J. Mason
L. Burton
K. Stacey

Thinking Mathematically

Second Edition

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John Mason
University of Oxford/Open University

with Leone Burton
and
Kaye Stacey
Melbourne Graduate School of Education
University of Melbourne

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Pearson Education Limited
Edinburgh Gate
Harlow
Essex CM20 2JE
England

and Associated Companies throughout the world

Visit us on the World Wide Web at:
www.pearsoned.co.uk

First published 1982
Second edition published 2010

ISBN: 978-0-273-72891-7

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library

Library of Congress Cataloging-in-Publication Data
A catalog record for this book is available from the Library of Congress

10 9 8 7 6 5 4 3 2 1
14 13 12 11 10

Typeset in 9.5/13 pt Avenir Book by 73.
Printed and bound in Great Britain by Henry Ling Ltd, Dorchester, Dorset

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Introduction to First Edition

Thinking Mathematically is about mathematical processes, and not about any particular branch of mathematics. Our aim is to show how to make a start on any question, how to attack it effectively and how to learn from the experience. Time and effort spent studying these processes of enquiry are wisely invested because doing so can bring you closer to realizing your full potential for mathematical thinking.

Experience in working with students of all ages has convinced us that mathematical thinking can be improved by

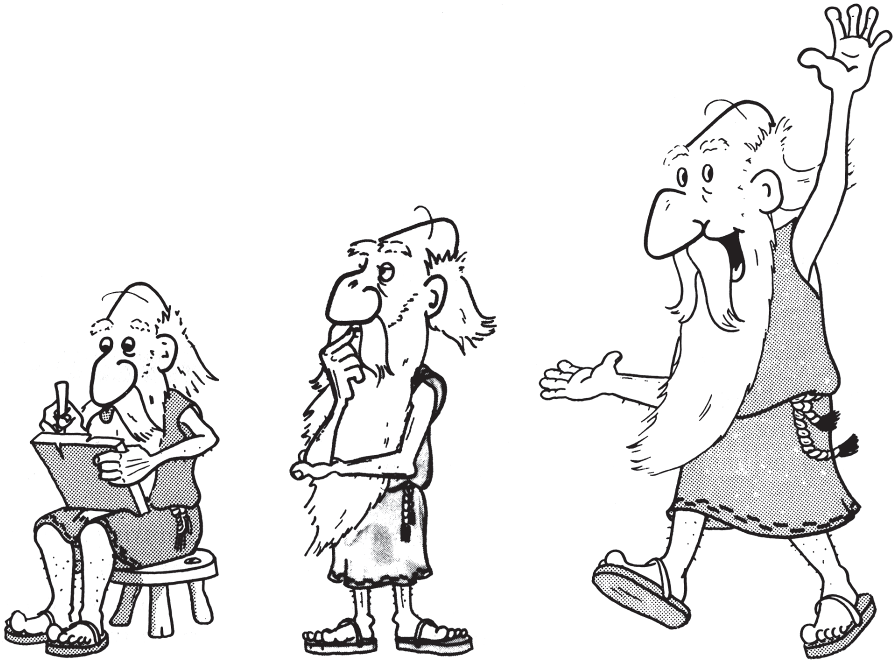
- tackling questions conscientiously;
- reflecting on this experience;
- linking feelings with action;
- studying the process of resolving problems; and
- noticing how what you learn fits in with your own experience.

Consequently while encouraging you to tackle questions, we show you how to reflect on that experience by drawing your attention to important features of the process of thinking mathematically.

How to use this book effectively!

Thinking Mathematically is a book to be used rather than read, so its value depends on how energetically the reader works through the questions posed throughout the text. Their purpose is to provide recent, vivid experience which will connect with the comments that are made. Failure to tackle the questions seriously will render the comments meaningless and empty, and it will be hard to use our advice when it is needed. Three kinds of involvement are required: physical, emotional and intellectual.

Probably the single most important lesson to be learned is that being stuck is an honourable state and an essential part of improving thinking. However, to get the most out of being stuck, it is not enough to think for a few minutes and then read on. Take time to ponder the question, and continue reading only when you are convinced that you have tried all possible alleys. Time taken to ponder the question and to try several approaches is time well spent. Each question is followed by suggestions under the heading STUCK? to provide signposts when progress seems blocked. Because different resolutions follow different paths, some of the suggestions might be mutually contradictory, or



irrelevant to your approach, so do not expect every suggestion to provide immediate insight!

Recalcitrant questions which resist resolution should not be permitted to produce disappointment. A great deal more can be learned from an unsuccessful attempt than from a question which is quickly resolved, provided you think about it earnestly, make use of techniques suggested in the book, and reflect on what you have done. Answers are irrelevant to the main purpose of this book. The important thing is to experience the processes being discussed.

