{\text{in}^2}$$

$$\sigma^{(2)} = E\left(\frac{u_3 - u_2}{\ell}\right) = \frac{(10.4 \times 10^6)(0.002210 - 0.001026)}{2.5} = 4925\,\frac{\text{lb}}{\text{in}^2}$$

$$\sigma^{(3)} = E\left(\frac{u_4 - u_3}{\ell}\right) = \frac{(10.4 \times 10^6)(0.003608 - 0.002210)}{2.5} = 5816\,\frac{\text{lb}}{\text{in}^2}$$

$$\sigma^{(4)} = E\left(\frac{u_5 - u_4}{\ell}\right) = \frac{(10.4 \times 10^6)(0.005317 - 0.003608)}{2.5} = 7109\,\frac{\text{lb}}{\text{in}^2}$$

In Figure 1.6, we note that for the given problem, regardless of where we cut a section through the bar, the internal force at the section is equal to 1000 lb. So,

$$\sigma^{(1)} = \frac{f}{A_{\text{avg}}} = \frac{1000}{0.234375} = 4267\,\frac{\text{lb}}{\text{in}^2}$$

$$\sigma^{(2)} = \frac{f}{A_{\text{avg}}} = \frac{1000}{0.203125} = 4923\,\frac{\text{lb}}{\text{in}^2}$$

$$\sigma^{(3)} = \frac{f}{A_{\text{avg}}} = \frac{1000}{0.171875} = 5818\,\frac{\text{lb}}{\text{in}^2}$$

$$\sigma^{(4)} = \frac{f}{A_{\text{avg}}} = \frac{1000}{0.140625} = 7111\,\frac{\text{lb}}{\text{in}^2}$$

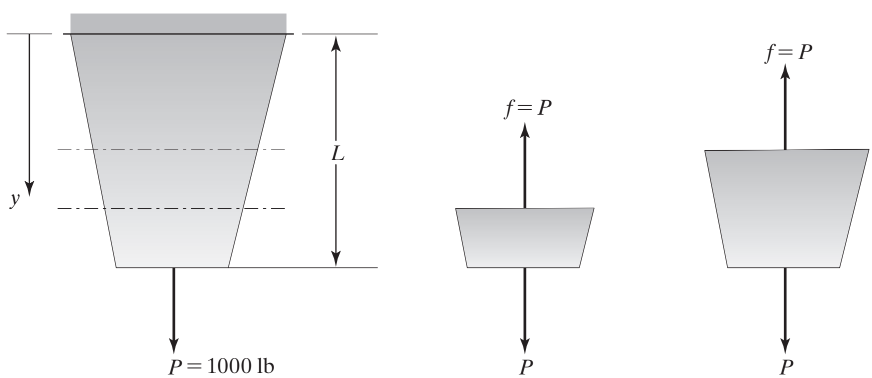


FIGURE 1.6 The internal forces in Example 1.1.

Ignoring the errors we get from rounding off our answers, we find that these results are identical to the element stresses computed from the displacement information. This comparison tells us that our displacement calculations are good for this problem.

Reaction Forces For Example 1.1, the reaction force may be computed in a number of ways. First, referring to Figure 1.4, we note that the statics equilibrium at node 1 requires

$$R_1 = k_1(u_2 - u_1) = 975 \times 10^3(0.001026 - 0) = 1000 \text{ lb}$$

The statics equilibrium for the entire bar also requires that

$$R_1 = P = 1000 \text{ lb}$$

As you may recall, we can also compute the reaction forces from the general reaction equation

$$\{\mathbf{R}\} = [\mathbf{K}]\{\mathbf{u}\} - \{\mathbf{F}\}$$

or

$$\{\textbf{reaction matrix}\} = [\textbf{stiffness matrix}]\{\textbf{displacement matrix}\} - \{\textbf{load matrix}\}$$

Because Example 1.1 is a simple problem, we do not actually need to go through the matrix operations in the aforementioned general equation to compute the reaction forces. However, as a demonstration, the procedure is shown here. From the general equation, we get

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} = 10^3 \begin{bmatrix} 975 & -975 & 0 & 0 & 0 \\ -975 & 1820 & -845 & 0 & 0 \\ 0 & -845 & 1560 & -715 & 0 \\ 0 & 0 & -715 & 1300 & -585 \\ 0 & 0 & 0 & -585 & 585 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.001026 \\ 0.002210 \\ 0.003608 \\ 0.005317 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 10^3 \end{Bmatrix}$$

where R_1, R_2, R_3, R_4, and R_5 represent the reactions forces at nodes 1 through 5 respectively. Performing the matrix operation, we have

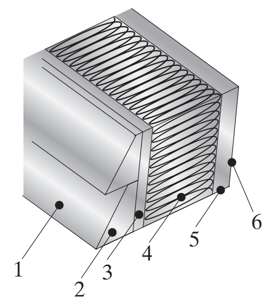
$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} = \begin{Bmatrix} -1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The negative value of R_1 simply means that the direction of the reaction force is up (because we assumed that the positive y-direction points down). Of course, as expected, the outcome is the same as in our earlier calculations because the rows of the above matrix represent the static equilibrium conditions at each node. When solving for reaction forces, it is important to note that you must use the complete stiffness matrix, without the influence of boundary conditions, as shown in Equations (1.11) and (1.12). Next, we will consider finite element formulation of a heat transfer problem. Example 1.1 also is solved using Excel. See Section 2.11.

EXAMPLE 1.2

A typical exterior frame wall (made up of 2 × 4 studs) of a house contains the materials shown in the table below. Let us assume an inside room temperature of 70°F and an outside air temperature of 20°F, with an exposed area of 150 ft^2. We are interested in determining the temperature distribution through the wall.

Items	Resistance hr · ft^2 · °F/Btu	*U*-factor Btu/hr · ft^2 · °F
1. Outside film resistance (winter, 15-mph wind)	0.17	5.88
2. Siding, wood (1/2 × 8 lapped)	0.81	1.23
3. Sheathing (1/2 in regular)	1.32	0.76
4. Insulation batt (3 − 3½ in)	11.0	0.091
5. Gypsum wall board (1/2 in)	0.45	2.22
6. Inside film resistance (winter)	0.68	1.47



Preprocessing Phase

1. *Discretize the solution domain into finite elements.*
We will represent this problem by a model that has seven nodes and six elements, as shown in Figure 1.7.
2. *Assume a solution that approximates the behavior of an element.*
For Example 1.2, there are two modes of heat transfer (conduction and convection) that we must first understand before we can proceed with formulating the conductance matrix and the thermal load matrix. The steady-state thermal behavior of the elements (2), (3), (4), and (5) may be modeled using Fourier's law. When there exists a temperature gradient in a medium, conduction heat transfer occurs, as shown in Figure 1.8. The energy is transported from the high-temperature region

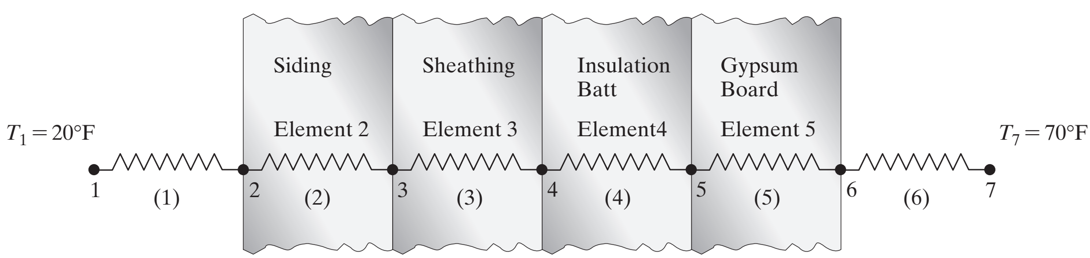


FIGURE 1.7 Finite element model of Example 1.2.

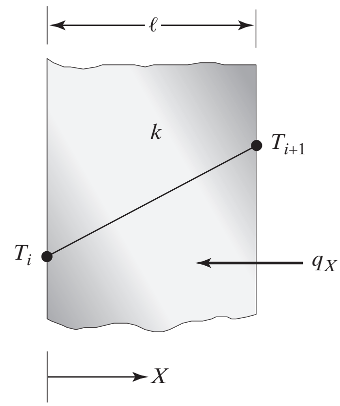


FIGURE 1.8 Heat transfer in a medium by conduction.

to the low-temperature region by molecular activities. The heat transfer rate is given by Fourier's law:

$$q_X = -kA\frac{\partial T}{\partial X} \tag{1.21}$$

q_X is the X-component of the heat transfer rate, k is the thermal conductivity of the medium, A is the area normal to heat flow, and $\frac{\partial T}{\partial X}$ is the temperature gradient. The minus sign in Eq. (1.21) is due to the fact that heat flows in the direction of decreasing temperature. Equation (1.21) can be written in a difference form in terms of the spacing between the nodes (length of the element) ℓ and the respective temperatures of the nodes i and $i + 1$, T_i and T_{i+1}, according to the equation

$$q = \frac{kA(T_{i+1} - T_i)}{\ell} \tag{1.22}$$

In the field of heat transfer, it is also common to write Eq. (1.22) in terms of the thermal transmittance coefficient U, or, as it is often called, the U-factor $(U = \frac{k}{\ell})$. The U-factor represents thermal transmission through a unit area and has the units of Btu/hr·ft^2·°F. It is the reciprocal of thermal resistance. So, Equation (1.22) becomes

$$q = UA(T_{i+1} - T_i) \tag{1.23}$$

The steady-state thermal behavior of elements (1) and (6) may be modeled using Newton's law of cooling. Convection heat transfer occurs when a fluid in motion comes into contact with a surface whose temperature differs from the moving fluid. The overall heat transfer rate between the fluid and the surface is governed by Newton's law of cooling, according to the equation

$$q = hA(T_s - T_f) \tag{1.24}$$

where h is the heat transfer coefficient, T_s is the surface temperature, and T_f represents the temperature of the moving fluid. Newton's law of cooling can also be written in terms of the U-factor, such that

$$q = UA(T_s - T_f) \tag{1.25}$$

where $U = h$, and it represents the reciprocal of thermal resistance due to convection boundary conditions. Under steady-state conduction, the application of energy balance to a surface, with a convective heat transfer, requires that the energy transferred to this surface via conduction must be equal to the energy transfer by convection. This principle,

$$-kA\frac{\partial T}{\partial X} = hA[T_s - T_f] \tag{1.26}$$

is depicted in Figure 1.9.

Now that we understand the two modes of heat transfer involved in this problem, we can apply the energy balance to the various surfaces of the wall, starting with the wall's exterior surface located at node 2. The heat loss through the wall due to conduction must equal the heat loss to the surrounding cold air by convection. That is,

$$U_2A(T_3 - T_2) = U_1A(T_2 - T_1)$$

The application of energy balance to surfaces located at nodes 3, 4, and 5 yields the equations

$$\begin{aligned} U_3A(T_4 - T_3) &= U_2A(T_3 - T_2) \\ U_4A(T_5 - T_4) &= U_3A(T_4 - T_3) \\ U_5A(T_6 - T_5) &= U_4A(T_5 - T_4) \end{aligned}$$

For the interior surface of the wall, located at node 6, the heat loss by convection of warm air is equal to the heat transfer by conduction through the gypsum board, according to the equation

$$U_6A(T_7 - T_6) = U_5A(T_6 - T_5)$$

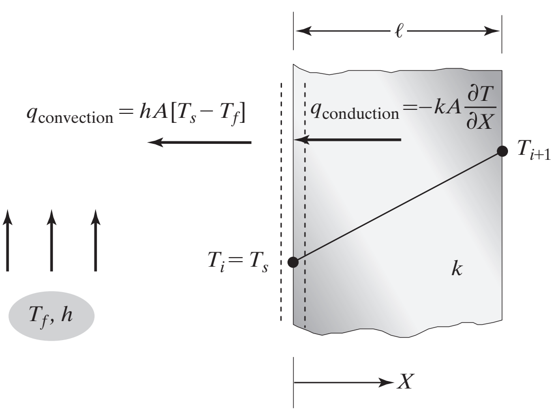


FIGURE 1.9 Energy balance at a surface with a convective heat transfer.

Separating the known temperatures from the unknown temperatures, we have

$$\begin{aligned}
+(U_1 + U_2)AT_2 \quad -U_2AT_3 \qquad\qquad\qquad\qquad\qquad\qquad &= U_1AT_1 \\
-U_2AT_2 \quad +(U_2 + U_3)AT_3 \quad -U_3AT_4 \qquad\qquad\qquad\qquad &= 0 \\
-U_3AT_3 \quad +(U_3 + U_4)AT_4 \quad -U_4AT_5 \qquad\qquad &= 0 \\
-U_4AT_4 \quad +(U_4 + U_5)AT_5 \quad -U_5AT_6 &= 0 \\
-U_5AT_5 \quad +(U_5 + U_6)AT_6 &= U_6AT_7
\end{aligned}$$

The above relationships can be represented in matrix form as

$$A\begin{bmatrix} U_1 + U_2 & -U_2 & 0 & 0 & 0 \\ -U_2 & U_2 + U_3 & -U_3 & 0 & 0 \\ 0 & -U_3 & U_3 + U_4 & -U_4 & 0 \\ 0 & 0 & -U_4 & U_4 + U_5 & -U_5 \\ 0 & 0 & 0 & -U_5 & U_5 + U_6 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} U_1AT_1 \\ 0 \\ 0 \\ 0 \\ U_6AT_7 \end{Bmatrix} \tag{1.27}$$

Note that the relationship given by Eq. (1.27) was developed by applying the conservation of energy to the surfaces located at nodes 2, 3, 4, 5, and 6. Next, we will consider the elemental formulation of this problem, which will lead to the same results.

3. *Develop equations for an element.*
In general, for conduction problems, the heat transfer rates q_i and q_{i+1} and the nodal temperatures T_i and T_{i+1} for an element are related according to the equations

$$\begin{aligned} q_i &= \frac{kA}{\ell}(T_i - T_{i+1}) \\ q_{i+1} &= \frac{kA}{\ell}(T_{i+1} - T_i) \end{aligned} \tag{1.28}$$

The heat flow through nodes i and $i + 1$ is depicted in Figure 1.10.

Because each of the elements in Example 1.2 has two nodes, and we have associated a temperature with each node, we want to create two equations for each element. These equations must involve nodal temperatures and the element's thermal conductivity or U-factor, based on Fourier's law. Under steady-state conditions, the application of the conservation of energy requires that the sum of q_i and q_{i+1} into an element be zero; that is, the energy flowing into node $i + 1$ must be equal to the energy flowing out of node i. Note that the sum of q_i and q_{i+1} is zero regardless of which representation of Figure 1.10 is selected. However, for the sake of consistency in the forthcoming derivation, we will use the representation given by Figure 1.10(b). Elemental description given by Eq. (1.28) can be expressed in matrix form by

$$\begin{Bmatrix} q_i \\ q_{i+1} \end{Bmatrix} = \frac{kA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_i \\ T_{i+1} \end{Bmatrix} \tag{1.29}$$

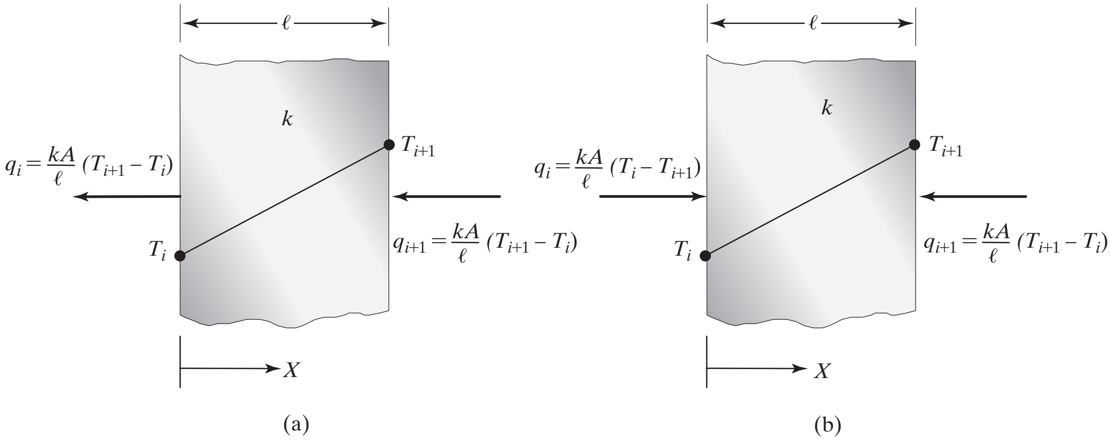


FIGURE 1.10 Heat flow through nodes i and $i + 1$.

