MATHEMATICS IN WESTERN CULTURE

MORRIS KLINE

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By MORRIS KLINE

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To Elizabeth and Judith

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Foreword

After an unbroken tradition of many centuries, mathematics has ceased to be generally considered as an integral part of culture in our era of mass education. The isolation of research scientists, the pitiful scarcity of inspiring teachers, the host of dull and empty commercial textbooks and the general educational trend away from intellectual discipline have contributed to the anti-mathematical fashion in education. It is very much to the credit of the public that a strong interest in mathematics is none the less alive.

Various attempts have recently been made to satisfy this interest. Together with H. Robbins I attempted in *What Is Mathematics?* to discuss the meaning of mathematics. Our book was, however, addressed to readers with a certain background of mathematical knowledge. More should be done on a less technical level for the large number of people who do not have this background, but still wish to acquire knowledge of the significance of mathematics in human culture.

For some time I have followed with great interest Professor Morris Kline's work on the present book. I believe that it will prove a major contribution and help to bring the mathematical sciences closer to people who have not as yet appreciated the fascination and scope of the subject.

R. Courant

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Preface

Now, when all these studies reach the point of inter-communion and connection with one another, and come to be considered in their mutual affinities, then, I think, but not till then, will the pursuit of them have a value for our objects; otherwise there is no profit in them.

PLATO

The object of this book is to advance the thesis that mathematics has been a major cultural force in Western civilization. Almost everyone knows that mathematics serves the very practical purpose of dictating engineering design. Fewer people seem to be aware that mathematics carries the main burden of scientific reasoning and is the core of the major theories of physical science. It is even less widely known that mathematics has determined the direction and content of much philosophic thought, has destroyed and rebuilt religious doctrines, has supplied substance to economic and political theories, has fashioned major painting, musical, architectural, and literary styles, has fathered our logic, and has furnished the best answers we have to fundamental questions about the nature of man and his universe. As the embodiment and most powerful advocate of the rational spirit, mathematics has invaded domains ruled by authority, custom, and habit, and supplanted them as the arbiter of thought and action. Finally, as an incomparably fine human achievement mathematics offers satisfactions and aesthetic values at least equal to those offered by any other branch of our culture.

Despite these by no means modest contributions to our life and thought, educated people almost universally reject mathematics as an intellectual interest. This attitude toward the subject is, in a sense, justified. School courses and books have presented 'mathematics' as a series of apparently meaningless technical procedures. Such material is as representative of the subject as an account of the name, position, and function of every bone in the human skeleton is representative of the living, thinking, and emotional being called man.

Just as a phrase either loses meaning or acquires an unintended meaning when removed from its context, so mathematics detached from its rich intellectual setting in the culture of our civilization and reduced to a series of techniques has been grossly distorted. Since the layman makes very little use of technical mathematics, he has objected to the naked and dry material usually presented. Consequently, a subject that is basic, vital, and elevating is neglected and even scorned by otherwise highly educated people. Indeed, ignorance of mathematics has attained the status of a social grace.

In this book we shall survey mathematics primarily to show how its ideas have helped to mold twentieth-century life and thought. The ideas will be in historical order so that our material will range from the beginnings in Babylonia and Egypt to the modern theory of relativity. Some people may question the pertinence of material belonging to earlier historical periods. Modern culture, however, is the accumulation and synthesis of contributions made by many preceding civilizations. The Greeks, who first appreciated the power of mathematical reasoning, graciously allowing the gods to use it in designing the universe, and then urging man to uncover the pattern of this design, not only gave mathematics a major place in their civilization but initiated patterns of thought that are basic in our own. As succeeding civilizations passed on their gifts to modern times, they handed on new and increasingly more significant roles for mathematics. Many of these functions and influences of mathematics are now deeply imbedded in our culture. Even the modern contributions of mathematics are best appreciated in the light of what existed previously.

Despite the historical approach, this book is not a history of mathematics. The historical order happens to be most convenient for the logical presentation of the subject and is the natural way of examining how the ideas arose, what the motivations for investigating these ideas were, and how these ideas influenced the course of other activities. An important by-product is that the reader may get some indication of how mathematics as a whole has developed, how its periods of activity and quiescence have been related to the general course of the history of Western civilization, and how the nature and contents of mathematics have been shaped by the civilizations that contributed to our modern Western one. It is hoped that new light will be shed on mathematics and on the dominant characteristics of our age by this account of mathematics as a fashioner of modern civilization.

We cannot, unfortunately, do more in one volume than merely illustrate the thesis. Limitations of space have necessitated a selection from a vast literature. The interrelationships of mathematics and art, for example, have been confined to the age of the Renaissance. The reader acquainted with modern science will notice that almost nothing has been said about the role of mathematics in atomic and nuclear theory. Some important modern philosophies of nature, notably Alfred North Whitehead's, have hardly been mentioned. Nevertheless, it is hoped that the illustrations chosen will be weighty enough to prove convincing as well as interesting.

The attempt to highlight a few episodes in the life of mathematics has also necessitated an over-simplification of history. In intellectual, as well as political, enterprises numerous forces and numerous individual contributions determine the outcomes. Galileo did not fashion the quantitative approach to modern science single-handed. Similarly, the calculus is almost as much the creation of Eudoxus, Archimedes, and a dozen major lights of the seventeenth century as it is that of Newton and Leibniz. It is especially true of mathematics that, while the creative work is done by individuals, the results are the fruition of centuries of thought and development.

There is no doubt that in invading the arts, philosophy, religion, and the social sciences the author has rushed in where angels—mathematical ones, of course—would fear to tread. The risk of errors, hopefully minor, must be undertaken if we are to see that mathematics is not a dry, mechanical tool but a body of living thought inseparably connected with, dependent on, and invaluable to other branches of our culture.