To stress our concentration on processes rather than answers, a 'solution' in the usual sense is rarely given. Instead, we offer sample 'resolutions' which include a good deal of commentary as well as many false starts, partially digested ideas and so on. Elegant solutions such as are found in most mathematics texts rarely spring fully formed from someone's brain. They are more often arrived at after a long and tortuous period of thinking and not thinking, with much modification and changing of understanding along the way, but most beginners do not realize this. By taking our informal approach, confidence can grow and progress made. Elegance can come later.

In summary, then, our approach rests on five important assumptions:

1 **You** can think mathematically.
2 Mathematical thinking can be **improved** by practice with reflection.
3 Mathematical thinking is **provoked** by contradiction, tension and surprise.

4 Mathematical thinking is **supported** by an atmosphere of questioning, challenging and reflecting.
5 Mathematical thinking helps in **understanding** yourself and the world.

You will notice that the text is written in the first person singular despite there being three authors. This reflects our way of working, as well as the fusion that has taken place during writing.

This book is addressed to students as a manual for developing mathematical thinking. It presents only one approach to the task and does not, for example, compare that approach with the schemes of earlier writers such as Pólya. A bibliography is provided for the reader who wishes to inspect personally the work which has most strongly influenced us.

Whilst some of the problems used in this book are original, many have come from the mathematical grapevine. We would like to thank the friends and colleagues who have introduced us to these questions and to thank especially their usually unknown authors for the enjoyment they have given us.

We are particularly indebted to:

George Pólya and J. G. Bennett for their inspiration;
Graham Read for the cartoons of PIX who first appeared in *Mathematics: A Psychological Perspective*, Open University Press, 1978;
Alan Schoenfeld for stressing the importance of the Monitor discussed in Chapter 7;
Mike Beetham for help in processing the text on the Cambridge computer;
numerous colleagues, most especially Joy Davis, Susie Groves, Peter Stacey and Collette Tasse; and
a myriad of students in three countries.

We offer the book as support to future thinking, particularly of

Quentin and Lydia Mason,
Mark Burton,
Carol and Andrew Stacey.

Introduction to Second Edition

Thinking Mathematically was published in 1982 and continues to find favour in many different countries. It is used by senior high school students, students going on to study mathematics at university, teacher preparation courses, and in courses for undergraduates in mathematics. Our aim in this new edition is to offer a range of questions for exploration appropriate to readers as pre-service primary and secondary teachers, and as mathematics undergraduates. These can be found in the new Chapter 11. Whereas the questions (problems) in the original book were chosen to illustrate the various 'processes', or, as we would now put it, the use of various natural powers, the questions in Chapter 11 have been chosen to make use of those powers to enrich and deepen appreciation of core ideas of various important mathematical topics.

A by-product is a demonstration of the way in which ordinary questions, designed to be approached routinely, can sometimes be transformed into intriguing questions. It also demonstrates that significant areas of higher mathematics and sometimes difficult mathematical questions often lie hidden just behind elementary topics. At the same time, we take the opportunity to rephrase the language of *thinking processes* used in the original book, into the language of *natural powers* which all human beings possess. This also provides an opportunity to include some insights and distinctions that have emerged in the period since the original was published.

Processes and natural powers

In the 1970s and early 1980s, there was great interest in the 'processes' by which things were done, and thinking mathematically is a fine example. However, while interest in delineating processes of thinking and creativity have become of renewed interest recently, the language in which they are described has changed considerably. We found that it made more sense to us, and to people with whom we engaged mathematically and pedagogically, to think in terms of natural powers that learners bring to the classroom. The task of teaching then becomes one of provoking learners to make use of and to develop those powers in the context of mathematical thinking.

We follow Caleb Gattegno in seeing awareness as the basis for action; without awareness, there is no action. However, some awarenesses may be so integrated into our functioning that we are not consciously aware of them operating. This is certainly the case when we suddenly find ourselves acting automatically out of habit. Again following Gattegno, mathematics as a discipline only arises when people become aware of the actions they are performing in certain contexts (relationships and properties in number and space) and articulate these awarenesses to produce 'mathematics'. So mathematics as a body of knowledge in books can be seen as formal recognition, expression and study of awarenesses that inform mathematical actions in problematic situations. To become a teacher requires becoming aware of the awarenesses that generate mathematical actions, because these are what trigger pedagogical actions. Consequently it is vital to educate one's awareness by engaging oneself in mathematical tasks which bring important mathematical awarenesses to the surface, so that they can inform future action.