The thermal conductance matrix for an element is

$$[\mathbf{K}]^{(e)} = \frac{kA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1.30}$$

The conductance matrix can also be written in terms of the U-factor $\left(U = \frac{k}{\ell}\right)$:

$$[\mathbf{K}]^{(e)} = UA\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1.31}$$

Similarly, under steady-state conditions, the application of the conservation of energy to the nodes of a convective element gives

$$\begin{aligned} q_i &= hA(T_i - T_{i+1}) \\ q_{i+1} &= hA(T_{i+1} - T_i) \end{aligned} \tag{1.32}$$

Equation (1.32) expressed in a matrix form is

$$\begin{Bmatrix} q_i \\ q_{i+1} \end{Bmatrix} = hA\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_i \\ T_{i+1} \end{Bmatrix}$$

The thermal conductance matrix for a convective element then becomes

$$[\mathbf{K}]^{(e)} = hA\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1.33}$$

Equation (1.33) can also be written in terms of the U-factor ($U = h$):

$$[\mathbf{K}]^{(e)} = UA\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1.34}$$

4. *Assemble the elements to present the entire problem.*
Applying the elemental description given by Eqs. (1.31) and (1.34) to all of the elements in Example 1.2 and assembling leads to the formation of the global stiffness matrix. So,

$$[\mathbf{K}]^{(1)} = A\begin{bmatrix} U_1 & -U_1 \\ -U_1 & U_1 \end{bmatrix}$$

and its position in the global matrix is

$$[\mathbf{K}]^{(1G)} = A\begin{bmatrix} U_1 & -U_1 & 0 & 0 & 0 & 0 & 0 \\ -U_1 & U_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{matrix}$$

The nodal temperature matrix is shown along with the global thermal conductance matrix to help you observe the contribution of a node to its neighboring elements:

$$[\mathbf{K}]^{(2)} = A\begin{bmatrix} U_2 & -U_2 \\ -U_2 & U_2 \end{bmatrix} \text{ and } [\mathbf{K}]^{(2G)} = A\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & U_2 & -U_2 & 0 & 0 & 0 & 0 \\ 0 & -U_2 & U_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{matrix}$$

$$[\mathbf{K}]^{(3)} = A\begin{bmatrix} U_3 & -U_3 \\ -U_3 & U_3 \end{bmatrix} \text{ and } [\mathbf{K}]^{(3G)} = A\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & U_3 & -U_3 & 0 & 0 & 0 \\ 0 & 0 & -U_3 & U_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{matrix}$$

$$[\mathbf{K}]^{(4)} = A\begin{bmatrix} U_4 & -U_4 \\ -U_4 & U_4 \end{bmatrix} \text{ and } [\mathbf{K}]^{(4G)} = A\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & U_4 & -U_4 & 0 & 0 \\ 0 & 0 & 0 & -U_4 & U_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{matrix}$$

$$[\mathbf{K}]^{(5)} = A\begin{bmatrix} U_5 & -U_5 \\ -U_5 & U_5 \end{bmatrix} \text{ and } [\mathbf{K}]^{(5G)} = A\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & U_5 & -U_5 & 0 \\ 0 & 0 & 0 & 0 & -U_5 & U_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{matrix}$$

$$[\mathbf{K}]^{(6)} = A\begin{bmatrix} U_6 & -U_6 \\ -U_6 & U_6 \end{bmatrix} \text{ and } [\mathbf{K}]^{(6G)} = A\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & U_6 & -U_6 \\ 0 & 0 & 0 & 0 & 0 & -U_6 & U_6 \end{bmatrix}\begin{matrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{matrix}$$

The global conductance matrix is

$$[\mathbf{K}]^{(G)} = [\mathbf{K}]^{(1G)} + [\mathbf{K}]^{(2G)} + [\mathbf{K}]^{(3G)} + [\mathbf{K}]^{(4G)} + [\mathbf{K}]^{(5G)} + [\mathbf{K}]^{(6G)}$$

$$[\mathbf{K}]^{(G)} = A\begin{bmatrix} U_1 & -U_1 & 0 & 0 & 0 & 0 & 0 \\ -U_1 & U_1 + U_2 & -U_2 & 0 & 0 & 0 & 0 \\ 0 & -U_2 & U_2 + U_3 & -U_3 & 0 & 0 & 0 \\ 0 & 0 & -U_3 & U_3 + U_4 & -U_4 & 0 & 0 \\ 0 & 0 & 0 & -U_4 & U_4 + U_5 & -U_5 & 0 \\ 0 & 0 & 0 & 0 & -U_5 & U_5 + U_6 & -U_6 \\ 0 & 0 & 0 & 0 & 0 & -U_6 & U_6 \end{bmatrix} \tag{1.35}$$

5. *Apply boundary conditions and thermal loads.*
For the given problem, the exterior of the wall is exposed to a known air temperature T_1, and the room temperature, T_7, is also known. Thus, we want the first row to read $T_1 = 20°\text{F}$ and the last row to read $T_7 = 70°\text{F}$. So, we have

$$A\begin{bmatrix} 1/A & 0 & 0 & 0 & 0 & 0 & 0 \\ -U_1 & U_1 + U_2 & -U_2 & 0 & 0 & 0 & 0 \\ 0 & -U_2 & U_2 + U_3 & -U_3 & 0 & 0 & 0 \\ 0 & 0 & -U_3 & U_3 + U_4 & -U_4 & 0 & 0 \\ 0 & 0 & 0 & -U_4 & U_4 + U_5 & -U_5 & 0 \\ 0 & 0 & 0 & 0 & -U_5 & U_5 + U_6 & -U_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/A \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{Bmatrix} = \begin{Bmatrix} 20°\text{F} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 70°\text{F} \end{Bmatrix} \tag{1.36}$$

Note that the finite element formulation of heat transfer problems will always lead to an equation of the form

$$[\mathbf{K}]\{\mathbf{T}\} = \{\mathbf{q}\}$$

$$[\textbf{conductance matrix}]\{\textbf{temperature matrix}\} = \{\textbf{heat flow matrix}\}$$

Also note that for Example 1.2, the heat transfer rate through each element was caused by temperature differences across the nodes of a given element. Thus, the external nodal heat flow values are zero in the heat flow matrix. An example of a situation in which external nodal heat values are not zero is a heating strip attached to a solid surface (e.g., the base of a pressing iron); for such a situation, the external nodal heat value is equal to the amount of heat being generated by the heating strip over the surface. Turning our attention to the matrices given by Eq. (1.36) and incorporating the known boundary conditions into rows 2 and 6 of the conductance matrix, we can reduce Eq. (1.36) to

$$A\begin{bmatrix} U_1+U_2 & -U_2 & 0 & 0 & 0 \\ -U_2 & U_2+U_3 & -U_3 & 0 & 0 \\ 0 & -U_3 & U_3+U_4 & -U_4 & 0 \\ 0 & 0 & -U_4 & U_4+U_5 & -U_5 \\ 0 & 0 & 0 & -U_5 & U_5+U_6 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} U_1AT_1 \\ 0 \\ 0 \\ 0 \\ U_6AT_7 \end{Bmatrix}$$

Keep in mind that the above matrix was obtained by assembling the elemental description and applying the boundary conditions. Moreover, the results of this approach are identical to the relations we obtained earlier by balancing the heat flows at the nodes, as given by Eq. (1.27). This equality in the outcome is expected because the elemental formulations are based on the application of energy balance as well.

Referring to the original global matrix, substituting for the U-values and employing the given boundary conditions, we have

$$150\begin{bmatrix} \frac{1}{150} & 0 & 0 & 0 & 0 & 0 & 0 \\ -5.88 & 5.88+1.23 & -1.23 & 0 & 0 & 0 & 0 \\ 0 & -1.23 & 1.23+0.76 & -0.76 & 0 & 0 & 0 \\ 0 & 0 & -0.76 & 0.76+0.091 & -0.091 & 0 & 0 \\ 0 & 0 & 0 & -0.091 & 0.091+2.22 & -2.22 & 0 \\ 0 & 0 & 0 & 0 & -2.22 & 2.22+1.47 & -1.47 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{150} \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{Bmatrix} = \begin{Bmatrix} 20°\text{F} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 70°\text{F} \end{Bmatrix}$$

Simplifying, we obtain

$$\begin{bmatrix} 7.11 & -1.23 & 0 & 0 & 0 \\ -1.23 & 1.99 & -0.76 & 0 & 0 \\ 0 & -0.76 & 0.851 & -0.091 & 0 \\ 0 & 0 & -0.091 & 2.311 & -2.22 \\ 0 & 0 & 0 & -2.22 & 3.69 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} (5.88)(20) \\ 0 \\ 0 \\ 0 \\ (1.47)(70) \end{Bmatrix}$$

Solution Phase

6. *Solve a system of algebraic equations simultaneously.*
Solving the previous matrix yields the temperature distribution along the wall:

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{Bmatrix} = \begin{Bmatrix} 20.00 \\ 20.59 \\ 23.41 \\ 27.97 \\ 66.08 \\ 67.64 \\ 70.00 \end{Bmatrix} ^\circ\text{C}$$

For problems similar to the type discussed here, the knowledge of temperature distribution within the wall is important in determining where condensation may occur in the wall and thus where one should place a vapor barrier to avoid moisture condensation. To demonstrate this concept, let us assume that moisture can diffuse through the gypsum board and that the inside air has a relative humidity of 40%. With the help of a psychometric chart, using a dry bulb temperature of 70°F and the value $\phi = 40\%$, we identify the condensation temperature to be 44°F. Therefore, the water vapor in the air at any surface whose temperature is 44°F or below will condense. In the absence of a vapor barrier, the water vapor in the air will condense somewhere between surface 5 and 4 for the assumed conditions in this problem.

Postprocessing Phase

7. *Obtain other information.*
For this example, we may be interested in obtaining other information, such as heat loss through the wall. Such information is important in computing the heat load for a building. Because we have assumed steady-state conditions, the heat loss through the wall should be equal to the heat transfer through each element. This value can be determined from the equation

$$q = UA(T_{i+1} - T_i) \tag{1.37}$$

The heat transfer through each element is

$$q = UA(T_{i+1} - T_i) = (1.47)(150)(70 - 67.64) = (2.22)(150)(67.64 - 66.08) = \cdots$$
$$= (5.88)(150)(20.59 - 20) = 520\frac{\text{Btu}}{\text{hr}}$$

We also could have calculated the heat loss through the wall using the overall U-factor in the following manner:

$$q = U_{\text{overall}}A(T_{\text{inside}} - T_{\text{outside}}) = \frac{1}{\Sigma\ \text{Resistance}}A(T_{\text{inside}} - T_{\text{outside}})$$
$$= (0.0693)(150)(70 - 20) = 520\frac{\text{Btu}}{\text{hr}}$$

This problem is just another example of how we can generate finite element models using the direct method.

A Torsional Problem: Direct Formulation

EXAMPLE 1.3

Consider the torsion of a circular shaft, shown in Figure 1.11. Recall from your previous study of the mechanics of materials that the angle of twist θ for a shaft with a constant cross-sectional area with a polar moment of inertia J and length ℓ, made of homogenous material with a shear modulus of elasticity G, subject to a torque T is given by

$$\theta = \frac{T\ell}{JG}$$

Using direct formulation, equilibrium conditions, and

$$\theta = \frac{T\ell}{JG}$$

we can show that for an element comprising two nodes, the stiffness matrix, the angle of twists, and the torques are related according to the equation

$$\frac{JG}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \tag{1.38}$$

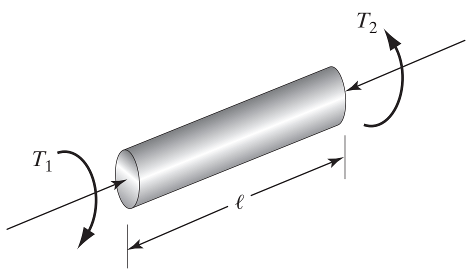


FIGURE 1.11 A torsion of circular shaft.

We will discuss torsional problems in much more detail in Chapter 10. For now, let us consider a shaft that is made of two parts, as shown in Figure 1.12. Part AB is made of material with a shear modulus of elasticity of $G_{AB} = 3.9 \times 10^6 \text{ lb/in}^2$ and has a diameter of 1.5 in. Segment BC is made of slightly different material with a shear modulus of elasticity of $G_{BC} = 4.0 \times 10^6 \text{ lb/in}^2$ and with a diameter of 1 in. The shaft is fixed at both ends. A torque of 200 lb · ft is applied at D. Using three elements, let us determine the angle of twist at D and B, and the torsional reactions at the boundaries.

We will represent this problem by a model that has four nodes at A, B, C, and D, respectively, and three elements (AD, DB, BC).

The polar moment of inertia for each element is given by

$$J_1 = J_2 = \frac{1}{2}\pi r^4 = \frac{1}{2}\pi\left(\frac{1.5}{2}\text{ in}\right)^4 = 0.497 \text{ in}^4$$

$$J_3 = \frac{1}{2}\pi r^4 = \frac{1}{2}\pi\left(\frac{1.0}{2}\text{ in}\right)^4 = 0.0982 \text{ in}^4$$

The stiffness matrix for each element is computed from Eq. (1.38) as

$$[\mathbf{K}]^{(e)} = \frac{JG}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

So, for element (1), the stiffness matrix is

$$[\mathbf{K}]^{(1)} = \frac{(0.497 \text{ in}^4)(3.9 \times 10^6 \text{ lb/in}^2)}{(12 \times 2.5)\text{ in}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 64610 & -64610 \\ -64610 & 64610 \end{bmatrix} \text{lb} \cdot \text{in}$$

and its position in the global stiffness matrix is

$$[\mathbf{K}]^{(1G)} = \begin{bmatrix} 64610 & -64610 & 0 & 0 \\ -64610 & 64610 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{matrix}$$

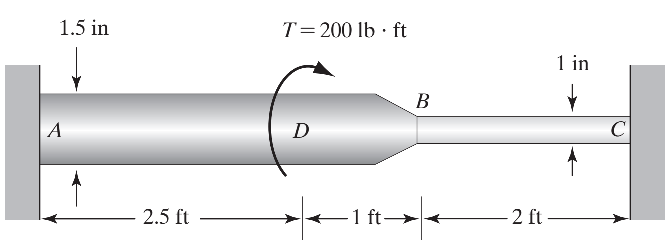


FIGURE 1.12 A schematic of the shaft in Example 1.3.

Similarly, for elements (2) and (3), their respective stiffness matrices and positions in the global stiffness matrix are as follows:

$$[\mathbf{K}]^{(2)} = \frac{(3.9 \times 10^6\,\text{lb/in}^2)(0.497\,\text{in}^4)}{(12 \times 1.0)\,\text{in}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 161525 & -161525 \\ -161525 & 161525 \end{bmatrix} \text{lb} \cdot \text{in}$$

$$[\mathbf{K}]^{(2G)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 161525 & -161525 & 0 \\ 0 & -161525 & 161525 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{matrix}$$

$$[\mathbf{K}]^{(3)} = \frac{(4.0 \times 10^6\,\text{lb/in}^2)(0.0982\,\text{in}^4)}{(12 \times 2.0)\,\text{in}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 16367 & -16367 \\ -16367 & 16367 \end{bmatrix} \text{lb} \cdot \text{in}$$

$$[\mathbf{K}]^{(3G)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16367 & -16367 \\ 0 & 0 & -16367 & 16367 \end{bmatrix} \begin{matrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{matrix}$$

The final global matrix is obtained simply by assembling, or adding, elemental descriptions:

$$[\mathbf{K}]^{(G)} = [\mathbf{K}]^{(1G)} + [\mathbf{K}]^{(2G)} + [\mathbf{K}]^{(3G)}$$

$$[\mathbf{K}]^{(G)} = \begin{bmatrix} 64610 & -64610 & 0 & 0 \\ -64610 & 64610 + 161525 & -161525 & 0 \\ 0 & -161525 & 161525 + 16367 & -16367 \\ 0 & 0 & -16367 & 16367 \end{bmatrix}$$

Applying the fixed boundary conditions at points A and C and applying the external torque, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -64610 & 226135 & -161525 & 0 \\ 0 & -161525 & 177892 & -16367 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -(200 \times 12)\,\text{lb} \cdot \text{in} \\ 0 \\ 0 \end{Bmatrix}$$

Solving the above set of equations, we obtain

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.03020\,\text{rad} \\ -0.02742\,\text{rad} \\ 0 \end{Bmatrix}$$

The reaction moments at boundaries A and C can be determined as follows:

$$\{\mathbf{R}\} = [\mathbf{K}]\{\boldsymbol{\theta}\} - \{\mathbf{T}\}$$

$$\begin{Bmatrix} R_A \\ R_D \\ R_B \\ R_C \end{Bmatrix} = \begin{bmatrix} 64610 & -64610 & 0 & 0 \\ -64610 & 226135 & -161525 & 0 \\ 0 & -161525 & 177892 & -16367 \\ 0 & 0 & -16367 & 16367 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.03020 \text{ rad} \\ -0.02742 \text{ rad} \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -(200 \times 12) \text{ lb} \cdot \text{in} \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} R_A \\ R_D \\ R_B \\ R_C \end{Bmatrix} = \begin{Bmatrix} 1951 \text{ lb} \cdot \text{in} \\ 0 \\ 0 \\ 449 \text{ lb} \cdot \text{in} \end{Bmatrix}$$

Note that the sum of R_A and R_C is equal to the applied torque of 2400 lb·in. Also note that the change in the diameter of the shafts will give rise to stress concentrations that are not accounted for by the model we used here.