Perhaps this account of the achievements of human reason may serve in some small measure to reinforce those ideals of our civilization which are in danger of destruction today. The burning problems of the hour may be political and economic. Yet it is not in those fields that we find evidence of man's ability to master his difficulties and build a desirable world. Confidence in man's power to solve his problems and indications of the method he has employed most successfully thus far can be gained by a study of his greatest and most enduring intellectual accomplishment—mathematics.

It is a pleasure to acknowledge help and favors received from many sources. I wish to thank numerous colleagues in the Washington Square College of Arts and Science of New York University for many helpful discussions, Professor Chester L. Riess of the Brooklyn College of Pharmacy for general criticism and particular suggestions concerning the literature of the Age of Reason, and Mr. John Begg of Oxford University Press for advice on the preparation of the figures and plates. Mrs. Beulah Marx is to be credited with the excellent illustrations. My wife, Helen, has aided me immeasurably by critical readings and preparation of the manuscript. I am especially grateful to Mr. Carroll G. Bowen and Mr. John A. S. Cushman for their advocacy of the idea of this book and for guiding the manuscript through the process of publication at Oxford.

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Morris Kline

New York City
August 1953

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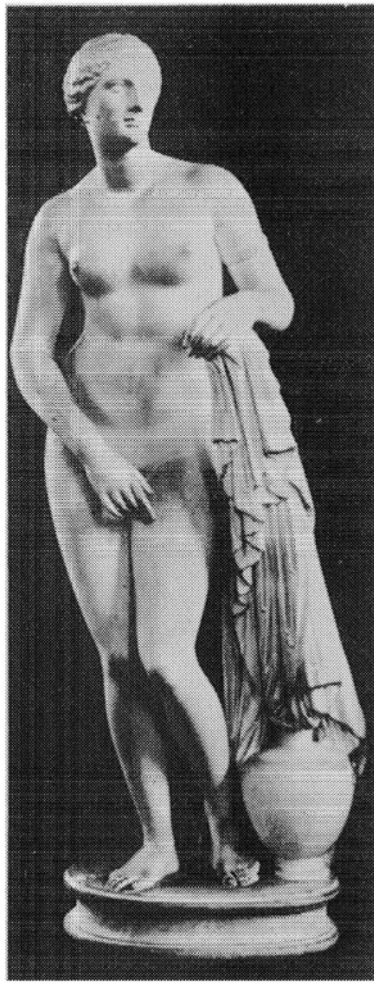

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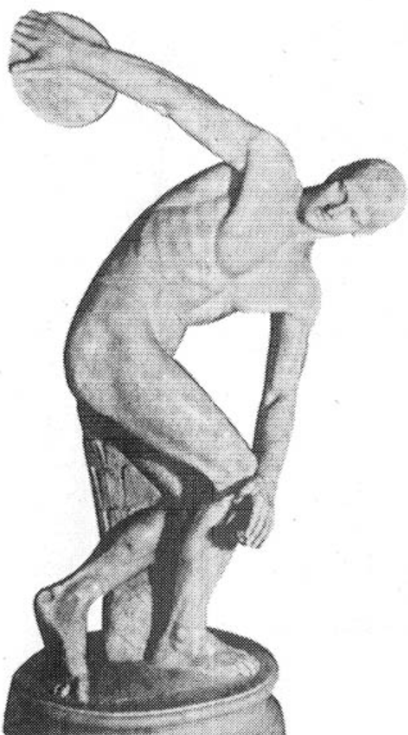

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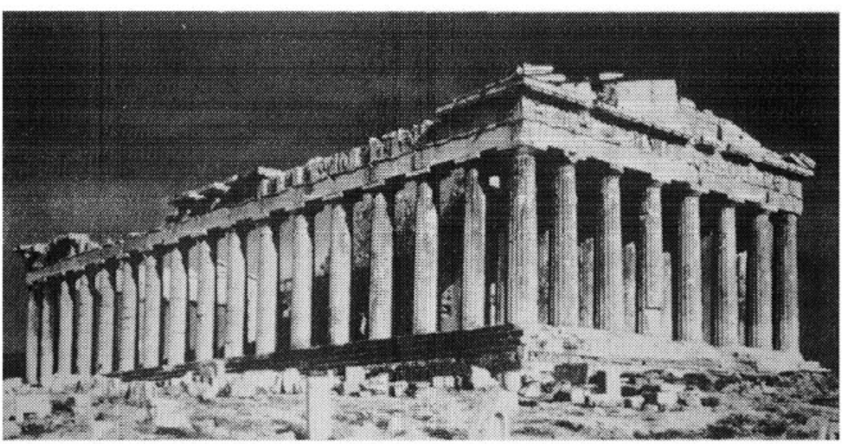