Awareness is closely related to cognition; action is closely related to behaviour. An often overlooked aspect of human psyche is the emotions or the domain of affect. The original book addressed this through suggesting that being stuck is an 'honourable state' from which it is possible to learn, and that expressing emotion-laden observations about being stuck and having an insight (AHA!), however transitory, releases energy which enables progress to be made. It celebrated positive emotions: the pleasure of making sense through use of your own powers, the excitement of discovery, the aesthetic pleasure in an interesting result, and the satisfaction of finding a resolution. We add here that developing a disposition to recognize problematic situations in the material world as well as in the world of mathematics, the 'questioning attitude' of Chapter 8, is also a significant contribution to the affective domain.

The current emphasis on collaborative activity as a necessary component of mathematics learning is a development from the value recognized and promulgated through the original book, that working together can be stimulating and can open up avenues that no single individual might have recognized by themselves. At the same time, it is vital to have periods of 'own thinking' during which possibilities are considered and either pursued or dropped. Some people like to start individually, and then, after a period, exchange possibilities; others like to have a period of collective idea-generation followed by own thinking before coming together again. Certainly it is helpful to have communal reflection as a force to bring to the surface and articulate insights and observations about salient moments in the exploration, even though these will often have occurred during individual thinking. The presence of significant others is an effective contribution to stimulating the impulse to express and so clarify your own thinking, as well as to connect it to the thinking of others.

We also take the opportunity afforded by this new edition to introduce persistent and ubiquitous mathematical themes which imbue mathematics. A brief description of powers, themes and related notions can be found in the new Chapter 12.

The power of an experiential approach

The original book was conceived as an exposition of our own experience as mathematical thinkers, profoundly influenced by the work of George Pólya. Indeed, JohnM had been shown his film *Let Us Teach Guessing* in 1967 as a graduate teaching assistant soon after it was made, and it released in him an approach to teaching which he later realized was moulded by his experience in high school, where he had been taught by Geoff Steele. To his surprise he discovered many years later that Geoff had never trained as a teacher, and was not primarily a mathematician, but rather a choir conductor! Nevertheless, his stimulation nurtured and sustained John through high school and into university where he arrived having internalized the elements of mathematical thinking.

On arriving at his first academic post at the Open University, John discovered that one of Pólya's books had been chosen as a set book. When he was asked to design a one-week summer school for up to 7,000 students over 11 weeks on three sites, he incorporated the film into the programme, accompanied by sessions called *active problem solving*. John naively assumed that all mathematics tutors would 'be mathematical with and in front of their students' and so would naturally get students specializing and generalizing, conjecturing and convincing and so on. It took some years before he realized that not all tutors were as self-aware of their own mathematical thinking as he had assumed. The result was a series of training sessions for tutors, designed to get them to experience mathematical thinking for themselves and to reflect on that experience so as to be able to draw student attention to important aspects. Meanwhile, the course had been redesigned and so the summer school programme was modified accordingly, with more stress on simpler problems which nevertheless highlighted a specific 'process' of mathematical thinking, or, put another way, which provoked learners to make spontaneous use of one or more natural powers which are important in thinking mathematically.

In 1979 John became involved in his first mathematics education in-service course with Leone, who had been teaching primary school teachers to work mathematically with their students, and researching the effects of this on children's learning. The course team wanted the course to be practical, so the experiential basis was extended by assigning a period of study each week to what was called 'own thinking'. The idea was that, in order to be alert and sensitized to students, it is necessary to be alerted to correspondingly relevant aspects of your own thinking. The problem was how to choose problems and commentary to put into the 'own thinking' sections. In order to make a sensible choice, Leone and John planned the book and were later joined by Kaye who from the other side of the world had also been inspired by Pólya's expositions of mathematical discovery and had for several years been fostering this for primary and secondary pre-service teachers through her innovative mathematical problem-solving courses run jointly with Susie Groves. *Thinking Mathematically* made use of one of the principles being proposed in the course, namely that doing and talking are vital activities to prepare for recording, and that recording helps to integrate doing and talking so as to make insights and experience available to inform action in the future. In our case, the writing of the book crystallized and organized our own thinking about what experiences would be most useful for teachers.

As authors, we attribute the continuing interest in and use of the book to its experiential basis, and that continues in the present edition. Indeed an intention of the new edition is to make it easier for teachers to bring the experience of thinking mathematically more centrally into all teaching. Developing your mathematical thinking, indeed engaging in discussion about any mathematics education issue, is greatly improved by engaging together in related mathematics first, and then looking for other shareable experiences on which to draw. Put another way, all of the great educational theorists who have addressed mathematics education agree that learning is enhanced when students are given tasks which spark off activity in which familiar actions are adapted and modified in order to meet the challenge. There is little use in rehearsing problems you can already do, using familiar actions, unless you are simply trying to gain speed. Activity produces experience, but as Immanuel Kant might have said,

> A succession of experiences does not add up to an experience of that succession.