EXAMPLE 1.4

A steel plate is subjected to an axial load, as shown in Figure 1.13. Approximate the deflections and average stresses along the plate. The plate is 1/16 in thick and has a modulus of elasticity $E = 29 \times 10^6$ lb/in^2.

We may model this problem using four nodes and four elements, as shown in Figure 1.13. Next, we compute the equivalent stiffness coefficient for each element:

$$k_1 = \frac{A_1 E}{\ell_1} = \frac{(5)(0.0625)(29 \times 10^6)}{1} = 9{,}062{,}500 \text{ lb/in}$$

$$k_2 = k_3 = \frac{A_2 E}{\ell_2} = \frac{(2)(0.0625)(29 \times 10^6)}{4} = 906{,}250 \text{ lb/in}$$

$$k_4 = \frac{A_4 E}{\ell_4} = \frac{(5)(0.0625)(29 \times 10^6)}{2} = 4{,}531{,}250 \text{ lb/in}$$

The stiffness matrix for element (1) is

$$[\mathbf{K}]^{(1)} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}$$

and its position in the global stiffness matrix is

$$[\mathbf{K}]^{(1G)} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix}$$

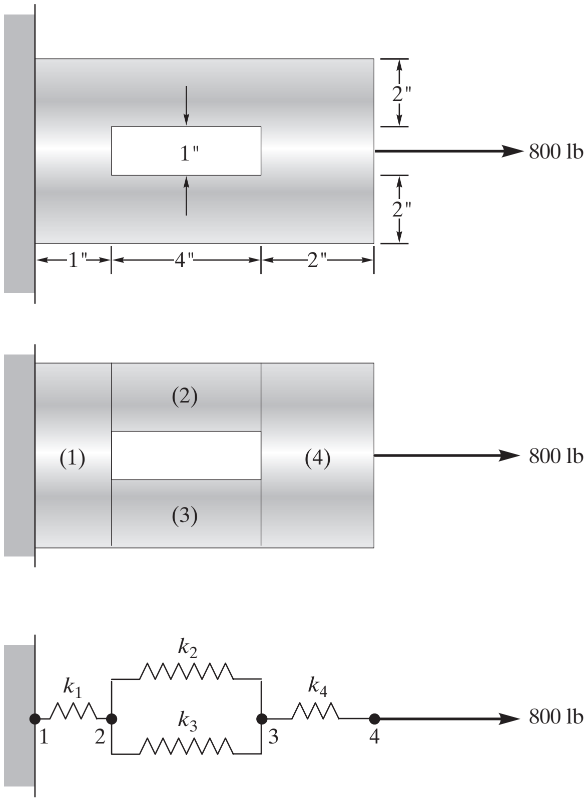


FIGURE 1.13 A schematic of the steel plate in Example 1.4.

Similarly, the respective stiffness matrices and positions in the global stiffness matrix for elements (2), (3), and (4) are

$$[\mathbf{K}]^{(2)} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \qquad [\mathbf{K}]^{(2G)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix}$$

$$[\mathbf{K}]^{(3)} = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \qquad [\mathbf{K}]^{(3G)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix}$$

$$[\mathbf{K}]^{(4)} = \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix} \qquad [\mathbf{K}]^{(4G)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix}$$

The final global matrix is obtained simply by assembling, or adding, the individual elemental matrices:

$$[\mathbf{K}]^{(G)} = [\mathbf{K}]^{(1G)} + [\mathbf{K}]^{(2G)} + [\mathbf{K}]^{(3G)} + [\mathbf{K}]^{(4G)}$$

$$[\mathbf{K}]^{(G)} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_2 - k_3 & 0 \\ 0 & -k_2 - k_3 & k_2 + k_3 + k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

Substituting for the elements' respective stiffness coefficients, the global stiffness matrix becomes

$$[\mathbf{K}]^{(G)} = \begin{bmatrix} 9{,}062{,}500 & -9{,}062{,}500 & 0 & 0 \\ -9{,}062{,}500 & 10{,}875{,}000 & -1{,}812{,}500 & 0 \\ 0 & -1{,}812{,}500 & 6{,}343{,}750 & -4{,}531{,}250 \\ 0 & 0 & -4{,}531{,}250 & 4{,}531{,}250 \end{bmatrix}$$

Applying the boundary condition $u_1 = 0$ and the load to node 4, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -9{,}062{,}500 & 10{,}875{,}000 & -1{,}812{,}500 & 0 \\ 0 & -1{,}812{,}500 & 6{,}343{,}750 & -4{,}531{,}250 \\ 0 & 0 & -4{,}531{,}250 & 4{,}531{,}250 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 800 \end{Bmatrix}$$

Solving the system of equations yields the displacement solution as

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 8.827 \times 10^{-5} \\ 5.296 \times 10^{-4} \\ 7.062 \times 10^{-4} \end{Bmatrix} \text{in}$$

and the stresses in each element are

$$\sigma^{(1)} = E\left(\frac{u_2 - u_1}{\ell}\right) = \frac{(29 \times 10^6)(8.827 \times 10^{-5} - 0)}{1} = 2560 \frac{\text{lb}}{\text{in}^2}$$

$$\sigma^{(2)} = \sigma^{(3)} = E\left(\frac{u_3 - u_2}{\ell}\right) = \frac{(29 \times 10^6)(5.296 \times 10^{-4} - 8{,}827 \times 10^{-5})}{4} = 3200 \frac{\text{lb}}{\text{in}^2}$$

$$\sigma^{(4)} = E\left(\frac{u_4 - u_3}{\ell}\right) = \frac{(29 \times 10^6)(7.062 \times 10^{-4} - 5.296 \times 10^{-4})}{2} = 2560 \frac{\text{lb}}{\text{in}^2}$$

Note that the model used to analyze this problem consisted of springs in parallel as well as in series. The two springs in parallel could have been combined and represented by a single spring having a stiffness equal to $k_2 + k_3$ (see Problem 25). Also note that because of the hole, the abrupt changes in the cross section of the strip will give rise to stress concentrations with values exceeding those average values we computed here. After you study plane-stress finite element formulation (discussed in Chapter 10), you

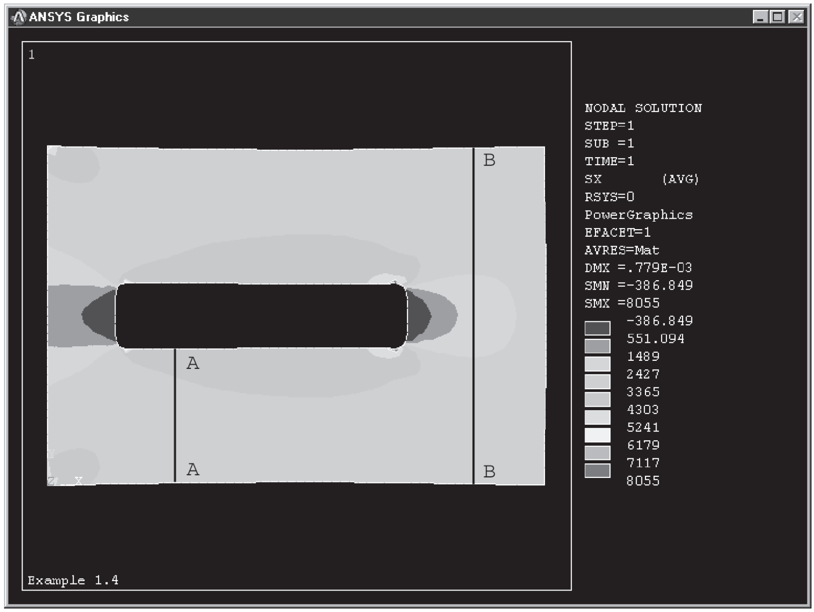


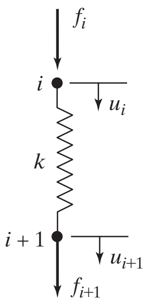
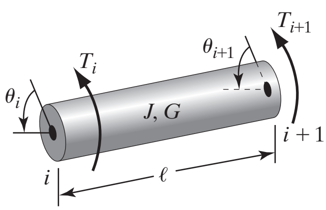
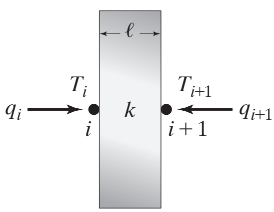
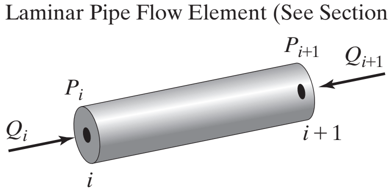
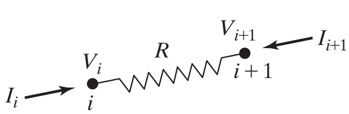
FIGURE 1.14 The *x*-component of stress distribution for the plate in Example 1.4, as computed by ANSYS.

will revisit this problem (see Problem 10.13) and be asked to solve it using ANSYS. Furthermore, you will be asked to plot the components of the stress distributions in the plate and thus identify the location and magnitude of maximum stresses.

To give you just a taste of what is to come in Chapter 10 and also to shed more light on our discussion about the stress concentration regions, we have solved Example 1.4 using ANSYS and have determined the *x*-component of the stress distribution in the plate, as shown in Figure 1.14. In the results shown in Figure 1.14, the load was applied as a pressure over the entire right surface of the bar. Note the variation of the stresses at section *A–A* from approximately 3000 psi to 3500 psi. At section *B–B,* the *x*-component of the stresses varies from approximately 2300 psi to 2600 psi. These values are not that far off from the average stress values obtained using the direct model. Also note that the maximum and minimum stress values given by ANSYS could change, depending upon how we apply the load to the bar, especially in the regions near the point of load application and the regions near the hole. Keeping in mind Example 1.4 and Figure 1.13, remember that in a real situation, the load would be applied over an area, not at a single point. Thus, remember that how you apply the external load to your finite element model will influence the stress distribution results, particularly in the region near where the load is applied. This principle is especially true in Example 1.4 because it deals with a short plate with a hole.

For the sake of convenience, the results of Section 1.5 is summarized in Table 1.6. In the table, carefully examine what constitutes an element, its degrees of freedom, and the physical balance requirements.

TABLE 1.6 Examples of elements and nodes

<table>
<tr><th colspan="2">Element</th><th>Degrees of Freedom</th><th>Physical Balance Requirement</th></tr>
<tr><td colspan="4">Linear Elastic Element (linear spring)</td></tr>
<tr><td></td><td>$f_i = k(u_i - u_{i+1})$
$f_{i+1} = k(u_{i+1} - u_i)$</td><td>Nodal displacements:
u_i, u_{i+1}</td><td>Force balance:
$f_i + f_{i+1} = 0$</td></tr>
<tr><td colspan="4">Torsional Elastic Element (torsional spring)</td></tr>
<tr><td></td><td>$T_i = \dfrac{JG}{\ell}(\theta_i - \theta_{i+1})$
$T_{i+1} = \dfrac{JG}{\ell}(\theta_{i+1} - \theta_i)$</td><td>Nodal angle of twist:
θ_i, θ_{i+1}</td><td>Torque balance:
$T_i + T_{i+1} = 0$</td></tr>
<tr><td colspan="4">Conduction Element</td></tr>
<tr><td></td><td>$q_i = \dfrac{kA}{\ell}(T_i - T_{i+1})$
$q_{i+1} = \dfrac{kA}{\ell}(T_{i+1} - T_i)$</td><td>Nodal temperatures:
T_i, T_{i+1}</td><td>Energy balance:
$q_i + q_{i+1} = 0$</td></tr>
<tr><td colspan="4">Laminar Pipe Flow Element (See Section 12.1)</td></tr>
<tr><td></td><td>$Q_i = C(P_i - P_{i+1})$
$Q_{i+1} = C(P_{i+1} - P_i)$</td><td>Nodal pressures:
P_i, P_{i+1}</td><td>Flow balance:
$Q_i + Q_{i+1} = 0$</td></tr>
<tr><td colspan="4">Electrical Resistance Element</td></tr>
<tr><td></td><td>$I_i = \dfrac{1}{R}(V_i - V_{i+1})$
$I_{i+1} = \dfrac{1}{R}(V_{i+1} - V_i)$</td><td>Nodal voltages:
V_i, V_{i+1}</td><td>Electric current balance:
$I_i + I_{i+1} = 0$</td></tr>
</table>

1.6 MINIMUM TOTAL POTENTIAL ENERGY FORMULATION

The minimum total potential energy formulation is a common approach in generating finite element models in solid mechanics. External loads applied to a body will cause the body to deform. During the deformation, the work done by the external forces is stored in the material in the form of elastic energy, called strain energy. Let us consider the strain energy in a solid member when it is subjected to a central force F, as shown in Figure 1.15.

Also shown in Figure 1.15 is a piece of material from the member in the form of differential volume and the normal stresses acting on the surfaces of this volume. Earlier, it was shown that the elastic behavior of the member may be modeled as a linear spring. In Figure 1.15 note that y' is a variable measuring deformation of the member and its value varies from 0 to $\Delta\ell$. When the member is stretched by a differential amount dy', the stored energy in the material is

$$\Lambda = \int_0^{y'} F dy' = \int_0^{y'} ky' dy' = \frac{1}{2} ky'^2 = \left(\frac{1}{2} ky'\right) y' \tag{1.39}$$

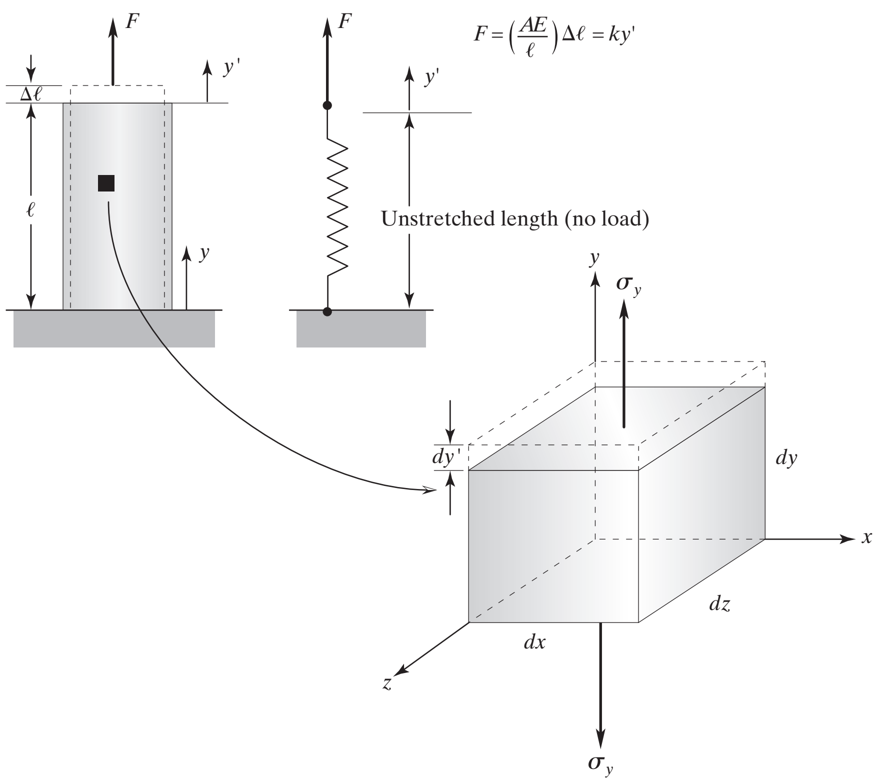


FIGURE 1.15 The elastic behavior of a member subjected to a central load.