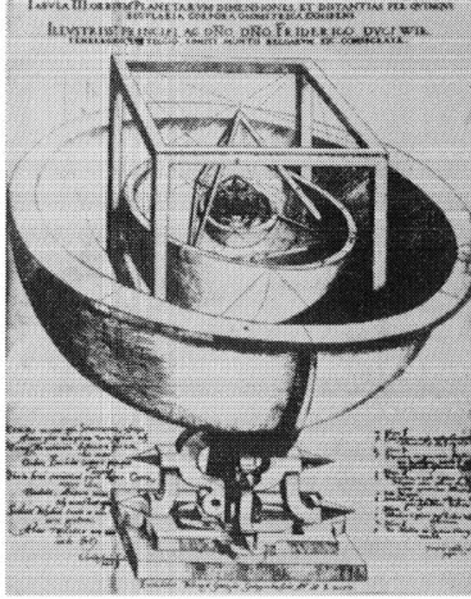

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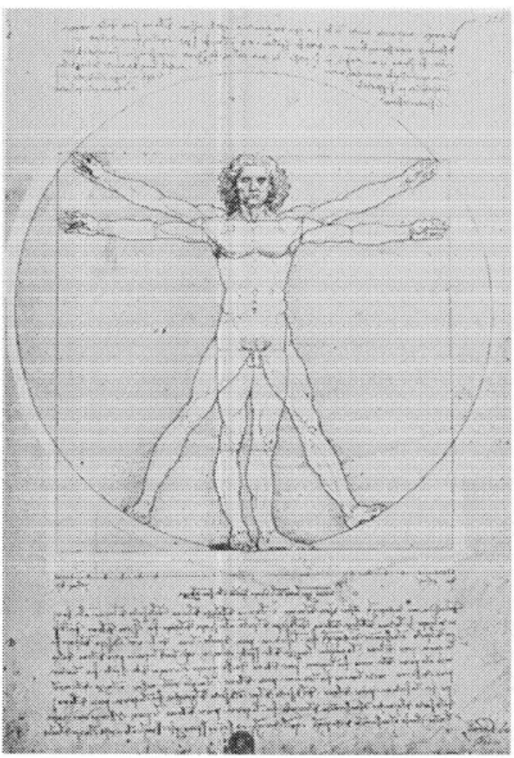

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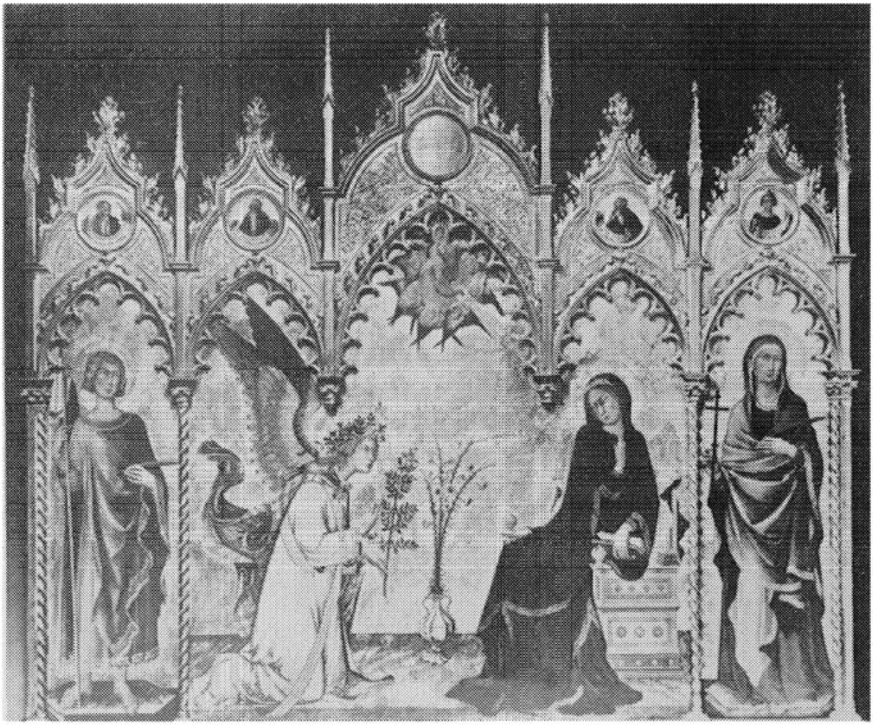

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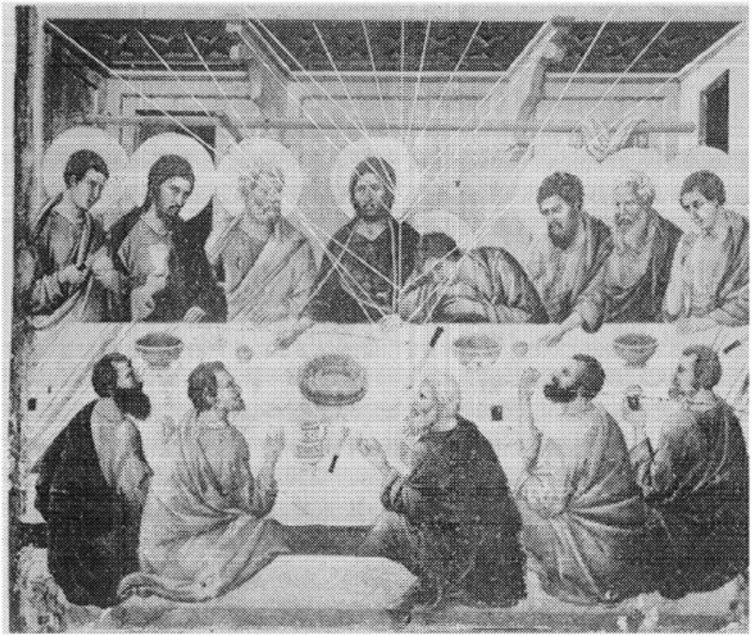

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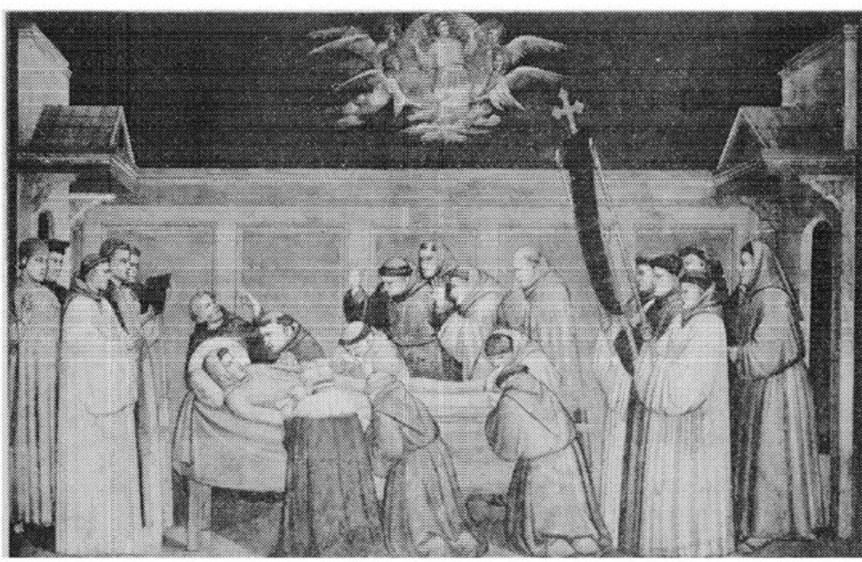