Something more is required. Pólya called it the stage of *looking back*. We chose to call it the Review phase, which is more precise than the term *reflecting* which we also use. *Reflection* has a multiplicity of meanings. Jim Wilson once said that this stage was the most talked about and the least used of Pólya's four stages. Most educationalists are agreed that some sort of drawing back from immersion in activity is necessary in order to learn from experience. After all,

> One thing we do not seem to learn from experience is that we do not often learn from experience alone.

Where educationalists differ is in the timing, the degree and the initiative of drawing back from the action that is required. Clearly drawing back too soon leaves everyone frustrated and is unlikely to have a lasting effect. On the other hand, leaving students to 'learn from their experience' is clearly unsatisfactory for all but the most gifted of mathematics students. For most students, learning to learn mathematics is a scientific endeavour rather than a natural endeavour, in the sense of Lev Vygotsky: most people need to be in the presence of someone more experienced, at least some of the time, in order to make sense of experience. Caleb Gattegno and others argue that learning really takes place during sleep, when the mind chooses what to forget, or at least to let go of. Be that as it may, where students have been engaged in practices of intentional reflection, review, reconstruction and rehearsal, they are much more likely to have access to insights in the future. The Discipline of Noticing which JohnM articulated based on his experience with J. G. Bennett was an attempt to provide a philosophically well-founded method for researching one's own practice, but it applies equally well to students learning.

In order to learn from experience, in order to have fresh possibilities come to mind when appropriate, it is necessary to sensitize yourself to notice opportunities to respond to situations rather than to react, to choose to act rather than to be caught up in and driven by old habits. Thus, by offering tasks and follow-up prompts it is possible to promote awareness of the use of people's natural powers. As they become more sensitized to and aware of their own use of these powers, those powers develop in flexibility and in usefulness. They become, in the language of Vygotsky 'actions for oneself', which can be self-initiated, rather than simply 'actions in oneself', which have to be triggered by a teacher or cued by some prompt in the task. All this provides some justification for the format both of the previous book and of this one: questions are provided to be worked on. They are followed by reflective prompts and commentary. They are of little or no use unless they are engaged in fully, perhaps over a considerable period of time, followed by reflection and looking out for resonance between comments provided and own experience. Our aim is to promote pondering and contemplating, getting stuck and restarting. Getting 'answers' is not the most valuable outcome of struggling. Rather, what is most important and valuable is what you notice happening in the way of getting stuck, making progress, making conjectures, modifying conjectures, using your powers, encountering mathematical themes, etc., and the little frissons of insight and excitement that come from using those powers and making some progress. The tasks are fodder for initiating activity; the mathematical results are not usually of significance. Put another way, this book does not try to teach any particular mathematical content, but rather to alert readers to the ways in which their own natural powers can be harnessed in the service of exploring and understanding mathematical topics and situations.

There is always an issue of level of challenge. Initial impressions of a question may lead to a sense of 'too challenging' or of 'not challenging enough'. One of the things to learn from working on questions is how to make something less challenging so that progress can be made (usually by specializing) and more challenging, by locating some *dimensions of possible variation* and varying them, or by extending the *range of permissible change* of those features. It is up to the reader to select the level of challenge appropriate to them at any given moment, with the hope that they will be inspired to return and tackle the more challenging tasks at a later date. The intention is that the questions posed will initiate mathematical experience and that readers and their teachers will adapt the degree of difficulty to make this experience productive, rather than focusing exclusively on getting answers to fixed problems.

Acknowledgements

When the original book was written, we believed that mathematical questions belonged in the world of mathematics, without the need to provide details of their origins. Age has brought with it interest in the origins of problems and how they are transformed over time. Where we believe we know of a specific origin, we have inserted that in this new edition. Where no reference is given, either its origins have been forgotten or it came to us from several different colleagues as it passed from person to person through the community of mathematical thinkers, or we believe that we have constructed it ourselves. We are grateful to comments on the new additions by Eva Knoll and Ami Mamolo.

Dedication

The original book was dedicated to our children, who of course have now grown up. Sadly Leone lost her fight against cancer before we got started on these revisions, so we offer this edition in her memory. In the words of her son, Mark:

> Leone Burton's books were always dedicated to me, her son. This book is dedicated to her, and her memory. Certainly problem solving, and thinking mathematically, have been the richest gift I have had; whether it came from her, or indeed another author of this book, John Mason, who first brought a computer to our house when I was a very young boy, so that I could 'play Logo'.