We can write Eq. (1.39)—for a piece of material from the member in the form of differential volume—in terms of the normal stress (σ) and strain (ε):

$$d\Lambda = \frac{1}{2}\overbrace{(ky')}^{\text{elastic force}}dy' = \frac{1}{2}\overbrace{(\sigma_y dxdz)}^{\text{elastic force}}\overbrace{\varepsilon dy}^{dy'} = \frac{1}{2}\sigma\varepsilon\, dV$$

Therefore, for a member or an element under axial loading, the strain energy $\Lambda^{(e)}$ is obtained by adding up the stored energy in all pieces (differential volumes) making up the member:

$$\Lambda^{(e)} = \int d\Lambda = \int_V \frac{\sigma\varepsilon}{2}dV = \int_V \frac{E\varepsilon^2}{2}dV \tag{1.40}$$

where V is the volume of the member and $\sigma = E\varepsilon$. The total potential energy Π for a body consisting of n elements and m nodes is the difference between the total strain energy and the work done by the external forces:

$$\Pi = \sum_{e=1}^{n}\Lambda^{(e)} - \sum_{i=1}^{m}F_i u_i \tag{1.41}$$

The minimum total potential energy principle simply states that for a stable system, the displacement at the equilibrium position occurs such that the value of the system's total potential energy is a minimum.

$$\frac{\partial\Pi}{\partial u_i} = \frac{\partial}{\partial u_i}\sum_{e=1}^{n}\Lambda^{(e)} - \frac{\partial}{\partial u_i}\sum_{i=1}^{m}F_i u_i = 0 \quad \text{for } i = 1, 2, 3, \ldots, n \tag{1.42}$$

The following examples offer insight into the physical meaning of Eq. (1.42).

EXAMPLE 1.5

Consider the following situations: (a) We have applied a force F to a linear spring as shown in Figure 1.16. Depending on the stiffness value of the spring, the spring stretches by a certain amount x. The static equilibrium requires that the applied force F be equal to the internal force in the spring kx.

$$F = kx \quad \text{or} \quad x = \frac{F}{k}$$

Now, let us consider the total potential energy of the system as defined by Eq. (1.41). The stored elastic energy in the spring is $\Lambda = \frac{1}{2}kx^2$ and the work done by the external force F is Fx (force times displacement). Thus, the total potential energy of the system is

$$\Pi = \frac{1}{2}kx^2 - Fx$$

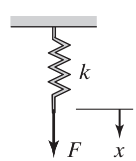


FIGURE 1.16 A linear spring subjected to a force F.

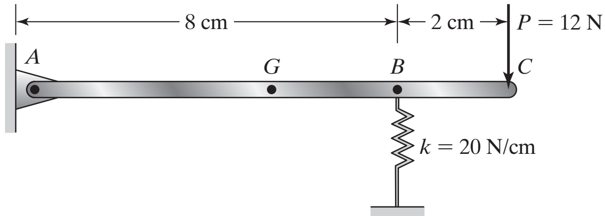


FIGURE 1.17 The rod of Example 1.5.

Minimizing Π with respect to *x*, we have

$$\frac{d\Pi}{dx} = \frac{d}{dx}\left(\frac{1}{2}kx^2 - Fx\right) = kx - F = 0$$

which results in $x = \frac{F}{k}$.

(b) The slender rod shown in Figure 1.17 weighs 8 *N* and is supported by a spring with a stiffness $k = 20\ N/\text{cm}$. A force $P = 12\ N$ is applied to the end of the rod at point C. We are interested in determining the deflection of the spring.

First, we solve this problem by applying the static equilibrium conditions and then apply the minimum total potential energy concept. Static equilibrium requires that sum of the moments of the forces acting on the rod about point A be zero. Considering the free-body diagram of the rod shown in Figure 1.18, we find

$$\circlearrowleft \sum M_A = 0 \qquad -(8N)(5\text{ cm}) + F_s(8\text{ cm}) - (12\ N)(10\text{ cm}) = 0$$

$$F_s = 20\ N \quad \text{and} \quad kx = (20\ N/\text{cm})(x) = 20\ N$$

$$x = 1\text{ cm}$$

Now, we solve the problem using the minimum total potential energy approach. We note that elastic energy stored in the system is predominantly due to elastic energy of the spring and is given by

$$\Lambda = \frac{1}{2}kx^2 = \frac{1}{2}(20\ N/\text{cm})(x^2) = 10x^2$$

The work done by the external forces is calculated by multiplying the weight of the rod by the displacement of point G, and force *P* by the displacement of endpoint C. Through

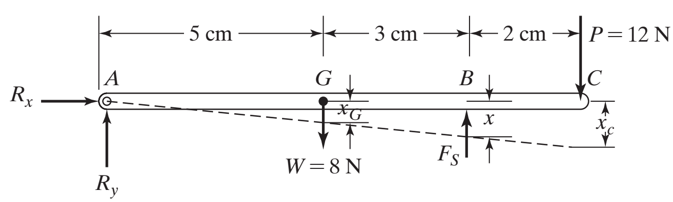


FIGURE 1.18 The free-body diagram of the rod in Example 1.5.

similar triangles, we can relate the displacements of points G and C to the displacement of the spring (point B) according to

$$\frac{x}{8} = \frac{x_G}{5} \quad \text{or} \quad x_G = \frac{5}{8}x$$
$$\frac{x}{8} = \frac{x_C}{10} \quad \text{or} \quad x_C = \frac{5}{4}x$$

Thus, the work done by the external forces is given by

$$\sum F_i u_i = (8\,N)\left(\frac{5}{8}x\right) + (12\,N)\left(\frac{5}{4}x\right) = 5x + 15x = 20x$$

The total potential energy of the system is

$$\Pi = \sum \Lambda - \sum F_i u_i = 10x^2 - 20x$$

and

$$\frac{d\Pi}{dx} = \frac{d}{dx}\left(10x^2 - 20x\right) = 20x - 20 = 0$$

Solving the above equation for x, we find $x = 1$ cm. Because there is only one unknown displacement, note that when we employed Eqs. (1.41) and (1.42), we replaced the displacement u_i with x and the partial derivative symbol with the ordinary symbol. We have plotted the total potential energy $\Pi = 10x^2 - 20x$ as a function of displacement x in Figure 1.19. It is clear from examining Figure 1.19 that the minimum total potential energy occurs at $x = 1$ cm.

Now, let us turn our attention back to Example 1.1. The strain energy for an arbitrary element (e) can be determined from Eq. (1.40) as

$$\Lambda^{(e)} = \int_V \frac{E\varepsilon^2}{2}\,dV = \frac{A_{\text{avg}}E}{2\ell}\left(u_{i+1}^2 + u_i^2 - 2u_{i+1}\,u_i\right) \tag{1.43}$$

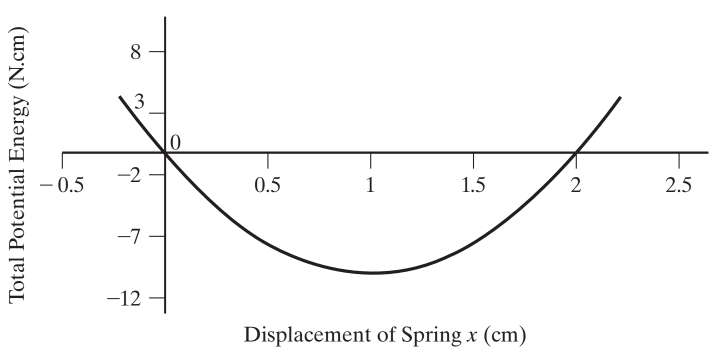


FIGURE 1.19 Total potential energy versus displacement x.

where $\varepsilon = (u_{i+1} - u_i)/\ell$, and $V = A_{avg}\ell$ were substituted for the axial strain and volume respectively. Minimizing the strain energy with respect to u_i and u_{i+1} leads to

$$\frac{\partial \Lambda^{(e)}}{\partial u_i} = \frac{A_{avg}E}{\ell}(u_i - u_{i+1}) \qquad (1.44)$$
$$\frac{\partial \Lambda^{(e)}}{\partial u_{i+1}} = \frac{A_{avg}E}{\ell}(u_{i+1} - u_i)$$

and, in matrix form,

$$\begin{Bmatrix} \dfrac{\partial \Lambda^{(e)}}{\partial u_i} \\ \dfrac{\partial \Lambda^{(e)}}{\partial u_{i+1}} \end{Bmatrix} = \begin{bmatrix} k_{eq} & -k_{eq} \\ -k_{eq} & k_{eq} \end{bmatrix} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix} \qquad (1.45)$$

where $k_{eq} = (A_{avg}E)/\ell$. Minimizing the work done by the external forces at nodes i and $i + 1$ of an arbitrary element (e), we get

$$\frac{\partial}{\partial u_i}(F_i u_i) = F_i$$
$$\frac{\partial}{\partial u_{i+1}}(F_{i+1} u_{i+1}) = F_{i+1} \qquad (1.46)$$

For Example 1.1, the minimum total potential energy formulation leads to a global stiffness matrix that is identical to the one obtained from direct formulation:

$$[\mathbf{K}]^{(G)} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

Furthermore, application of the boundary condition and the load results in

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P \end{Bmatrix} \qquad (1.47)$$

The displacement results will be identical to the ones obtained earlier from the direct method, as given by Eq. (1.17). The concepts of strain energy and minimum total potential energy will be used to formulate solid mechanics problems in Chapters 4, 10, and 13. Therefore, spending a little extra time now to understand the basic ideas will benefit you enormously later.

Example 1.1: Exact Solution*

In this section, we will derive the exact solution to Example 1.1 and compare the finite element formulation displacement results for this problem to the exact displacement solutions. As shown in Figure 1.20, the statics equilibrium requires the sum of the forces in the y-direction to be zero. This requirement leads to the relation

$$P - (\sigma_{avg})A(y) = 0 \tag{1.48}$$

Once again, using Hooke's law ($\sigma = E\varepsilon$) and substituting for the average stress in terms of the strain, we have

$$P - E\varepsilon A(y) = 0 \tag{1.49}$$

Recall that the average normal strain is the change in length du per unit original length of the differential segment dy. So,

$$\varepsilon = \frac{du}{dy}$$

If we substitute this relationship into Eq. (1.49), we now have

$$P - EA(y)\frac{du}{dy} = 0 \tag{1.50}$$

Rearranging Eq. (1.50), we get

$$du = \frac{Pdy}{EA(y)} \tag{1.51}$$

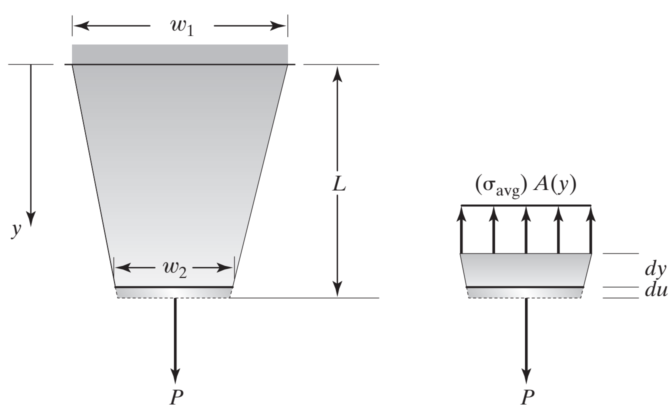


FIGURE 1.20 The relationship between the external force P and the average stresses for the bar in Example 1.1.

*The contribution of shear stresses is neglected.

TABLE 1.7 Comparison of displacement results

Location of a Point Along the Bar (in)	Results from the Exact Displacement Method (in) Eq. (1.53)	Results from the Direct Method (in)	Results from the Energy Method (in)
$y = 0$	0	0	0
$y = 2.5$	0.001027	0.001026	0.001026
$y = 5.0$	0.002213	0.002210	0.002210
$y = 7.5$	0.003615	0.003608	0.003608
$y = 10$	0.005333	0.005317	0.005317

The exact solution is then obtained by integrating Eq. (1.51) over the length of the bar

$$\int_0^u du = \int_0^L \frac{Pdy}{EA(y)}$$

$$u(y) = \int_0^y \frac{Pdy}{EA(y)} = \int_0^y \frac{Pdy}{E\left(w_1 + \left(\frac{w_2 - w_1}{L}\right)y\right)t} \tag{1.52}$$

where the area is

$$A(y) = \left(w_1 + \left(\frac{w_2 - w_1}{L}\right)y\right)t$$

The deflection profile along the bar is obtained by integrating Eq. (1.52), resulting in

$$u(y) = \frac{PL}{Et(w_2 - w_1)}\left[\ln\left(w_1 + \left(\frac{w_2 - w_1}{L}\right)y\right) - \ln w_1\right] \tag{1.53}$$

Equation (1.53) can be used to generate displacement values at various points along the bar. It is now appropriate to examine the accuracy of the direct and potential energy methods by comparing their displacement results with the values. Table 1.7 shows nodal displacements computed using direct and energy methods.

It is clear from examination of Table 1.7 that all of the results are in agreement with each other.

1.7 WEIGHTED RESIDUAL FORMULATIONS

The *weighted residual methods* are based on assuming an approximate solution for the governing differential equation. The assumed solution must satisfy the initial and boundary conditions of the given problem. Because the assumed solution is not exact, substitution of the solution into the differential equation will lead to some *residuals* or *errors*. Simply stated, each residual method requires the error to vanish over some selected intervals or at some points. To demonstrate this concept, let's turn our attention to Example 1.1. The governing differential equation and the corresponding boundary condition for this problem are as follows:

$$A(y)E\frac{du}{dy} - P = 0 \quad \text{subject to the boundary condition } u(0) = 0 \tag{1.54}$$

Next, we need to assume an approximate solution. Again, keep in mind that the assumed solution must satisfy the boundary condition. We choose

$$u(y) = c_1 y + c_2 y^2 + c_3 y^3 \tag{1.55}$$

where c_1, c_2, and c_3 are unknown coefficients. Equation (1.55) certainly satisfies the fixed boundary condition represented by $u(0) = 0$. Substitution of the assumed solution, Eq. (1.55), into the governing differential equation, Eq. (1.54), yields the error function $\mathcal{R}$:

$$\overbrace{\left(w_1 + \left(\frac{w_2 - w_1}{L}\right)y\right)}^{A(y)} tE\overbrace{(c_1 + 2c_2 y + 3c_3 y^2)}^{\frac{du}{dy}} - P = \mathcal{R} \tag{1.56}$$

Substituting for values of w_1, w_2, L, t, and E in Example 1.1 and simplifying, we get

$$\mathcal{R}/E = (0.25 - 0.0125y)(c_1 + 2c_2 y + 3c_3\, y^2) - 96.154 \times 10^{-6}$$

Collocation Method

In the *collocation method* the error, or residual function $\mathcal{R}$ is forced to be zero at as many points as there are unknown coefficients. Because the assumed solution in this example has three unknown coefficients, we will force the error function to equal zero at three points. We choose the error function to vanish at $y = L/3$, $y = 2L/3$, and $y = L$:

$$\mathcal{R}(c, y)\Big|_{y=\frac{L}{3}} = 0$$

$$\mathcal{R} = \left(0.25 - 0.0125\left(\frac{10}{3}\right)\right)\left(c_1 + 2c_2\left(\frac{10}{3}\right) + 3c_3\left(\frac{10}{3}\right)^2\right) - 96.154 \times 10^{-6} = 0$$

$$\mathcal{R}(c, y)\Big|_{y=\frac{2L}{3}} = 0$$

$$\mathcal{R} = \left(0.25 - 0.0125\left(\frac{20}{3}\right)\right)\left(c_1 + 2c_2\left(\frac{20}{3}\right) + 3c_3\left(\frac{20}{3}\right)^2\right) - 96.154 \times 10^{-6} = 0$$

$$\mathcal{R}(c, y)\Big|_{y=L} = 0$$

$$\mathcal{R} = (0.25 - 0.0125(10))(c_1 + 2c_2(10) + 3c_3(10)^2) - 96.154 \times 10^{-6} = 0$$

This procedure creates three linear equations that we can solve to obtain the unknown coefficients c_1, c_2, and c_3:

$$c_1 + \frac{20}{3}c_2 + \frac{100}{3}c_3 = 461.539 \times 10^{-6}$$
$$c_1 + \frac{40}{3}c_2 + \frac{400}{3}c_3 = 576.924 \times 10^{-6}$$
$$c_1 + 20c_2 + 300c_3 = 769.232 \times 10^{-6}$$

Solving the above equations yields $c_1 = 423.0776 \times 10^{-6}$, $c_2 = 21.65 \times 10^{-15}$, and $c_3 = 1.153848 \times 10^{-6}$. Substitution of the c-coefficients into Eq. (1.55) yields the approximate displacement profile:

$$u(y) = 423.0776 \times 10^{-6}y + 21.65 \times 10^{-15}y^2 + 1.153848 \times 10^{-6}y^3 \qquad (1.57)$$

In order to get an idea of how accurate the collocation approximate results are, we will compare them to the exact results later in this chapter.