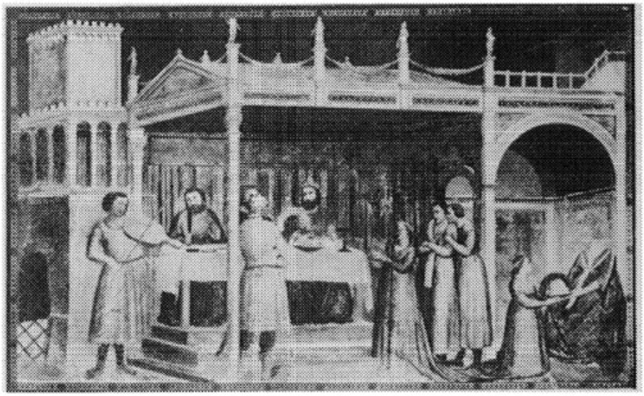

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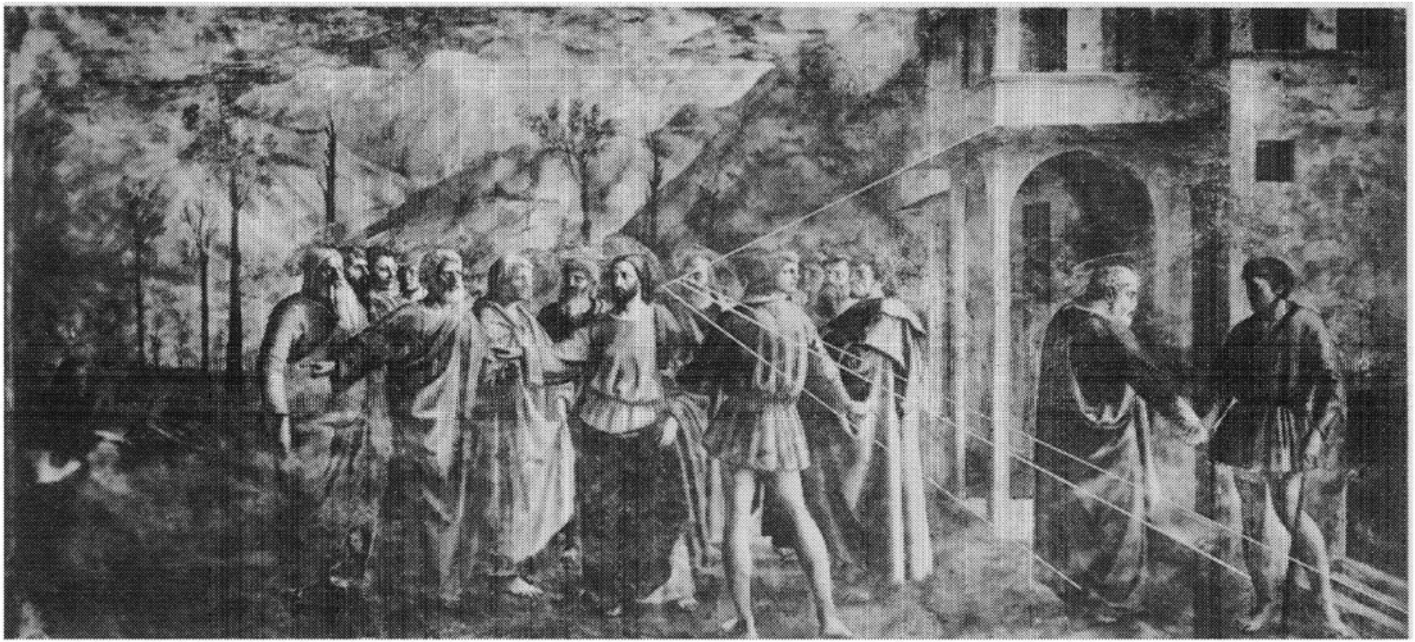

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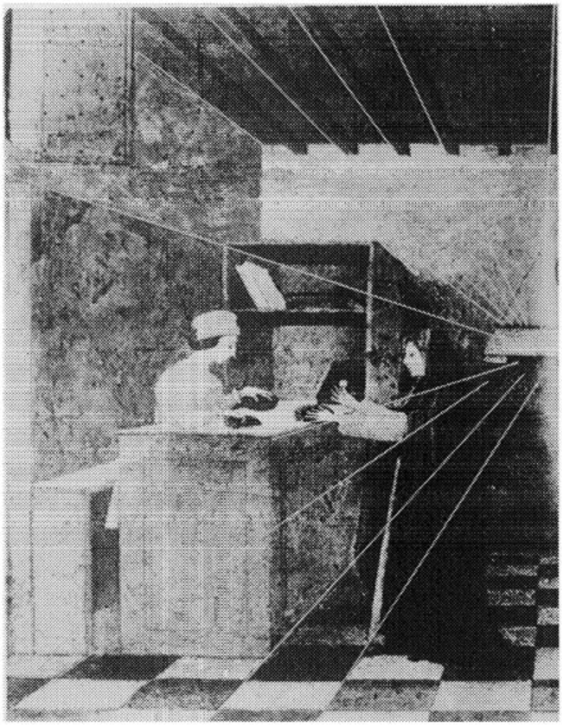