John Mason, Oxford, April 2010
Kaye Stacey, Melbourne, April 2010

1

Everyone can start

This chapter introduces the activities which will get your thinking started on any question. There is no need to shy away from a mathematical question, and no reason to stare at a blank piece of paper feeling hopeless. Driving straight down the first path that appears hoping brute force will succeed is not a good tactic either. However, there are productive things you can do.

Specializing

The best place to begin is to work on a question:

> ***Warehouse***
>
> In a warehouse you obtain 20% discount but you must pay a 15% sales tax. Which would you prefer to have calculated first: discount or tax?

How can you get to grips with such a question? To make progress, you must be clear what the question is asking, but this may not fully emerge until you've done a bit of doodling. The best way to start is by trying some specific cases. I hope you spontaneously want to try it with an item priced at say £100.

DO SO NOW IF YOU HAVE NOT ALREADY

Surprised by the result? Most people are, and it is that surprise which fuels mathematical thinking. Now, will the same thing happen for a price of say £120?.

TRY IT AND SEE!

Write down your calculations and your insights. It is the only way to develop your thinking skills.

TRY IT AND SEE!

Now, perhaps using a calculator, try other examples. Your aim in doing this is two-fold: to get an idea of what the answer to the question might be, and at the same time to develop a sense of why your answer might be correct. Put another way, by doing examples you make the question meaningful to yourself and you may also begin to see an underlying pattern in all the special cases which will be the clue to resolving the question completely.

What might be the underlying pattern in this question? Perhaps you have experience of questions like this and know what to do. If so, think how you would encourage someone less experienced to tackle it, then read my suggestions. It is important to work through my discussions because that is where important points about mathematical thinking will be introduced and illustrated.

How does the final price depend on the order of calculating discount and tax? There should be a pattern in the examples you have tried. If not, check your calculations! Will this result be true for other prices? If you are not certain, try some more examples. When you are sure, search for an explanation (or read further).

TRY EXAMPLES UNTIL YOU ARE SURE

A lot depends on the form in which you do your calculations. The usual form for doing discount followed by tax is to

calculate the discount:	on £100 discount is £20
subtract it from the price:	£100 − £20 = £80
calculate the tax:	15% of £80 is £12
add the tax on to get the final price:	£80 + £12 = £92

Try to find other ways of doing the calculation until you hit upon one which reveals why your result is always true. As a suggestion, you want to find a form of calculation which is independent of the initial price. To do this, try calculating what percentage of an original price you pay when the discount has been subtracted, and what percentage of an original price you pay when tax has been added.

DO IT NOW

With any luck you will have found that

(i) subtracting 20% from a price is the same as paying 80% of it, that is you pay 0.80 times the price:

(ii) adding 15% to a price is the same as paying 115% of it, that is you pay 1.15 times the price.

Then, for any initial price of say £100, calculating

discount first: you pay 1.15 × (0.80 × £100)
tax first: you pay 0.80 × (1.15 × £100)

By writing the calculation in this form you can see that the order of calculation does not matter, because all that is involved is multiplying the original price by two numbers, in either order. If the original price is £*P* then calculating

discount first: you pay 1.15 × 0.80 × £*P*
tax first: you pay 0.80 × 1.15 × £*P*

and these are always equal.

Notice the value of standing back from the detail of the calculation and looking at its form or shape. This sort of reflective activity is fundamental to developing your mathematical thinking.

Warehouse illustrates several important aspects of mathematical thinking, two of which I want to draw to your attention. Firstly there are specific processes which aid mathematical thinking. In this case the process being emphasized is SPECIALIZING which means turning to examples to learn about the question. The examples you choose are special in the sense that they are particular instances of a more general situation in the question. Secondly, being STUCK is a natural state of affairs, and something can usually be done about it. Here, the something being suggested is SPECIALIZING. This is a simple technique which everyone can use, and when people find themselves unable to proceed with a question, suggestions like

Have you tried an example?

and

What happens in this particular case?

are what gets them going again.

The next question, taken from Banwell, Saunders, and Tahta (1986), illustrates other forms of specializing.

Paper Strip

Imagine a long thin strip of paper stretched out in front of you, left to right. Imagine taking the ends in your hands and placing the right hand end on top of the left. Now press the strip flat so that it is folded in half and has a crease. Repeat the whole operation on the new strip two more times. How many creases are there? How many creases will there be if the operation is repeated 10 times in total?

TRY IT NOW

STUCK?

➤ Specialize mentally by counting the creases after two folds.
➤ Perhaps a diagram will steady your mental image.
➤ Specialize by trying it on a strip of paper.
➤ Try three folds and four folds. Look for a pattern.
➤ What do you want to find? Be clear and precise.
➤ Is there something related to the creases that you can count more easily?
➤ Check any conjectures on new examples!