Subdomain Method

In the *subdomain method,* the integral of the error function over some selected subintervals is forced to be zero. The number of subintervals chosen must equal the number of unknown coefficients. Thus, for our assumed solution, we will have three integrals:

$$\int_0^{\frac{L}{3}} \mathcal{R}\, dy = 0 \qquad (1.58)$$

$$\int_0^{\frac{L}{3}} [(0.25 - 0.0125y)(c_1 + 2c_2y + 3c_3y^2) - 96.154 \times 10^{-6}]dy = 0$$

$$\int_{\frac{L}{3}}^{\frac{2L}{3}} \mathcal{R}\, dy = 0$$

$$\int_{\frac{L}{3}}^{\frac{2L}{3}} [(0.25 - 0.0125y)(c_1 + 2c_2y + 3c_3y^2) - 96.154 \times 10^{-6}]dy = 0$$

$$\int_{\frac{2L}{3}}^{L} \mathcal{R}\, dy = 0$$

$$\int_{\frac{2L}{3}}^{L} [(0.25 - 0.0125y)(c_1 + 2c_2y + 3c_3y^2) - 96.154 \times 10^{-6}]dy = 0$$

Integration of equations given by Eq. (1.58) results in three linear equations that we can solve to obtain the unknown coefficients c_1, c_2, and c_3:

$$763.88889 \times 10^{-3}c_1 + 2.4691358c_2 + 8.1018519c_3 = 320.513333 \times 10^{-6}$$
$$0.625c_1 + 6.1728395c_2 + 47.4537041c_3 = 3.2051333 \times 10^{-4}$$
$$0.4861111c_1 + 8.0246917c_2 + 100.694444c_3 = 3.2051333 \times 10^{-4}$$

Solving the above equations yields $c_1 = 391.35088 \times 10^{-6}$, $c_2 = 6.075 \times 10^{-6}$, and $c_3 = 809.61092 \times 10^{-9}$. Substitution of the c-coefficients into Eq. (1.55) yields the approximate displacement profile:

$$u(y) = 391.35088 \times 10^{-6}y + 6.075 \times 10^{-6}y^2 + 809.61092 \times 10^{-9}y^3 \qquad (1.59)$$

We will compare the displacement results obtained from the subdomain method to the exact results later in this chapter.

Galerkin Method

The *Galerkin method* requires the error to be orthogonal to some weighting functions Φ_i, according to the integral

$$\int_a^b \Phi_i \mathcal{R}\, dy = 0 \qquad i = 1, 2, \ldots, N \qquad (1.60)$$

The weighting functions are chosen to be members of the approximate solution. Because there are three unknowns in the assumed approximate solution for Example 1.1, we need to generate three equations. Recall that the assumed solution is $u(y) = c_1y + c_2y^2 + c_3y^3$; thus, the weighting functions are selected to be $\Phi_1 = y$, $\Phi_2 = y^2$, and $\Phi_3 = y^3$. This selection leads to the following equations:

$$\int_0^L y[(0.25 - 0.0125y)(c_1 + 2c_2y + 3c_3y^2) - 96.154 \times 10^{-6}]dy = 0 \qquad (1.61)$$

$$\int_0^L y^2[(0.25 - 0.0125y)(c_1 + 2c_2y + 3c_3y^2) - 96.154 \times 10^{-6}]dy = 0$$

$$\int_0^L y^3[0.25 - 0.0125y)(c_1 + 2c_2y + 3c_3y^2) - 96.154 \times 10^{-6}]dy = 0$$

Integration of Eq. (1.61) results in three linear equations that we can solve to obtain the unknown coefficients c_1, c_2, and c_3:

$$8.333333c_1 + 104.1666667c_2 + 1125c_3 = 0.0048077$$

$$52.083333c_1 + 750c_2 + 8750c_3 = 0.0320513333$$

$$375c_1 + 5833.3333c_2 + 71428.57143c_3 = 0.240385$$

Solving the above equations yields $c_1 = 400.642 \times 10^{-6}$, $c_2 = 4.006 \times 10^{-6}$, and $c_3 = 0.935 \times 10^{-6}$. Substitution of the c-coefficients into Eq. (1.55) yields the approximate displacement profile:

$$u(y) = 400.642 \times 10^{-6}y + 4.006 \times 10^{-6}y^2 + 0.935 \times 10^{-6}y^3 \qquad (1.62)$$

We will compare the displacement results obtained from the Galerkin method to the exact results later in this chapter.

Least-Squares Method

The *least-squares method* requires the error to be minimized with respect to the unknown coefficients in the assumed solution, according to the relationship

$$\text{Minimize}\left(\int_a^b \mathcal{R}^2 dy\right)$$

which leads to

$$\int_a^b \mathcal{R}\frac{\partial \mathcal{R}}{\partial c_i} dy = 0 \qquad i = 1, 2, \ldots, N \tag{1.63}$$

Because there are three unknowns in the approximate solution of Example 1.1, Eq. (1.63) generates three equations. Recall that the error function is

$$\mathcal{R}/E = (0.25 - 0.0125y)(c_1 + 2c_2y + 3c_3y^2) - 96.154 \times 10^{-6}$$

Differentiating the error function with respect to c_1, c_2, and c_3 and substituting into Eq. (1.63), we have:

$$\int_0^{10} \overbrace{[(0.25 - 0.0125y)(c_1 + 2c_2y + 3c_3y^2) - 96.154 \times 10^{-6}]}^{\mathcal{R}} \overbrace{(0.25 - 0.0125y)}^{\frac{\partial \mathcal{R}}{\partial c_1}} dy = 0$$

$$\int_0^{10} \overbrace{[(0.25 - 0.0125y)(c_1 + 2c_2y + 3c_3y^2) - 96.154 \times 10^{-6}]}^{\mathcal{R}} \overbrace{(0.25 - 0.0125y)2y}^{\frac{\partial \mathcal{R}}{\partial c_2}} dy = 0$$

$$\int_0^{10} \overbrace{[(0.25 - 0.0125y)(c_1 + 2c_2y + 3c_3y^2) - 96.154 \times 10^{-6}]}^{\mathcal{R}} \overbrace{(0.25 - 0.0125y)3y^2}^{\frac{\partial \mathcal{R}}{\partial c_3}} dy = 0$$

Integration of the above equations results in three linear equations that we can solve to obtain the unknown coefficients c_1, c_2, and c_3:

$$0.364583333c_1 + 2.864583333c_2 + 25c_3 = 0.000180289$$
$$2.864583333c_1 + 33.333333c_2 + 343.75c_3 = 0.001602567$$
$$25c_1 + 343.75c_2 + 3883.928571c_3 = 0.015024063$$

Solving the set of equations simultaneously yields $c_1 = 389.773 \times 10^{-6}$, $c_2 = 6.442 \times 10^{-6}$, and $c_3 = 0.789 \times 10^{-6}$. Substitution of the c-coefficients into Eq. (1.55) yields the approximate displacement profile:

$$u(y) = 389.733 \times 10^{-6}y + 6.442 \times 10^{-6}y^2 + 0.789 \times 10^{-6}y^3 \tag{1.64}$$

Next, we will compare the displacement results obtained from the least-squares method and the other weighted residual methods to the exact results.

TABLE 1.8 Comparison of weighted residual results

Location of a Point Along the Bar (in)	Displacement Results from the Exact Solution Eq. (1.53) (in)	Displacement Results from the Collocation Method Eq. (1.57) (in)	Displacement Results from the Subdomain Method Eq. (1.59) (in)	Displacement Results from the Galerkin Method Eq. (1.62) (in)	Displacement Results from the Least-Squares Method Eq. (1.64) (in)
$y = 0$	0	0	0	0	0
$y = 2.5$	0.001027	0.001076	0.001029	0.001041	0.001027
$y = 5.0$	0.002213	0.002259	0.002209	0.002220	0.002208
$y = 7.5$	0.003615	0.003660	0.003618	0.003624	0.003618
$y = 10$	0.005333	0.005384	0.005330	0.005342	0.005331

Comparison of Weighted Residual Solutions

Now we will examine the accuracy of weighted residual methods by comparing their displacement results with the exact values. Table 1.8 shows nodal displacements computed using the exact, collocation, subdomain, Galerkin, and least-squares methods.

It is clear from an examination of Table 1.8 that the results are in agreement with each other. It is also important to note here that the primary purpose of Section 1.7 was to introduce you to the general concepts of weighted residual methods and the basic procedures in the simplest possible way. Because the Galerkin method is one of the most commonly used procedures in finite element formulations, more detail and an in-depth view of the Galerkin method will be offered later in Chapters 6 and 9. We will employ the Galerkin method to formulate one- and two-dimensional problems once you have become familiar with the ideas of one- and two-dimensional elements. Also note that in the above examples of the use of weighted residual methods, we assumed a solution that was to provide an approximate solution over the entire domain of the given problem. As you will see later, we will use piecewise solutions with the Galerkin method. That is to say, we will assume linear or nonlinear solutions that are valid only over each element and then combine, or assemble, the elemental solutions.

1.8 VERIFICATION OF RESULTS

In recent years, the use of finite element analysis as a design tool has grown rapidly. Easy-to-use, comprehensive packages such as ANSYS have become a common tool in the hands of design engineers. Unfortunately, many engineers without the proper training or a solid understanding of the underlying concepts have been using finite element analysis. Engineers who use finite element analysis must understand the limitations of the finite element procedures. There are various sources of error that can contribute to incorrect results. They include

1. *Wrong input data, such as physical properties and dimensions*

   This mistake can be corrected by simply listing and verifying physical properties and coordinates of nodes or keypoints (points defining the vertices of an object;

they are covered in more detail in Chapters 8 and 13) before proceeding any further with the analysis.

2. *Selecting inappropriate types of elements*
Understanding the underlying theory will benefit you the most in this respect. You need to fully grasp the limitations of a given type of element and understand to which type of problems it applies.

3. *Poor element shape and size after meshing*
This area is a very important part of any finite element analysis. Inappropriate element shape and size will influence the accuracy of your results. It is important that the user understands the difference between free meshing (using mixed-area element shapes) and mapped meshing (using all quadrilateral area elements or all hexahedral volume elements) and the limitations associated with them. These concepts will be explained in more detail in Chapters 8 and 13.

4. *Applying wrong boundary conditions and loads*
This step is usually the most difficult aspect of modeling. It involves taking an actual problem and estimating the loading and the appropriate boundary conditions for a finite element model. This step requires good judgment and some experience.

You must always find ways to check your results. While experimental testing of your model may be the best way to do so, it may be expensive or time consuming. You should always start by applying equilibrium conditions and energy balance to different portions of a model to ensure that the physical laws are not violated. For example, for static models, the sum of the forces acting on a free-body diagram of your model must be zero. This concept will allow you to check for the accuracy of computed reaction forces. You may want to consider defining and mapping stresses along an arbitrary cross section and integrating this information. The resultant internal forces computed in this manner must balance against external forces. In a heat transfer problem under steady-state conditions, apply conservation of energy to a control volume surrounding an arbitrary node. Are the energies flowing into and out of a node balanced? At the end of particular chapters in this text, a section is devoted to verifying the results of your models. In these sections, problems will be solved using ANSYS, and the steps for verifying results will be shown.

1.9 UNDERSTANDING THE PROBLEM

You can save lots of time and money if you first spend a little time with a piece of paper and a pencil to try to understand the problem you are planning to analyze. Before initiating numerical modeling on the computer and generating a finite element model, it is imperative that you develop a sense of or a feel for the problem. There are many questions that a good engineer will ask before proceeding with the modeling process: Is the material under axial loading? Is the body under bending moments or twisting moments or a combination of the two? Do we need to worry about buckling? Can we approximate the behavior of the material with a two-dimensional model? Does heat transfer play a significant role in the problem? Which modes of heat transfer are influential? If you choose to employ FEA, "back-of-the-envelope" calculations will greatly enhance your understanding of the problem, in turn helping you to develop a good, reasonable finite element model, particularly in terms of your selection of element types. Some practicing engineers still use finite element analysis to solve a problem that could have been solved more easily by hand by someone with a good grasp of the fundamental concepts of the mechanics of materials and heat transfer. To shed more light on this very important point, consider the following examples.

EXAMPLE 1.6

Imagine that by mistake, an empty coffee pot has been left on a heating element. Assuming that the heater puts approximately 20 Watts (typically, a heater creates a lower wattage) into the bottom of the pot, determine the temperature distribution within the glass if the surrounding air is at 25°C, with a corresponding heat transfer coefficient $h = 15\ \mathrm{W/m^2 \cdot K}$. The pot is cylindrical in shape, with a diameter of 14 cm and height of 14 cm, and the glass is 3 mm thick.

This problem is first analyzed using a finite element model. After you study three-dimensional thermal-solid elements (discussed in Chapter 13), you will revisit this problem (see Problem 13.11) and be asked to solve it using ANSYS. As you will learn later, a solid model of the pot is created and meshed and the appropriate boundary conditions are applied and the temperature solutions is then obtained. The results of this analysis is shown in Figure 1.21.

From the results of finite element analysis we find that the maximum temperature of 113.18°C occurs at the bottom of the pot in the center location, as shown in Figure 1.21. This is a good example of a problem that could have been solved more easily by hand by someone with a good grasp of the fundamental concepts of heat transfer. We can approximate the temperature of the glass by applying the energy balance to the bottom of the pot and assuming a one-dimensional model. Because the pot is made of thin glass, we can neglect the spatial temperature variation within the glass. Under steady-state

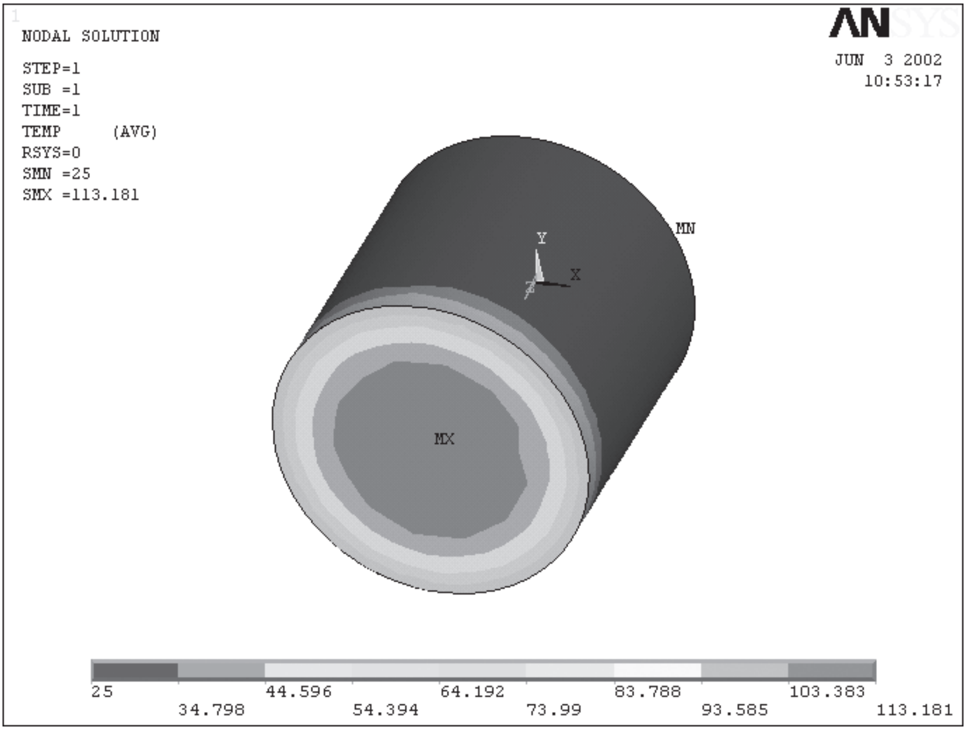


FIGURE 1.21 The temperature distribution in the pot of Example 1.6.

conditions, the heat flux added into the bottom of the glass is approximately equal to the rate of energy convected away by air. Thus, we employ Newton's law of cooling

$$q'' = h(T_s - T_f) \tag{1.65}$$

where

q'' = heat flux, W/m^2
h = heat transfer coefficient, W/m$^2 \cdot$°C (W/m$^2 \cdot$K)
T_s = surface temperature of the coffee pot,°C
T_f = surrounding air temperature,°C

We can estimate the heat flux into the bottom of the pot:

$$q'' = \frac{20 \text{ W}}{\frac{\pi}{4}(0.14 \text{ m})^2} = 1299 \text{ W/m}^2$$

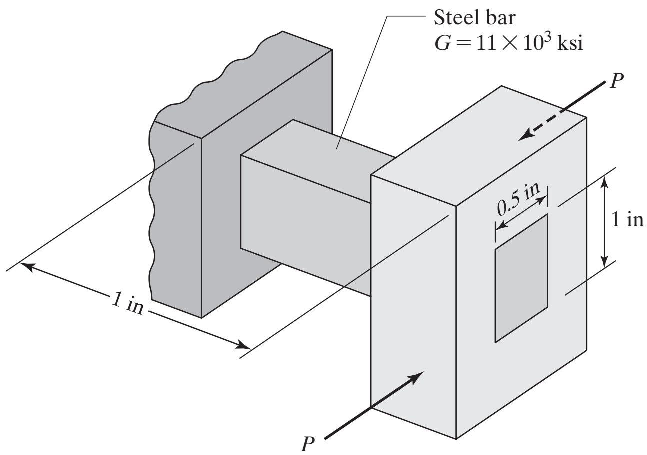
and substituting for heat flux, h, and T_f into Eq. (1.65), and solving for T_s,

$$1299 \text{ W/m}^2 = (15 \text{ W/m} \cdot \text{°C})(T_s - 25) \qquad \rightarrow \qquad T_s = 111.6\text{°C}$$

As you can see, the temperature result obtained by hand calculation ($T_s = 111.6°\text{C}$) is very close to the result of our finite element model ($T_{\max} = 113.18°\text{C}$). Thus, there was no need to resort to finite element formulation to solve the above problem.