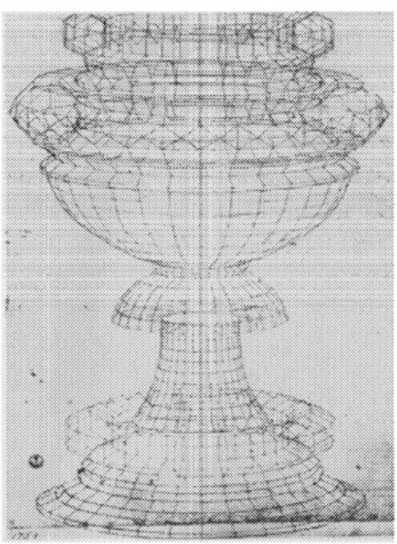

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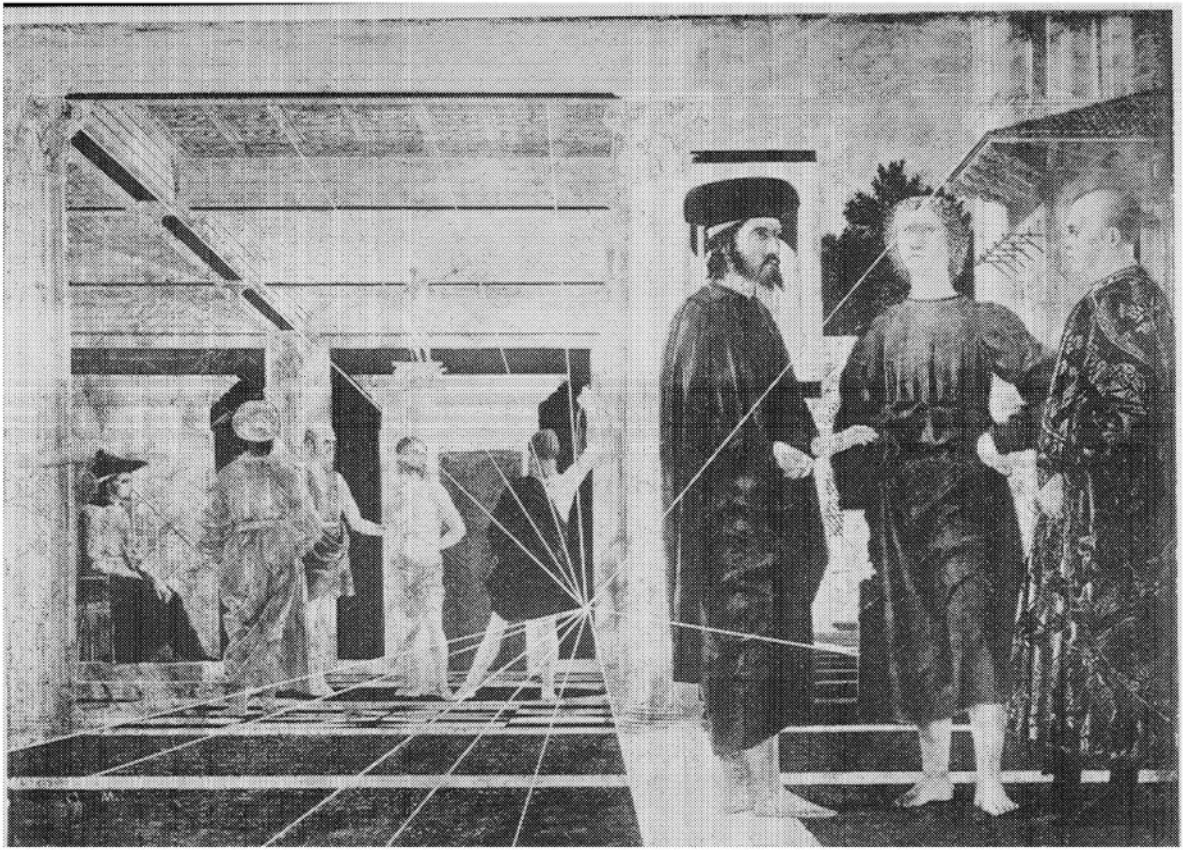

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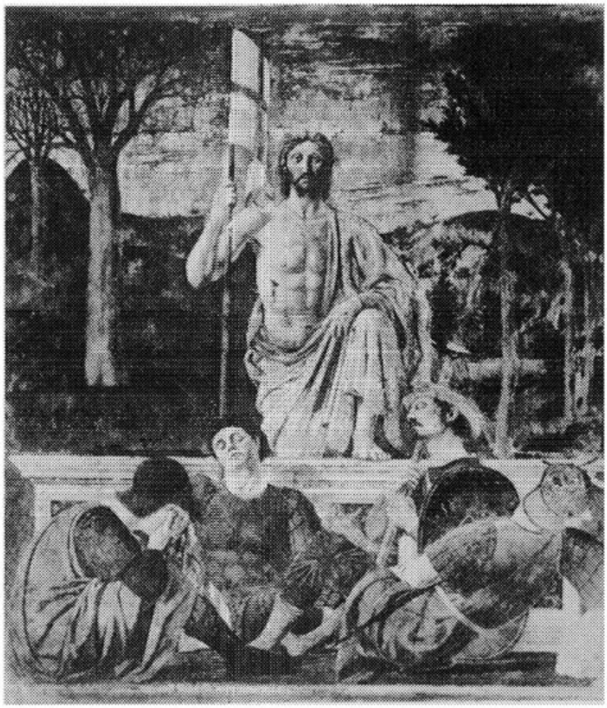

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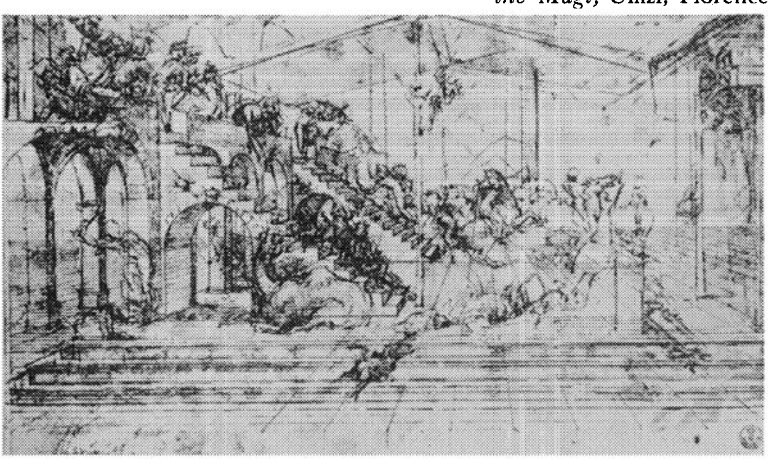