I am not going to give a full resolution of this question. If you are STUCK, do not be upset. Being stuck is fine, as long as you look on it as an opportunity to learn. Perhaps you can return to the question with renewed vigour as you read the next chapter! Before you set it aside, try up to five folds either mentally, with diagrams, or with real paper. Count the creases and draw up a table of the results. Whereas with *Warehouse* specializing means turning to numerical examples to get to grips with the question, specializing for *Paper Strip* means turning to diagrams or pieces of paper and experimenting. It is important to turn to objects which you are confidently able to manipulate. These may be physical objects, or mathematical ones such as diagrams, numbers or algebraic symbols.

Specializing alone is unlikely to resolve a question for you, but it does get you started and involved. The question loses its forbidding exterior and becomes less intimidating. Furthermore, the specific cases should help you to get a sense of what the question is really about, enabling you to make an informed guess. Further careful specializing with an eye on the 'why' rather than the 'what' may lead to insight into what is really happening.

The next question is on more familiar ground.

Palindromes

A number like 12321 is called a palindrome because it reads the same backwards as forwards. A friend of mine claims that all palindromes with four digits are exactly divisible by 11. Are they?

TRY IT NOW

STUCK?

➤ Find some palindromes with four digits.
➤ Do you believe my friend?
➤ What do you want to show?

A resolution

Remember that a resolution is not intended to be polished, and is only one way of thinking about it. The only sensible way to begin is to specialize. I want to get a feel for the kinds of numbers involved. What are some palindromes?

> 747 is one
> 88 and 6 are others

The question only mentions palindromes with four digits, which means numbers like

> 1221, 3003, 6996 and 7557.

What do I want? I want to find out if all such numbers are divisible by 11.

TEST IT NOW

By trying specific numerical examples, I convinced myself that the result seems plausible. Notice however that I cannot be sure my result is **always** correct just by specializing unless I am prepared to test **every** four-digit palindrome. As there are some 90 of them, it is better to try to get some idea of the underlying pattern.

DO SO NOW

I tried four specific cases:

> $1221/11 = 111$
> $3003/11 = 273$
> $6996/11 = 636$
> $7557/11 = 687$

but I could see no obvious pattern in them. This brings up an extremely important point about specializing. Choosing examples randomly is a good way of getting an idea of what is involved in a question and seeing if a statement or guess is likely to be true, but when searching for patterns, success is more likely if the specializing is done systematically. How can I be systematic in this case?

TRY IT NOW

STUCK?

➤ What is the smallest four-digit palindrome?
➤ What is the next smallest?
➤ How can one palindrome be changed into another one?

One way is to start with the smallest four-digit palindrome (that is 1001) and work upwards in numerical order:

> 1001, 1111, 1221, 1331, . . .

Checking my friend's statement:

$$1001/11 = 91$$
$$1111/11 = 101$$
$$1221/11 = 111$$
$$1331/11 = 121$$

This not only supports my friend's claim, but also suggests more. Notice that the palindromes are rising by 110 each time, and the quotients are rising by 10 each time.

AHA! Now I can see why my friend's claim is true. The difference between successive palindromes is always 110. The smallest palindrome (1001) is exactly divisible by 11 and so is 110. As all the other palindromes are obtained from 1001 by adding on 110, all the palindromes with four digits must be exactly divisible by 11.

So apart from tidying up and expressing it nicely, the question is resolved.

Or is it? Does the resolution cover all the specific cases I have used? Look more closely! If all palindromes can be constructed by successively adding 110 to 1001, they will all have one as the units digit. But they do not! For example, 7557 is a palindrome with 7 as its units digit. What has gone wrong? Specializing led to a pattern (that successive palindromes differ by 110) on which I based my resolution. But this pattern cannot hold for all palindromes because it predicts something that is false (all palindromes do not end in one). The fault lies in jumping too quickly from the three differences to a general result. Fortunately, specializing can help again; this time to pinpoint the weakness in the pattern. Look further along the list of palindromes:

Palindromes	1881		1991		2002		2112		2222		2332
Differences		110		11		110		110		110	

This time I will proceed more cautiously, perhaps in a mood of disbelief rather than of belief. The pattern seems to be that successive palindromes differ by 110 except when the thousands digit changes and then the difference is 11. Further specializing gives results in agreement with this and increases confidence that this is indeed the underlying pattern. Thus specializing has again provided insight into what pattern might be valid. Now it is time to seek a general reason why the new pattern is valid, finally arriving at something like this:

> Successive palindromes that have the same thousands digit must have the same units in order to be palindromes. Thus the numbers differ only in the second and third digits which are each greater by one. The difference is therefore 110.
>
> Successive palindromes that differ in the thousands digit arise by adding 1001 (to increase the thousands and unit digits) and subtracting 990 (to reduce the second and third digits from nines to zeros). But $1001 - 990 = 11$, as observed in the examples.