EXAMPLE 1.7

Consider the torsion of a steel bar ($G = 11 \times 10^3$ ksi) having a rectangular cross section, as shown in the accompanying figure. Under the loading shown, the angle of twist is measured to be $\theta = 0.0005$ rad/in. We are interested in determining the location(s) and magnitude of the maximum shear stress.



Again, we have analyzed this problem using a finite element model. All of the steps leading to the ANSYS solution are given in Example 10.1 (revisited). The results of this analysis are shown in Figure 1.22.

The results of finite element analysis show that the maximum shear stress of 2558 lb/in^2 occurs at the midsection of the rectangle. This is another example of a problem that could have been solved more easily by hand by someone with a good grasp of the fundamental concepts of mechanics of materials.

As you will learn in Chapter 10, Section 10.1, there are analytical solutions that we could employ to solve problems dealing with torsion of members with rectangular cross-sectional area. When a torque is applied to a straight bar with a rectangular cross-sectional area, within the elastic region of the material, the maximum shearing stress and an angle of twist caused by the torque are given by

$$\tau_{\max} = \frac{T}{c_1 w h^2}$$

where

$\tau_{\max}$ = maximum shear stress, lb/in^2

T = applied torque, lb · in

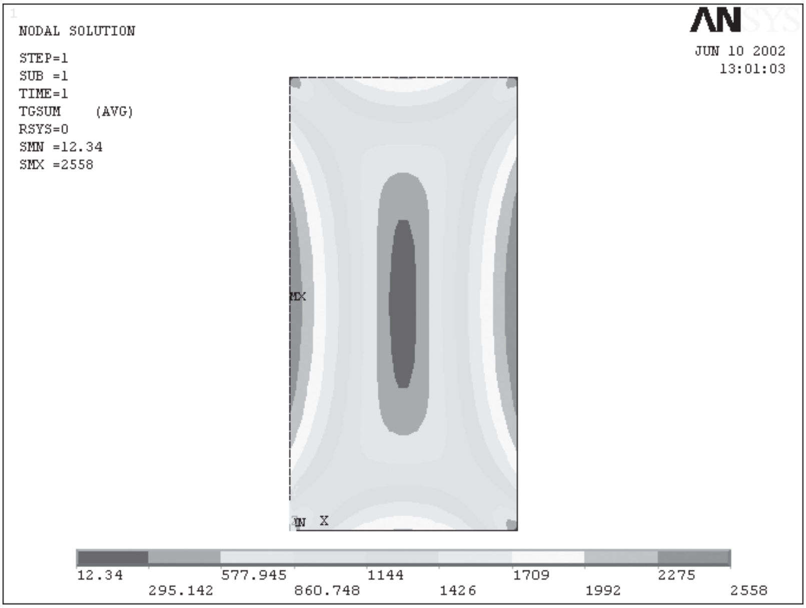


FIGURE 1.22 The shear stress distribution for the steel bar of Example 1.7.

w = width of the rectangular cross-section, in

h = height of the rectangular cross-section, in

c_1 = a constant coefficient that depends on aspect ratio of the cross section, 0.246; see Table 10.1

and

$$\theta = \frac{TL}{c_2\,Gwh^3}$$

L = length of the bar, in

G = shear modulus or modulus of rigidity of material, lb/in^2

c_2 = a constant coefficient that depends on aspect ratio of the cross section, 0.229; see Table 10.1

Substituting into the above equations appropriate values, we get

$$\theta = \frac{TL}{c_2\,Gwh^3} = 0.0005\text{ rad/in} = \frac{T(1\text{ in})}{0.229(11 \times 10^6\text{ lb/in}^2)(1\text{ in})(0.5\text{ in})^3} \Rightarrow T = 157.5\text{ lb}\cdot\text{in}$$

$$\tau_{\max} = \frac{T}{c_1\,wh^2} = \frac{157.5\text{ lb}\cdot\text{in}}{0.246(1\text{ in})(0.5\text{ in})^2} = 2560\text{ lb/in}^2$$

When comparing 2560 lb/in^2 to the FEA results of 2558 lb/in^2, you see that we could have saved lots of time by calculating the maximum shear stress using the analytical solution and avoided generating a finite element model.

SUMMARY

At this point you should

1. have a good understanding of the physical properties and the parameters that characterize the behavior of an engineering system. Examples of these properties and design parameters are given in Tables 1.2 and 1.3.
2. realize that a good understanding of the fundamental concepts of the finite element method will benefit you by enabling you to use ANSYS more effectively.
3. know the seven basic steps involved in any finite element analysis, as discussed in Section 1.4.
4. understand the differences among direct formulation, minimum total potential energy formulation, and the weighted residual methods (particularly the Galerkin formulation).
5. know that it is wise to spend some time to gain a full understanding of a problem before initiating a finite element model of the problem. There may even exist a reasonable closed-form solution to the problem, and thus you can save lots of time and money.
6. realize that you must always find a way to verify your FEA results.

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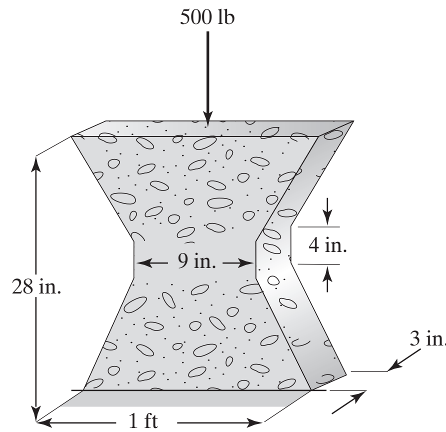
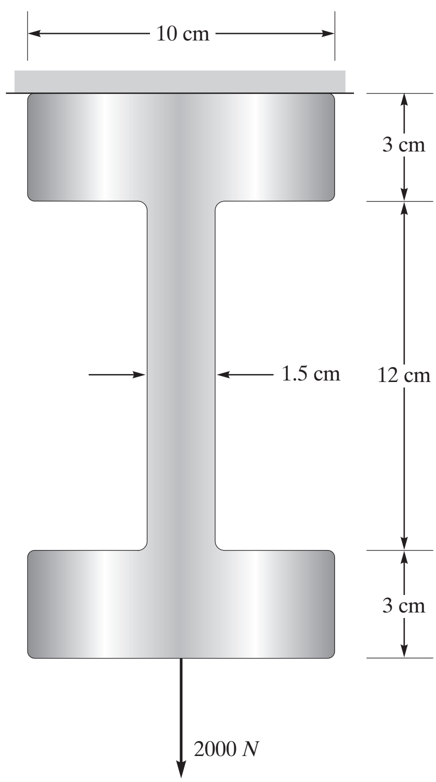
PROBLEMS

1. Solve Example 1.1 using: (a) two elements, and (b) eight elements. Compare your results to the exact values.
2. A concrete table column-support with the profile shown in the accompanying figure is to carry a load of approximately 500 lb. Using the direct method discussed in Section 1.5,

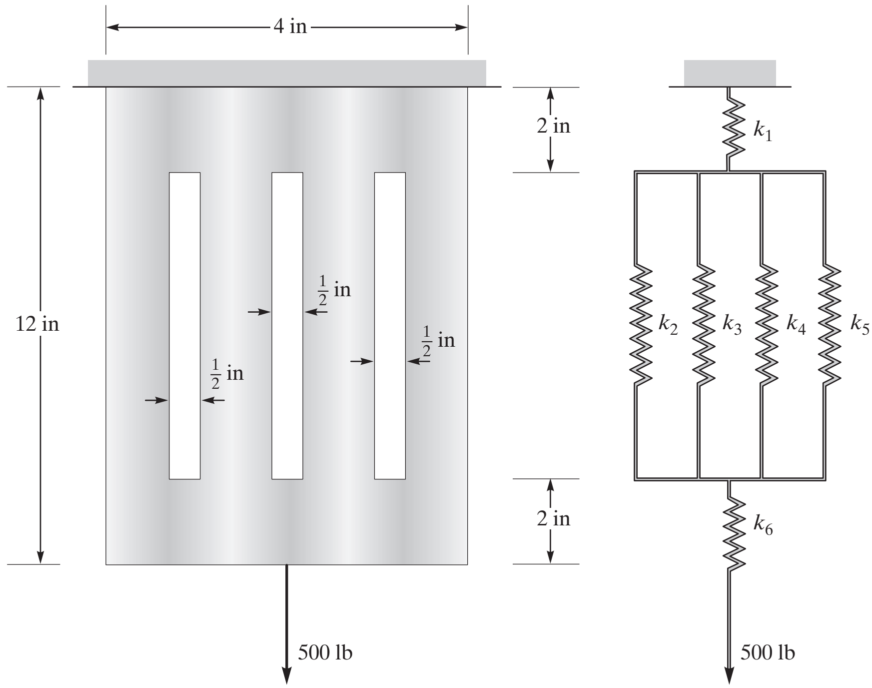
determine the deflection and average normal stresses along the column. Divide the column into five elements. ($E = 3.27 \times 10^3$ ksi)



3. An aluminum strap with a thickness of 5 mm and the profile shown in the accompanying figure is to carry a load of 2000 N. Using the direct method discussed in Section 1.5, determine the deflection and the average normal stress along the strap. Divide the strap into three elements. This problem may be revisited again in Chapter 10, where a more in-depth analysis may be sought. ($E = 70$ GPa)

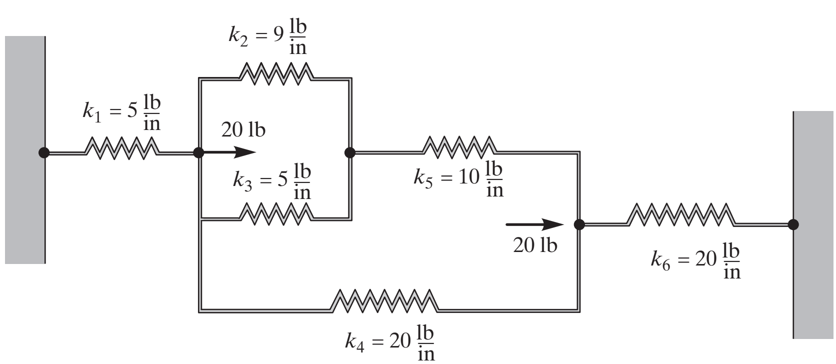


4. A thin steel plate with the profile shown in the accompanying figure is subjected to an axial load. Approximate the deflection and the average normal stresses along the plate using the model shown in the figure. The plate has a thickness of 0.125 in and a modulus of elasticity $E = 28 \times 10^3$ ksi. You will be asked to use ANSYS to analyze this problem again in Chapter 10.



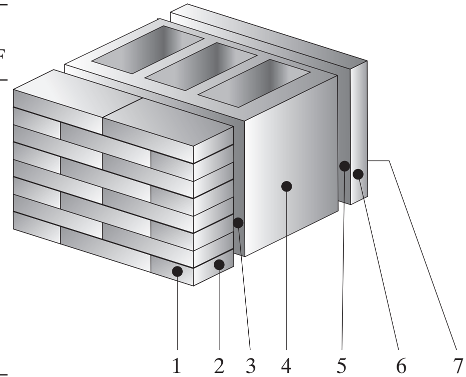
5. Apply the statics equilibrium conditions directly to each node of the thin steel plate (using a finite element model) in Problem 4.

6. For the spring system shown in the accompanying figure, determine the displacement of each node. Start by identifying the size of the global matrix. Write down elemental stiffness matrices, and show the position of each elemental matrix in the global matrix. Apply the boundary conditions and loads. Solve the set of linear equations. Also compute the reaction forces.



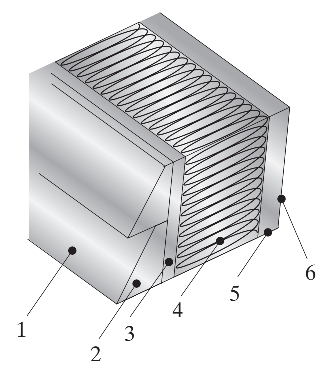
7. A typical exterior masonry wall of a house, shown in the accompanying figure, consists of the items in the accompanying table. Assume an inside room temperature of 68°F and an outside air temperature of 10°F, with an exposed area of 150 ft^2. Determine the temperature distribution through the wall. Also calculate the heat loss through the wall.

Items	Resistance hr · ft^2 · °F/Btu	*U*-factor Btu/hr · ft^2 · °F
1. Outside film resistance (winter, 15-mph wind)	0.17	5.88
2. Face brick (4 in)	0.44	2.27
3. Cement mortar (1/2 in)	0.1	10.0
4. Cinder block (8 in)	1.72	0.581
5. Air space (3/4 in)	1.28	0.781
6. Gypsum board (1/2 in)	0.45	2.22
7. Inside film resistance (winter)	0.68	1.47



8. In order to increase the thermal resistance of a typical exterior frame wall, such as the one in Example 1.2, it is customary to use 2 × 6 studs instead of 2 × 4 studs to allow for placement of more insulation within the wall cavity. A typical exterior (2 × 6) frame wall of a house consists of the materials shown in the accompanying figure. Assume an inside room temperature of 68°F and an outside air temperature of 20°F, with an exposed area of 150 ft^2. Determine the temperature distribution through the wall.

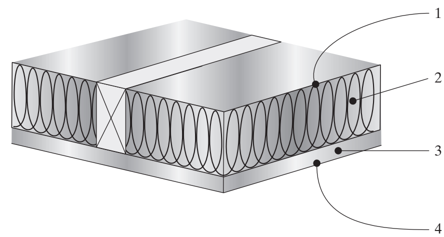
Items	Resistance hr · ft^2 · °F/Btu	*U*-factor Btu/hr · ft^2 · °F
1. Outside film resistance (winter, 15-mph wind)	0.17	5.88
2. Siding, wood (1/2 × 8 lapped)	0.81	1.23
3. Sheathing (1/2 in regular)	1.32	0.76
4. Insulation batt ($5\frac{1}{2}$ in)	19.0	0.053
5. Gypsum wall board (1/2 in)	0.45	2.22
6. Inside film resistance (winter)	0.68	1.47



9. Assuming the moisture can diffuse through the gypsum board in Problem 8, where should you place a vapor barrier to avoid moisture condensation? Assume an indoor air temperature of 68°F with relative humidity of 40%.

10. A typical ceiling of a house consists of the items in the accompanying table. Assume an inside room temperature of 75°F and an attic air temperature of 25°F, with an exposed area of 1000 ft^2. Determine the temperature distribution through the ceiling. Also calculate heat loss through the ceiling.

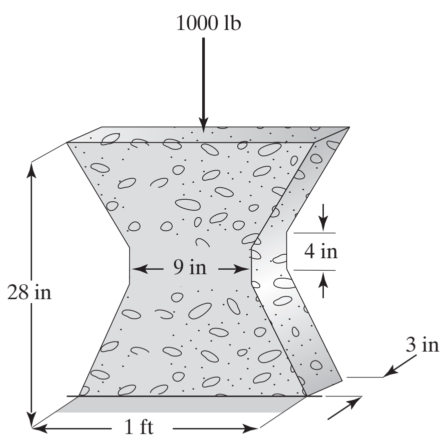
Items	Resistance hr · ft² · °F/Btu	*U*-factor Btu/hr · ft² · °F
1. Inside attic film resistance	0.7	1.5
2. Insulation batt (6 in)	19.0	0.05
3. Gypsum board (1/2 in)	0.5	2.3
4. Inside film resistance (winter)	0.7	1.5

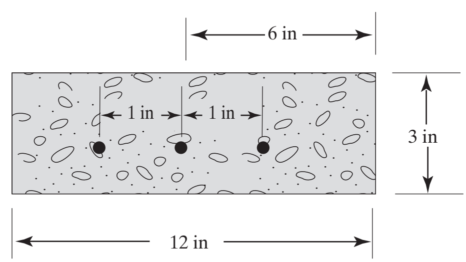


11. A typical $1\frac{3}{8}$-in solid wood core door exposed to winter conditions has the characteristics shown in the accompanying table. Assume an inside room temperature of 50°F and an outside air temperature of 25°F, with an exposed area of 25 ft². (a) Determine the inside and outside temperatures of the door's surface. (b) Determine heat loss through the door.