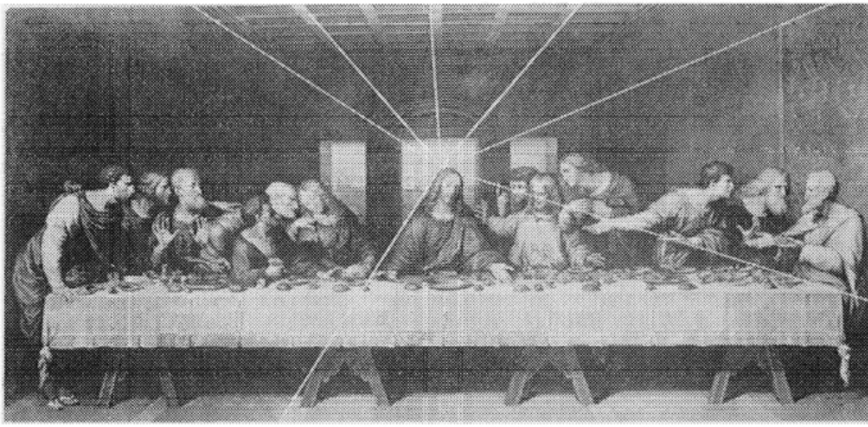

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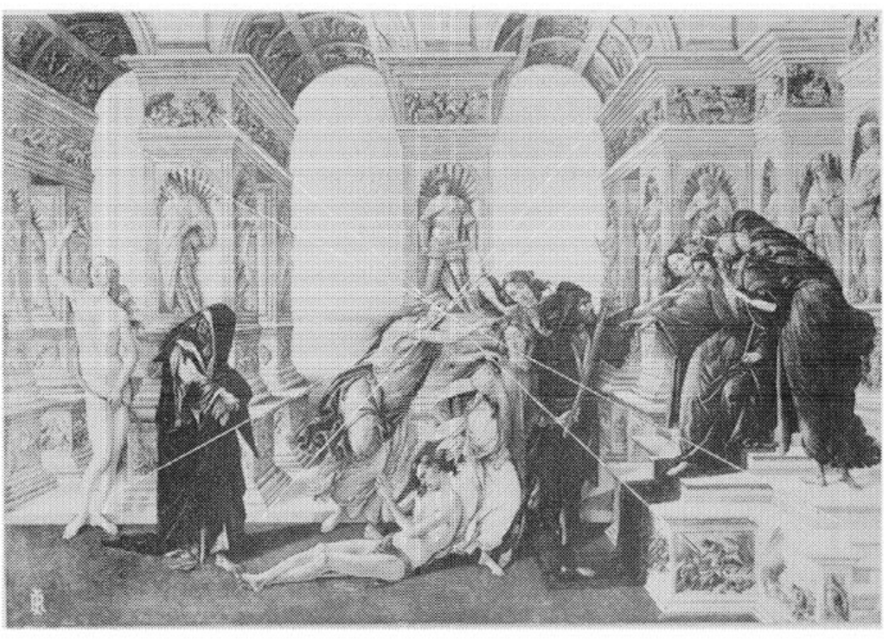

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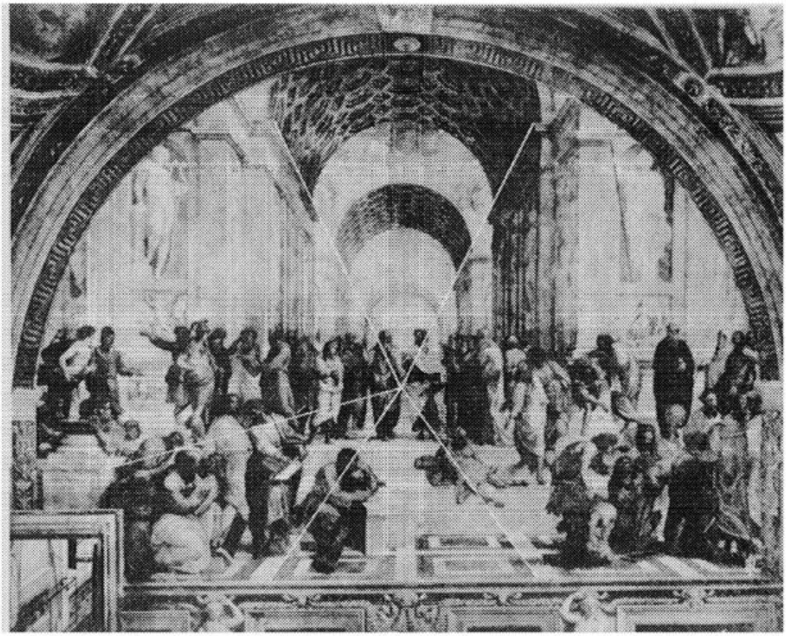

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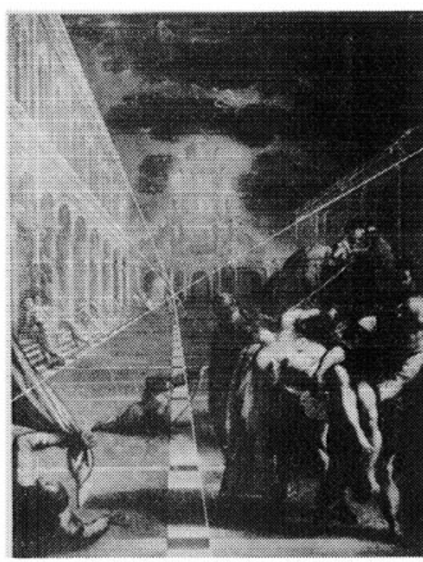