In both cases, the differences are divisible by 11, so as long as the smallest four digit palindrome (1001) is divisible by 11 (it is), all of them are.

Now look back at the ways in which specializing has been used:

- It helped me to understand the question by forcing me to clarify the idea of a palindrome.
- It also led me to discover the form of a four-digit palindrome.
- I used it to convince myself that what my friend claimed was indeed likely to be true.
- Later on, systematic specializing exposed a pattern and so gave me an idea of why the result was true.
- Testing whether that pattern was correct (it was not) involved further specializing.

It is because it can be used so effectively, so easily, and in so many ways that specializing is basic to mathematical thinking.

The argument given in my resolution is by no means the most elegant, but then my aim is not for elegance in the first instance. The first attempt is rarely like the solutions printed in text books. If you are more mathematically sophisticated and confident with letters standing for arbitrary numbers, then you may easily have reached a resolution more quickly. You may for example have noticed that every four-digit palindrome has the shape *ABBA* where *A* and *B* are digits. Such a number has the value

$$\begin{aligned} 1000A + 100B + 10B + A &= (1000 + 1)A + (100 + 1)B \\ &= 1001A + 110B \\ &= 11 \times 91A + 11 \times 10B \\ &= 11\,(91A + 10B) \end{aligned}$$

(If you find this symbolic argument hard to follow, specialize and follow it through with $A = 3$ and $B = 4$. Then use other values for *A* and *B* until you have a sense of the patterns being expressed by the symbols.)

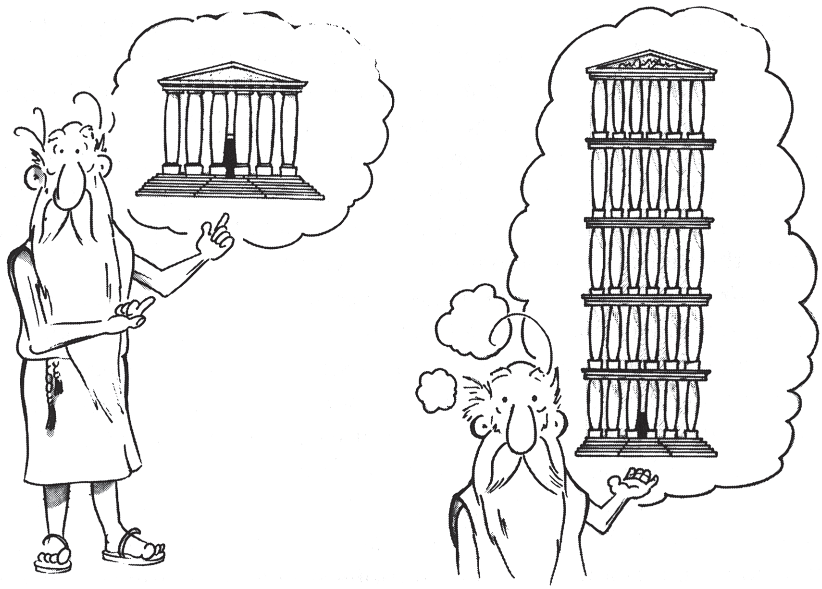
Elegant resolutions like that one apparently show no evidence of specializing since, by means of the symbols, a general argument applying to all four-digit palindromes is given. However, in order to create this argument, I must be sufficiently familiar with the entities involved (namely four-digit palindromes, *A*s and *B*s and decimal notation) that the general form *ABBA* is concrete and confidence inspiring. I must be at ease manipulating both the palindromes and the symbols standing for them. This is the essence of specializing. Turning to familiar, confidence inspiring entities and using them to explore what the question is about creates feelings of confidence and ease in otherwise unfamiliar situations.

Generalizing

In the discussion of specializing, it was impossible to avoid the other side of the coin, the process of generalizing: moving from a few instances to making guesses about a wide class of cases.

Generalizations are the life-blood of mathematics. Whereas specific results may in themselves be useful, the characteristically mathematical result is the general one. For example, knowing what happens for an article priced at £100 in *Warehouse* is less powerful than knowing that the final price is always independent of the order of calculation of discount and tax.