Items	Resistance hr · ft² · °F/Btu	*U*-factor Btu/hr · ft² · °F
1. Outside film resistance (winter, 15-mph wind)	0.15	6
2. $1\frac{3}{8}$-in solid wood core	0.4	2.5
3. Inside film resistance (winter)	0.68	1.5

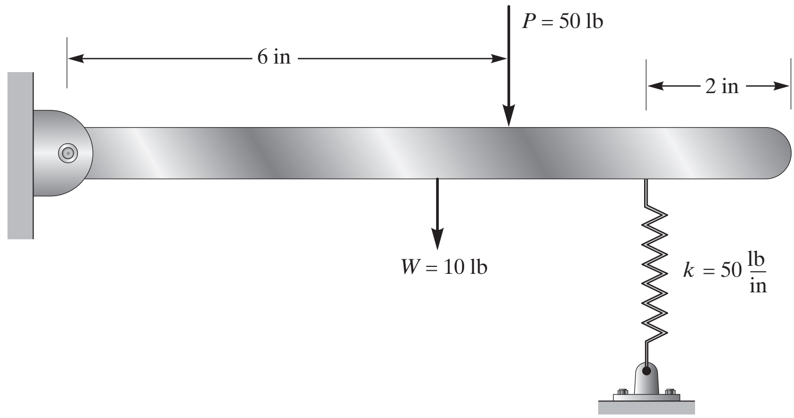
12. The concrete table column-support in Problem 2 is reinforced with three $\frac{1}{2}$-in steel rods, as shown in the accompanying figure. Determine the deflection and average normal stresses along the column under a load of 1000 lb. Divide the column into five elements. ($E_C = 3.27 \times 10^3$ ksi; $E_s = 29 \times 10^3$ ksi)



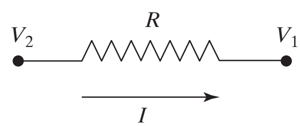


13. Compute the total strain energy for the concrete table column-support in Problem 12.

14. A 10-in slender rod weighing 10 lb is supported by a spring with a stiffness $k = 50$ lb/in. A force $P = 50$ lb is applied to the rod at the location shown in the accompanying figure. Determine the deflection of the spring (a) by drawing a free-body diagram of the rod and applying the statics equilibrium conditions, and (b) by applying the minimum total potential energy concept.



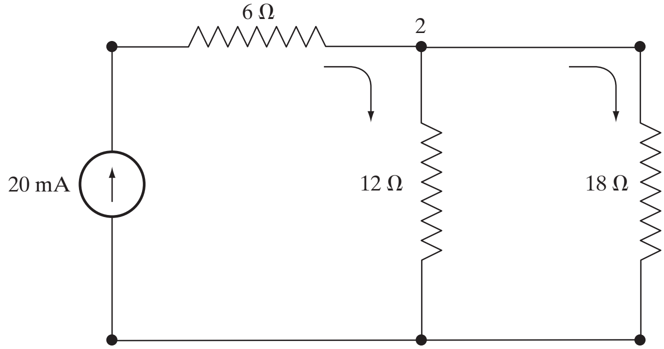
15. In a DC electrical circuit, Ohm's law relates the voltage drop $V_2 - V_1$ across a resistor to a current I flowing through the element and the resistance R according to the equation $V_2 - V_1 = RI$.



Using direct formulation, show that for a resistance element comprising two nodes, the conductance matrix, the voltage drop, and the currents are related according to the equation

$$\frac{1}{R}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = \begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix}$$

16. Use the results of Problem 15 to set up and solve for the voltage drop in each branch of the circuit shown in the accompanying figure.

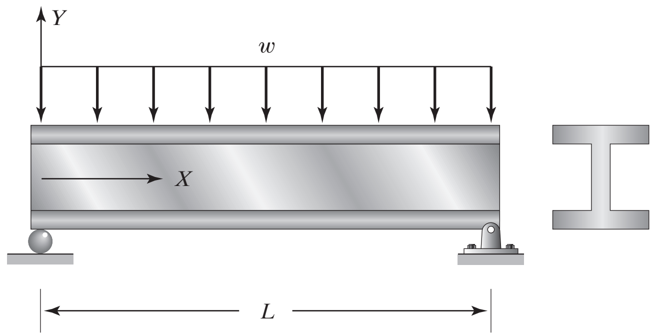


17. The deformation of a simply supported beam under a distributed load, shown in the accompanying figure, is governed by the relationship

$$\frac{d^2Y}{dX^2} = \frac{M(X)}{EI}$$

where $M(X)$ is the internal bending moment and is given by

$$M(X) = \frac{wX(L - X)}{2}$$



Derive the equation for the exact deflection. Assume an approximate deflection solution of the form

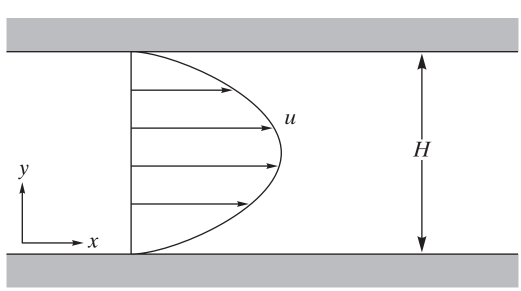
$$Y(X) = c_1\left[\left(\frac{X}{L}\right)^2 - \left(\frac{X}{L}\right)\right]$$

Use the following methods to evaluate c_1: (a) the collocation method and (b) the subdomain method. Also, using the approximate solutions, determine the maximum deflection of the beam if a W27 $\times$ 84 (wide flange shape) with a span of $L = 22$ ft supports a distributed load of $w = 6$ kips/ft.

18. For the example problem used throughout Section 1.7, assume an approximate solution of the form $u(y) = c_1y + c_2y^2 + c_3y^3 + c_4y^4$. Using the collocation, subdomain, Galerkin, and least-squares methods, determine the unknown coefficients c_1, c_2, c_3, and c_4. Compare your results to those obtained in Section 1.7.

19. The leakage flow of hydraulic fluid through the gap between a piston–cylinder arrangement may be modeled as laminar flow of fluid between infinite parallel plates, as shown in the accompanying figure. This model offers reasonable results for relatively small gaps. The differential equation governing the flow is

$$\mu\frac{d^2u}{dy^2} = \frac{dp}{dx}$$



where μ is the dynamic viscosity of the hydraulic fluid, u is the fluid velocity, and $\frac{dp}{dx}$ is the pressure drop and is constant. Derive the equation for the exact fluid velocities. Assume an approximate fluid velocity solution of the form $u(y) = c_1\left[\sin\left(\frac{\pi y}{H}\right)\right]$. Use the following methods to evaluate c_1: (a) the collocation method and (b) the subdomain method. Compare the approximate results to the exact solution.

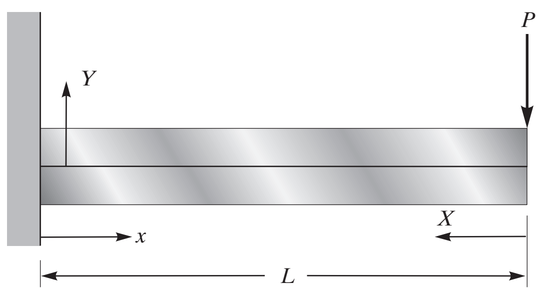
20. Use the Galerkin and least-squares methods to solve Problem 19. Compare the approximate results to the exact solution.

21. For the cantilever beam shown in the accompanying figure, the deformation of the beam under a load P is governed by the relationship

$$\frac{d^2Y}{dX^2} = \frac{M(X)}{EI}$$

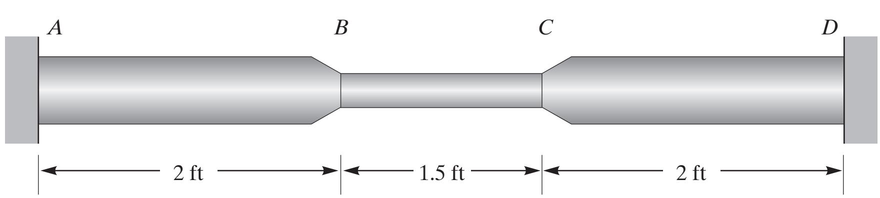
where $M(X)$ is the internal bending moment and is

$$M(X) = -PX$$



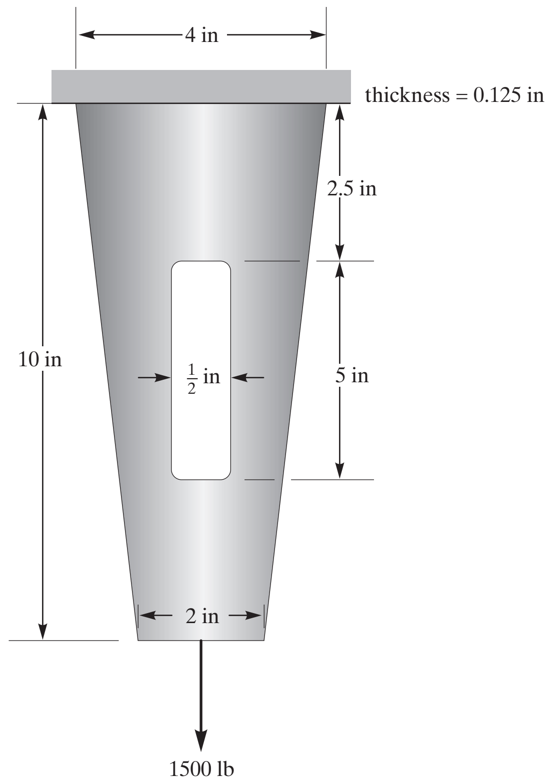
Derive the equation for the exact deflection. Assume an appropriate form of a polynomial function. Keep in mind that the assumed solution must satisfy the given boundary conditions. Use the subdomain method and the Galerkin method to solve for the unknown coefficients of the assumed solution.

22. A shaft is made of three parts, as shown in the accompanying figure. Parts AB and CD are made of the same material with a modulus of rigidity of $G = 9.8 \times 10^3$ ksi, and each has a diameter of 1.5 in. Segment BC is made of a material with a modulus of rigidity of $G = 11.2 \times 10^3$ ksi and has a diameter of 1 in. The shaft is fixed at both ends. A torque of 2400 lb · in is applied at C. Using three elements, determine the angle of twist at B and C and the torsional reactions at the boundaries.



23. For the shaft in Problem 22, replace the torque at C by two equal torques of 1500 lb · in at B and C. Compute the angle of twist at B and C and the torsional reactions at the boundaries.

24. Consider a plate with a variable cross section supporting a load of 1500 lb, as shown in the accompanying figure. Using direct formulation, determine the deflection of the bar at locations $y = 2.5$ in, $y = 7.5$ in, and $y = 10$ in. The plate is made of a material with a modulus of elasticity $E = 10.6 \times 10^3$ ksi.

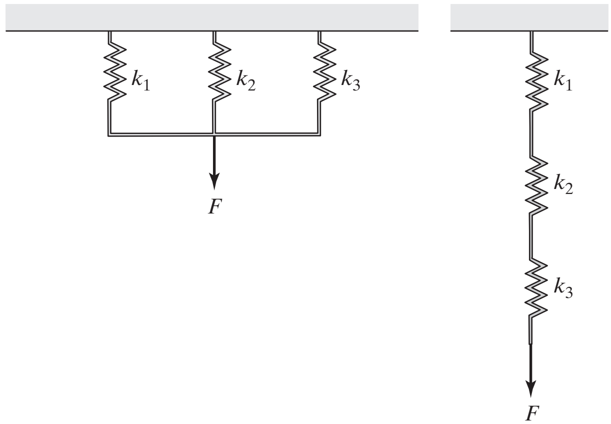


25. Consider the springs in parallel and in series, as shown in the accompanying figure. Realizing that deformation of each spring in parallel is the same, and the applied force must equal the sum of forces in individual springs, show that for the springs in parallel the equivalent spring constant k_e is

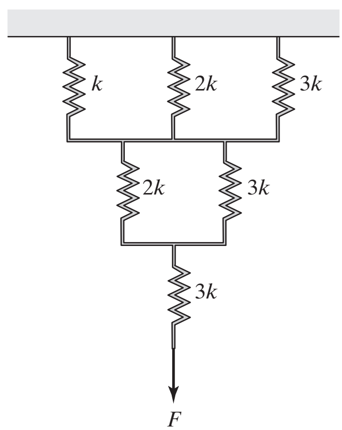
$$k_e = k_1 + k_2 + k_3$$

For the springs in series, realizing that the total deformation of the springs is the sum of the deformations of the individual springs, and the force in each spring equals the applied force, show that for the springs in series, the equivalent spring constant is

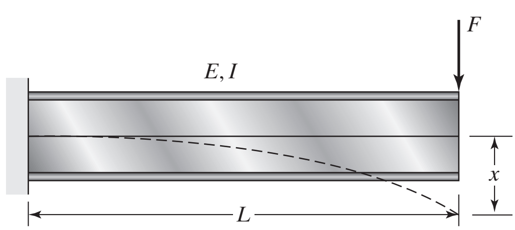
$$k_e = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}}$$



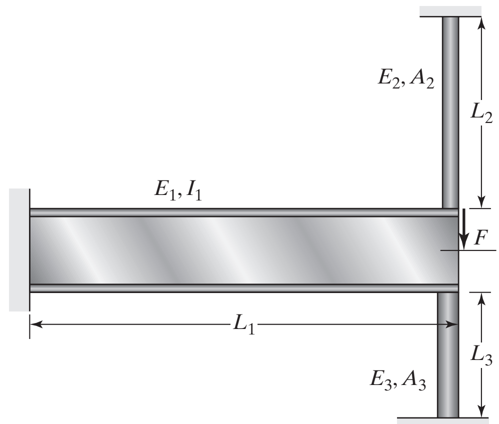
26. Use the results of Problem 25 and determine the equivalent spring constant for the system of the springs shown in the accompanying figure.



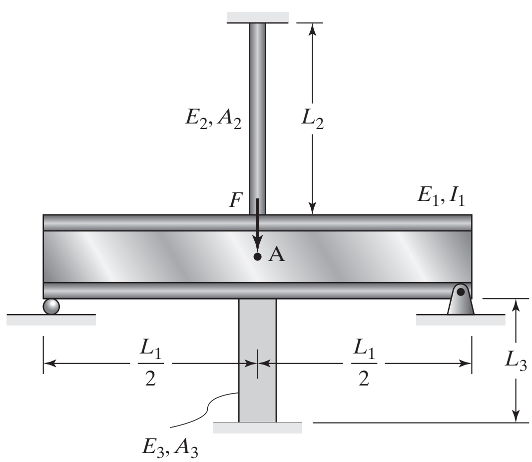
27. Determine the equivalent spring constant for the cantilever beam shown in the accompanying figure.



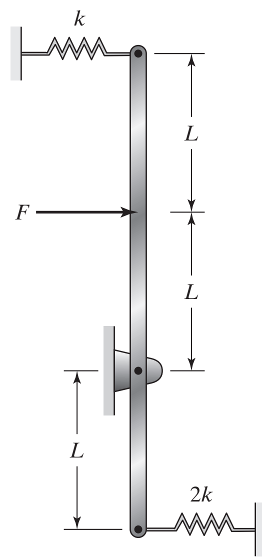
28. Use the results of Problem 27 and Eq. (1.5) to determine the equivalent spring constant for the system shown in the accompanying figure.



29. Determine the equivalent spring constant for the system shown. Determine the deflection of point A, using the minimum total potential energy concept.



30. Neglect the mass of the connecting rod and determine the deflection of each spring for the system shown in the accompanying figure (a) by applying the statics equilibrium conditions, and (b) by applying the minimum total potential energy concept.