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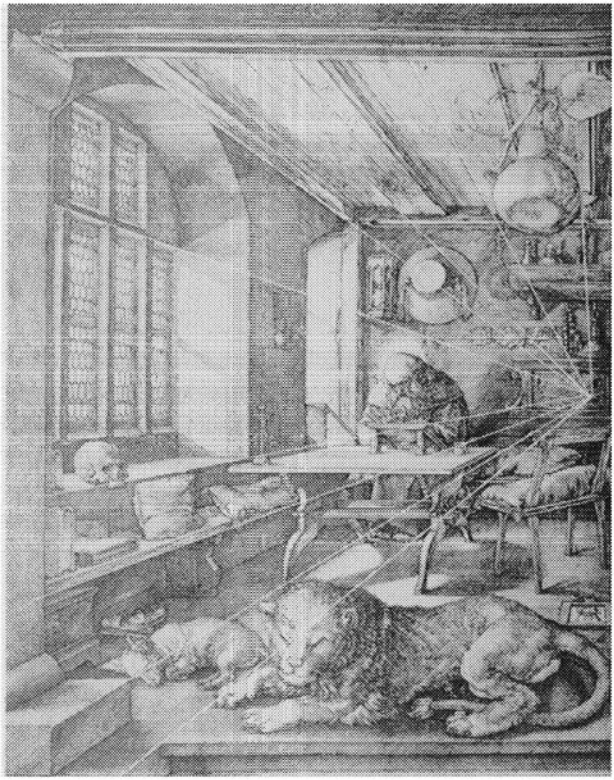

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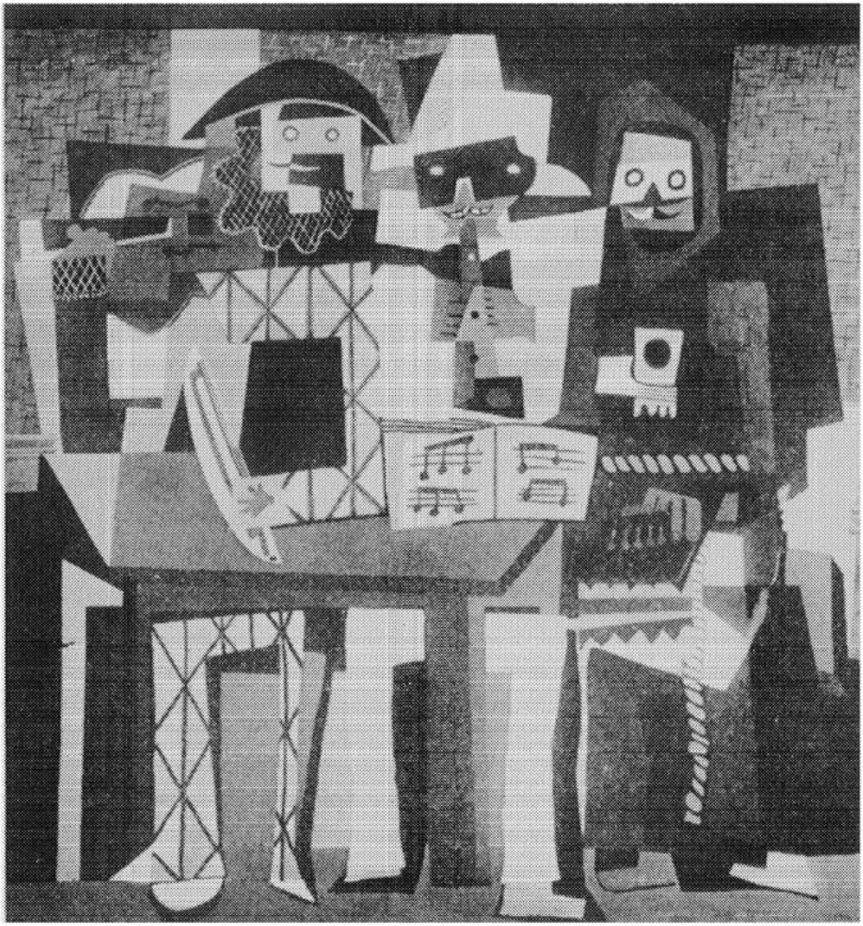

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WHEN *first I applied my mind to Mathematics I read straight away most of what is usually given by the mathematical writers, and I paid special attention to Arithmetic and Geometry because they were said to be the simplest and so to speak the way to all the rest. But in neither case did I then meet with authors who fully satisfied me. I did indeed learn in their works many propositions about numbers which I found on calculation to be true. As to figures, they in a sense exhibited to my eyes a great number of truths and drew conclusions from certain consequences. But they did not seem to make it sufficiently plain to the mind itself why these things are so, and how they discovered them. Consequently I was not surprised that many people, even of talent and scholarship, should, after glancing at these sciences, have either given them up as being empty and childish or, taking them to be very difficult and intricate, been deterred at the very outset from learning them. . . But when I afterwards bethought myself how it could be that the earliest pioneers of Philosophy in bygone ages refused to admit to the study of wisdom any one who was not versed in Mathematics . . . I was confirmed in my suspicion that they had knowledge of a species of Mathematics very different from that which passes current in our time.*

René Descartes

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I

Introduction. True and False Conceptions

Stay your rude steps, or e'er your feet invade
The Muses' haunts, ye sons of War and Trade!
Nor you, ye legion fiends of Church and Law,
Pollute these pages with unhallow'd paw!
Debased, corrupted, grovelling, and confin'd,
No definitions touch your senseless mind;
To you no Postulates prefer their claim,
No ardent Axioms your dull souls inflame;
For you no Tangents touch, no Angles meet,
No Circles join in osculation sweet!