Generalizing starts when you sense an underlying pattern, even if you cannot articulate it. After the *Warehouse* calculations had been carried out for a few prices I noticed that, in each case, the order of calculation did not affect the result. This is the underlying pattern, the generalization. I conjectured that the order of calculation would never alter the result. When the calculation was put into a helpful form it was easy to introduce the symbol P for the original price and thereby show that the generalization was true.



Generalizing need not stop here. What if the discount and tax rates change? Does the order of calculation sometimes make a difference?

IF YOU HAVE NOT ALREADY DONE SO, TRY IT NOW

I hope you can see from the shape of the calculation derived earlier that the actual percentages are irrelevant to the argument. Part of the power of symbols in mathematics is to express such a general pattern. In this case denote the discount rate as a decimal or fraction by *D*, denote the tax rate as a decimal or fraction by *V*, and denote the original price by *P*. Then with

discount first: you pay $P(1 - D)(1 + V)$
tax first: you pay $P(1 + V)(1 - D)$

These are always equal because the order in which we multiply numbers (and hence symbols representing numbers) does not change the outcome. The use of symbols enables the argument to be presented concisely and whole classes of examples (in this case, all possible prices, tax rates and discount rates) can be treated at once. However, exploiting symbols is by no means as straightforward as is popularly imagined – it depends on the symbols becoming as familiar and meaningful as the numbers they replace.

Warehouse illustrates in a simple form the constant interplay between specializing and generalizing that makes up a large part of mathematical thinking. Specializing is used to gather the evidence upon which a generalization is to be made. Articulating the pattern that has been sensed produces a conjecture (a shrewd or informed guess) which further specializing can support or demolish. The process of justifying the conjecture involves more generalizing, with a shift in emphasis from guessing what may be true to seeing why it may be true. In *Warehouse* I first generalized the result by conjecturing that changing the order of calculation does not alter the final price (the 'what'). To justify this I had to study the method of calculation (the 'why').

Palindromes illustrates two other important aspects of generalizing. Being systematic in specializing is often an important aid to generalizing because pattern is more likely to be evident among related examples than with randomly chosen ones. There is, however, an inherent danger. Whilst a pattern may stick out, it is easy to be misled into believing the pattern is right when it is too simple and only partly correct. In *Palindromes*, the difference of 11 between some successive palindromes was overlooked because no examples had been tried where the thousands digit changed. Being cautious about believing an observed pattern or generalization reminds you to test it with a variety of examples. This is the bread and butter of mathematical thinking. Being trigger-happy with conjectures is as dangerous as being reticent to guess. The sometimes delicate balance that has to be struck between being too willing to believe a generalization and too sceptical to make any leaps into darkness is discussed in Chapters 5 and 6.

Writing yourself notes

Before we go on to look at more examples of specializing and generalizing, I wish to introduce a technique for recording mathematical experience. The reason for introducing it now is that you should start recording your experiences so that they are not lost, but can be analysed and studied later. Recording your experiences will also help you to notice them and this contributes to developing your mathematical thinking. Aim to record three things:

- all the significant ideas that occur to you as you search for a resolution to a question;
- what you are trying to do;
- your feelings about it.

Obviously this is a tall order, but it is well worth attempting. In particular, it gives you something to do when you get stuck – write down STUCK! Recognizing that you are stuck is the first step towards getting out of it.

Writing down the feelings you have and the mathematical ideas that occur to you will destroy the stark whiteness of the piece of paper that confronts you as you begin a question.

Once a start has been made, ideas often begin to flow more freely. Then it is important to write down what you are trying to do as it is easy to lose track of your approach or the reasons for embarking on some long calculation. There is nothing worse than surfacing from a bit of work and having no idea what you are doing or why!

I suggest that you get into the habit of writing notes to yourself when working on any of the questions in this book. **Do not be put off** by the large variety of things to note down. As the chapters progress I shall be making suggestions about the most useful things to record. The best time to begin is now, so try making notes as you work on the next question. **Avoid describing** what you do. Brief notes which help you recall the moment are all that is needed. Remember to specialize and generalize, and compare your account with mine only when you have done all you can. My account is necessarily more formal than yours will be, and for later reference I have put certain words in capitals.

Patchwork

Take a square and draw a straight line right across it. Draw several more lines in any arrangement so that the lines all cross the square, and the square is divided into several regions. The task is to colour the regions in such a way that adjacent regions are never coloured the same. (Regions having only one point in common are not considered adjacent.) How few different colours are needed to colour any such arrangement?

TRY IT NOW. NOTE DOWN IDEAS AND FEELINGS, RESORTING TO MY COMMENTS ONLY WHEN YOU GET STUCK

STUCK?

➤ Clarify the question by specializing – try colouring an arrangement.
➤ What do you KNOW? How is an arrangement constructed?