C H A P T E R 2

Matrix Algebra

In Chapter 1 we discussed the basic steps involved in any finite element analysis. These steps include discretizing the problem into elements and nodes, assuming a function that represents behavior of an element, developing a set of equations for an element, assembling the elemental formulations to present the entire problem, and applying the boundary conditions and loading. These steps lead to a set of linear (nonlinear for some problems) algebraic equations that must be solved simultaneously. A good understanding of matrix algebra is essential in formulation and solution of finite element models. As is the case with any topic, matrix algebra has its own terminology and follows a set of rules. We provide an overview of matrix terminology and matrix algebra in this chapter. The main topics discussed in Chapter 2 include

2.1 Basic Definitions
2.2 Matrix Addition or Subtraction
2.3 Matrix Multiplication
2.4 Partitioning of a Matrix
2.5 Transpose of a Matrix
2.6 Determinant of a Matrix
2.7 Solutions of Simultaneous Linear Equations
2.8 Inverse of a Matrix
2.9 Eigenvalues and Eigenvectors
2.10 Using MATLAB to Manipulate Matrices
2.11 Using Excel to Manipulate Matrices

2.1 BASIC DEFINITIONS

A matrix is an array of numbers or mathematical terms. The numbers or the mathematical terms that make up the matrix are called the *elements of matrix*. The *size* of a matrix is defined by its number of rows and columns. A matrix may consist of m rows and n

columns. For example,

$$[\boldsymbol{N}] = \begin{bmatrix} 6 & 5 & 9 \\ 1 & 26 & 14 \\ -5 & 8 & 0 \end{bmatrix} \quad [\boldsymbol{T}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\{\boldsymbol{L}\} = \begin{Bmatrix} \dfrac{\partial f(x,y,z)}{\partial x} \\ \dfrac{\partial f(x,y,z)}{\partial y} \\ \dfrac{\partial f(x,y,z)}{\partial z} \end{Bmatrix} \quad [\boldsymbol{I}] = \begin{bmatrix} \displaystyle\int_0^L x\,dx & \displaystyle\int_0^W y\,dy \\ \displaystyle\int_0^L \frac{x^2}{L}\,dx & \displaystyle\int_0^W \frac{y^2}{L}\,dy \end{bmatrix}$$

Matrix $[\boldsymbol{N}]$ is a 3 by 3 (or 3×3) matrix whose elements are numbers, $[\boldsymbol{T}]$ is a 4×4 that has *sine* and *cosine* terms as its elements, $\{\boldsymbol{L}\}$ is a 3×1 matrix with its elements representing partial derivatives, and $[\boldsymbol{I}]$ is a 2×2 matrix with integrals for its elements. The $[\boldsymbol{N}]$, $[\boldsymbol{T}]$, and $[\boldsymbol{I}]$ are square matrices. A *square* matrix has the same number of rows and columns. The element of a matrix is denoted by its location. For example, the element in the first row and the third column of a matrix $[\boldsymbol{B}]$ is denoted by b_{13}, and an element occurring in matrix $[\boldsymbol{A}]$ in row 2 and column 3 is denoted by the term a_{23}. In this book, we denote the matrix by a **bold-face letter** in brackets [] and { }, for example: [**K**], [**T**], {**F**}, and the elements of matrices are represented by regular lowercase letters. The { } is used to distinguish a column matrix.

Column Matrix and Row Matrix

A column matrix is defined as a matrix that has one column but could have many rows. On the other hand, a row matrix is a matrix that has one row but could have many columns. Examples of column and row matrices follow.

$$\{\boldsymbol{A}\} = \begin{Bmatrix} 1 \\ 5 \\ -2 \\ 3 \end{Bmatrix}, \ \{\boldsymbol{X}\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \text{ and } \{\boldsymbol{L}\} = \begin{Bmatrix} \dfrac{\partial f(x,y,z)}{\partial x} \\ \dfrac{\partial f(x,y,z)}{\partial y} \\ \dfrac{\partial f(x,y,z)}{\partial z} \end{Bmatrix} \text{ are examples of column matrices,}$$

whereas $[\boldsymbol{C}] = [5 \quad 0 \quad 2 \quad -3]$ and $[\boldsymbol{Y}] = [y_1 \quad y_2 \quad y_3]$ are examples of row matrices.

Diagonal, Unit, and Band (Banded) Matrix

A diagonal matrix is one that has elements only along its principal diagonal; the elements are zero everywhere else ($a_{ij} = 0$ when $i \neq j$). An example of a 4×4 diagonal matrix follows:

$$[\boldsymbol{A}] = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix}$$

The diagonal along which a_1, a_2, a_3, and a_4 lies is called the *principal diagonal*. An *identity* or *unit matrix* is a diagonal matrix whose elements consist of a value 1. An example of an identity matrix follows:

$$[\boldsymbol{I}] = \begin{bmatrix} 1 & 0 & 0 & . & . & 0 & 0 \\ 0 & 1 & 0 & . & . & 0 & 0 \\ 0 & 0 & 1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 & 0 \\ 0 & 0 & 0 & . & . & 0 & 1 \end{bmatrix}$$

A *banded matrix* is a matrix that has a band of nonzero elements parallel to its principal diagonal. As shown in the example that follows, all other elements outside the band are zero.

$$[\boldsymbol{B}] = \begin{bmatrix} b_{11} & b_{12} & 0 & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 & 0 & 0 \\ 0 & b_{32} & b_{33} & b_{34} & 0 & 0 & 0 \\ 0 & 0 & b_{43} & b_{44} & b_{45} & 0 & 0 \\ 0 & 0 & 0 & b_{54} & b_{55} & b_{56} & 0 \\ 0 & 0 & 0 & 0 & b_{65} & b_{66} & b_{67} \\ 0 & 0 & 0 & 0 & 0 & b_{76} & b_{77} \end{bmatrix}$$

Upper and Lower Triangular Matrix

An *upper triangular matrix* is one that has zero elements below the principal diagonal ($u_{ij} = 0$ when $i > j$), and the *lower triangular matrix* is one that has zero elements above the principal diagonal ($l_{ij} = 0$ when $i < j$). Examples of upper triangular and lower triangular matrices are shown below.

$$[\boldsymbol{U}] = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} \qquad [\boldsymbol{L}] = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix}$$

2.2 MATRIX ADDITION OR SUBTRACTION

Two matrices can be added together or subtracted from each other provided that they are of the same size—each matrix must have the same number of rows and columns. We can add matrix $[\boldsymbol{A}]_{m \times n}$ of dimension m by n to matrix $[\boldsymbol{B}]_{m \times n}$ of the same dimension by adding the like elements. Matrix subtraction follows a similar rule, as shown.

$$[\boldsymbol{A}] \pm [\boldsymbol{B}] = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} & . & . & b_{1n} \\ b_{21} & b_{22} & . & . & b_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ b_{m1} & b_{m2} & . & . & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11} \pm b_{11)} & (a_{12} \pm b_{12}) & . & . & (a_{1n} \pm b_{1n}) \\ (a_{21} \pm b_{21}) & (a_{22} \pm b_{22}) & . & . & (a_{2n} \pm b_{2n}) \\ . & . & . & . & . \\ . & . & . & . & . \\ (a_{m1} \pm b_{m1}) & (a_{m2} \pm b_{m2}) & . & . & (a_{mn} \pm b_{mn}) \end{bmatrix}$$

The rule for matrix addition or subtraction can be generalized in the following manner. Let us denote the elements of matrix $[\boldsymbol{A}]$ by a_{ij} and the elements of matrix $[\boldsymbol{B}]$ by b_{ij}, where the number of rows i varies from 1 to m and the number of columns j varies from 1 to n. If we were to add matrix $[\boldsymbol{A}]$ to matrix $[\boldsymbol{B}]$ and denote the resulting matrix by $[\boldsymbol{C}]$, it follows that

$$[\boldsymbol{A}] + [\boldsymbol{B}] = [\boldsymbol{C}]$$

and

$$c_{ij} = a_{ij} + b_{ij} \text{ for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n \qquad (2.1)$$

2.3 MATRIX MULTIPLICATION

In this section we discuss the rules for multiplying a matrix by a scalar quantity and by another matrix.

Multiplying a Matrix by a Scalar Quantity

When a matrix $[\boldsymbol{A}]$ of size $m \times n$ is multiplied by a scalar quantity such as β, the operation results in a matrix of the same size $m \times n$, whose elements are the product of elements in the original matrix and the scalar quantity. For example, when we multiply matrix $[\boldsymbol{A}]$ of size $m \times n$ by a scalar quantity β, this operation results in another matrix of size $m \times n$, whose elements are computed by multiplying each element of matrix $[\boldsymbol{A}]$ by β, as shown below.

$$\beta[\boldsymbol{A}] = \beta \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} = \begin{bmatrix} \beta a_{11} & \beta a_{12} & . & . & \beta a_{1n} \\ \beta a_{21} & \beta a_{22} & . & . & \beta a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ \beta a_{m1} & \beta a_{m2} & . & . & \beta a_{mn} \end{bmatrix} \tag{2.2}$$

Multiplying a Matrix by Another Matrix

Whereas any size matrix can be multiplied by a scalar quantity, matrix multiplication can be performed only when the number of columns in the *premultiplier* matrix is equal to the number of rows in the *postmultiplier* matrix. For example, matrix $[\boldsymbol{A}]$ of size $m \times n$ can be premultiplied by matrix $[\boldsymbol{B}]$ of size $n \times p$ because the number of columns n in matrix $[\boldsymbol{A}]$ is equal to number of rows n in matrix $[\boldsymbol{B}]$. Moreover, the multiplication results in another matrix, say $[\boldsymbol{C}]$, of size $m \times p$. Matrix multiplication is carried out according to the following rule:

$$[\boldsymbol{A}]_{m\times n}[\boldsymbol{B}]_{n\times p} = [\boldsymbol{C}]_{m\times p}$$

must match

$$[\boldsymbol{A}][\boldsymbol{B}] = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & . & . & b_{1p} \\ b_{21} & b_{22} & . & . & b_{2p} \\ . & . & . & . & . \\ . & . & . & . & . \\ b_{n1} & b_{n2} & . & . & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & . & . & c_{1p} \\ c_{21} & c_{22} & . & . & c_{2p} \\ . & . & . & . & . \\ . & . & . & . & . \\ c_{m1} & c_{m2} & . & . & c_{mp} \end{bmatrix}$$

where the elements in the first column of the $[\boldsymbol{C}]$ matrix are computed from

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1n}b_{n1} \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} + \ldots + a_{2n}b_{n1} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ c_{m1} &= a_{m1}b_{11} + a_{m2}b_{21} + \ldots + a_{mn}b_{n1} \end{aligned}$$

and the elements in the second column of the $[\boldsymbol{C}]$ matrix are

$$\begin{aligned} c_{12} &= a_{11}b_{12} + a_{12}b_{22} + \ldots + a_{1n}b_{n2} \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22} + \ldots + a_{2n}b_{n2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ c_{m2} &= a_{m1}b_{12} + a_{m2}b_{22} + \ldots + a_{mn}b_{n2} \end{aligned}$$

and similarly, the elements in the other columns are computed, leading to the last column of the $[\boldsymbol{C}]$ matrix

$$\begin{aligned} c_{1p} &= a_{11}b_{1p} + a_{12}b_{2p} + \ldots + a_{1n}b_{np} \\ c_{2p} &= a_{21}b_{1p} + a_{22}b_{2p} + \ldots + a_{2n}b_{np} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ c_{mp} &= a_{m1}b_{1p} + a_{m2}b_{2p} + \ldots + a_{mn}b_{np} \end{aligned}$$

The multiplication procedure that leads to the values of the elements in the $[\boldsymbol{C}]$ matrix may be represented in a compact summation form by

$$c_{mp} = \sum_{k=1}^{n} a_{mk} b_{kp} \tag{2.3}$$

When multiplying matrices, keep in mind the following rules. Matrix multiplication is not commutative except for very special cases.

$$[\boldsymbol{A}][\boldsymbol{B}] \neq [\boldsymbol{B}][\boldsymbol{A}] \tag{2.4}$$

Matrix multiplication is associative; that is

$$[\boldsymbol{A}]([\boldsymbol{B}][\boldsymbol{C}]) = ([\boldsymbol{A}][\boldsymbol{B}])[\boldsymbol{C}] \tag{2.5}$$

The distributive law holds true for matrix multiplication; that is

$$([\boldsymbol{A}] + [\boldsymbol{B}])[\boldsymbol{C}] = [\boldsymbol{A}][\boldsymbol{C}] + [\boldsymbol{B}][\boldsymbol{C}] \tag{2.6}$$

or

$$[\boldsymbol{A}]([\boldsymbol{B}] + [\boldsymbol{C}]) = [\boldsymbol{A}][\boldsymbol{B}] + [\boldsymbol{A}][\boldsymbol{C}] \tag{2.7}$$

For a square matrix, the matrix may be raised to an integer power n in the following manner:

$$[\boldsymbol{A}]^n = \overbrace{[\boldsymbol{A}][\boldsymbol{A}]\ldots[\boldsymbol{A}]}^{n \text{ times}} \tag{2.8}$$

This may be a good place to point out that if $[\boldsymbol{I}]$ is an identity matrix and $[\boldsymbol{A}]$ is a square matrix of matching size, then it can be readily shown that the product of $[\boldsymbol{I}][\boldsymbol{A}] = [\boldsymbol{A}][\boldsymbol{I}] = [\boldsymbol{A}]$. See Example 2.1 for the proof.

EXAMPLE 2.1

Given matrices

$$[\boldsymbol{A}] = \begin{bmatrix} 0 & 5 & 0 \\ 8 & 3 & 7 \\ 9 & -2 & 9 \end{bmatrix}, [\boldsymbol{B}] = \begin{bmatrix} 4 & 6 & -2 \\ 7 & 2 & 3 \\ 1 & 3 & -4 \end{bmatrix}, \text{and } \{\boldsymbol{C}\} = \begin{Bmatrix} -1 \\ 2 \\ 5 \end{Bmatrix}$$

perform the following operations:

a. $[\boldsymbol{A}] + [\boldsymbol{B}] = ?$
b. $[\boldsymbol{A}] - [\boldsymbol{B}] = ?$
c. $3[\boldsymbol{A}] = ?$
d. $[\boldsymbol{A}][\boldsymbol{B}] = ?$
e. $[\boldsymbol{A}]\{\boldsymbol{C}\} = ?$
f. $[\boldsymbol{A}]^2 = ?$
g. Show that $[\boldsymbol{I}][\boldsymbol{A}] = [\boldsymbol{A}][\boldsymbol{I}] = [\boldsymbol{A}]$

We will use the operation rules discussed in the preceding sections to answer these questions.

a. $[A] + [B] = ?$

$$[A] + [B] = \begin{bmatrix} 0 & 5 & 0 \\ 8 & 3 & 7 \\ 9 & -2 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 6 & -2 \\ 7 & 2 & 3 \\ 1 & 3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} (0+4) & (5+6) & (0+(-2)) \\ (8+7) & (3+2) & (7+3) \\ (9+1) & (-2+3) & (9+(-4)) \end{bmatrix} = \begin{bmatrix} 4 & 11 & -2 \\ 15 & 5 & 10 \\ 10 & 1 & 5 \end{bmatrix}$$

b. $[A] - [B] = ?$

$$[A] - [B] = \begin{bmatrix} 0 & 5 & 0 \\ 8 & 3 & 7 \\ 9 & -2 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 6 & -2 \\ 7 & 2 & 3 \\ 1 & 3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} (0-4) & (5-6) & (0-(-2)) \\ (8-7) & (3-2) & (7-3) \\ (9-1) & (-2-3) & (9-(-4)) \end{bmatrix} = \begin{bmatrix} -4 & -1 & 2 \\ 1 & 1 & 4 \\ 8 & -5 & 13 \end{bmatrix}$$

c. $3[A] = ?$

$$3[A] = 3\begin{bmatrix} 0 & 5 & 0 \\ 8 & 3 & 7 \\ 9 & -2 & 9 \end{bmatrix} = \begin{bmatrix} 0 & (3)(5) & 0 \\ (3)(8) & (3)(3) & (3)(7) \\ (3)(9) & (3)(-2) & (3)(9) \end{bmatrix} = \begin{bmatrix} 0 & 15 & 0 \\ 24 & 9 & 21 \\ 27 & -6 & 27 \end{bmatrix}$$

d. $[A][B] = ?$

$$[A][B] = \begin{bmatrix} 0 & 5 & 0 \\ 8 & 3 & 7 \\ 9 & -2 & 9 \end{bmatrix}\begin{bmatrix} 4 & 6 & -2 \\ 7 & 2 & 3 \\ 1 & 3 & -4 \end{bmatrix} =$$

$$\begin{bmatrix} (0)(4)+(5)(7)+(0)(1) & (0)(6)+(5)(2)+(0)(3) & (0)(-2)+(5)(3)+(0)(-4) \\ (8)(4)+(3)(7)+(7)(1) & (8)(6)+(3)(2)+(7)(3) & (8)(-2)+(3)(3)+(7)(-4) \\ (9)(4)+(-2)(7)+(9)(1) & (9)(6)+(-2)(2)+(9)(3) & (9)(-2)+(-2)(3)+(9)(-4) \end{bmatrix}$$

$$= \begin{bmatrix} 35 & 10 & 15 \\ 60 & 75 & -35 \\ 31 & 77 & -60 \end{bmatrix}$$

e. $[A]\{C\} = ?$

$$[A]\{C\} = \begin{bmatrix} 0 & 5 & 0 \\ 8 & 3 & 7 \\ 9 & -2 & 9 \end{bmatrix}\begin{Bmatrix} -1 \\ 2 \\ 5 \end{Bmatrix} = \begin{Bmatrix} (0)(-1)+(5)(2)+(0)(5) \\ (8)(-1)+(3)(2)+(7)(5) \\ (9)(-1)+(-2)(2)+(9)(5) \end{Bmatrix} = \begin{Bmatrix} 10 \\ 33 \\ 32 \end{Bmatrix}$$