JOHN HOOKHAM FRERE, GEORGE CANNING,
and GEORGE ELLIS

The assertion that mathematics has been a major force in the molding of modern culture, as well as a vital element of that culture, appears to many people incredible or, at best, a rank exaggeration. This disbelief is quite understandable and results from a very common but erroneous conception of what mathematics really is.

Influenced by what he was taught in school, the average person regards mathematics as a series of techniques of use only to the scientist, the engineer, and perhaps the financier. The reaction to such teachings is distaste for the subject and a decision to ignore it. When challenged on this decision a well-read person can obtain the support of authorities. Did not St. Augustine say: 'The good Christian should beware of mathematicians and all those who make empty prophecies. The danger already exists that the mathematicians have made a covenant with the devil to darken the spirit and to confine man in the bonds of Hell.' And did not the Roman jurists rule, 'concerning evil-doers, mathematicians, and the like,' that, 'To learn the art of geometry and to take part in public exercises, an art as damnable as mathematics, are forbidden.' No less a personage than the distinguished modern philosopher, Schopenhauer, described arithmetic as the lowest activity of the spirit, as is shown by the fact that it can be performed by a machine.

Despite such authoritative judgments and despite common opinion, justified as it may be in view of the teachings in the schools, the layman's decision to ignore mathematics is wrong. The subject is not a series of techniques. These are indeed the least important aspect, and they fall as far short of representing mathematics as color mixing does of painting. The techniques are mathematics stripped of motivation, reasoning, beauty, and significance. If we acquire some understanding of the nature of mathematics, we shall see that the assertion of its importance in modern life and thought is at least plausible.

Let us, therefore, consider briefly at this point the twentieth-century view of the subject. Primarily, mathematics is a method of inquiry known as postulational thinking. The method consists in carefully formulating definitions of the concepts to be discussed and in explicitly stating the assumptions that shall be the basis for reasoning. From these definitions and assumptions conclusions are deduced by the application of the most rigorous logic man is capable of using. This characterization of mathematics was expressed somewhat differently by a famous seventeenth-century writer on mathematics and science: 'Mathematicians are like lovers. . . Grant a mathematician the least principle, and he will draw from it a consequence which you must also grant him, and from this consequence another.'

To describe mathematics as only a method of inquiry is to describe da Vinci's 'Last Supper' as an organization of paint on canvas. Mathematics is, also, a field for creative endeavor. In divining what can be proved, as well as in constructing methods of proof, mathematicians employ a high order of intuition and imagination. Kepler and Newton, for example, were men of wonderful imaginative powers, which enabled them not only to break away from age-long and rigid tradition but also to set up new and revolutionary concepts. The extent to which the creative faculties of man are exercised in mathematics could be determined only by an examination of the creations themselves. While some of these will appear in subsequent discussion it must suffice here to state that there are now some eighty extensive branches of the subject.

If mathematics is indeed a creative activity, what driving force causes men to pursue it? The most obvious, though not necessarily the most important, motive for mathematical investigations has been to answer questions arising directly out of social needs. Commercial and financial transactions, navigation, calendar reckoning, the construction of bridges, dams, churches, and palaces, the design of fortifications and weapons of warfare, and numerous other human pursuits involve problems that can best be resolved by mathematics. It is especially true of our engineering age that mathematics is a universal tool.

Another basic use of mathematics, indeed one that is especially prominent in modern times, has been to provide a rational organization of natural phenomena. The concepts, methods, and conclusions of mathematics are the substratum of the physical sciences. The success of these fields has been dependent on the extent to which they have entered into partnership with mathematics. Mathematics has brought life to the dry bones of disconnected facts and, acting as connective tissue, has bound series of detached observations into bodies of science.

Intellectual curiosity and a zest for pure thought have started many mathematicians in pursuit of properties of numbers and geometric figures and have produced some of the most original contributions. The whole subject of probability, important as it is today, began with a question arising in a game of cards, namely, the proper division of a gambling stake in a game interrupted before its close. Another most decisive contribution in no way connected with social needs or science was made by the Greeks of the classical period who converted mathematics into an abstract, deductive, and axiomatic system of thought. In fact, some of the greatest contributions to the subject matter of mathematics—projective geometry, the theory of numbers, the theory of infinite quantities, and non-Euclidean geometry, to mention only those that will be within our purview—constitute responses to purely intellectual challenges.

Over and above all other drives to create is the search for beauty. Bertrand Russell, the master of abstract mathematical thought, speaks without qualification:

> Mathematics, rightly viewed, possesses . . . supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet

sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

In addition to the beauty of the completed structure, the indispensable use of imagination and intuition in the creation of proofs and conclusions affords high aesthetic satisfaction to the creator. If insight and imagination, symmetry and proportion, lack of superfluity, and exact adaption of means to ends are comprehended in beauty and are characteristic of works of art, then mathematics is an art with a beauty of its own.

Despite the clear indications of history that all of the factors above have motivated the creation of mathematics there have been many erroneous pronouncements. There are the charges—often made to excuse neglect of the subject—that mathematicians like to indulge in pointless speculations or that they are silly and useless dreamers. To these charges a crushing reply can readily be made. Even purely abstract studies, let alone those motivated by scientific and engineering needs, have proved immensely useful. The discovery of the conic sections (parabolas, ellipses, and hyperbolas) which, for two thousand years, amounted to no more than 'the unprofitable amusement of a speculative brain,' ultimately made possible modern astronomy, the theory of projectile motion, and the law of universal gravitation.

On the other hand, it is a mistake to assert, as some 'socially minded' writers do rather sweepingly, that mathematicians are stimulated entirely or even largely by practical considerations, by the desire to build bridges, radios, and airplanes. Mathematics has made these conveniences possible, but the great mathematicians rarely have them in mind while pursuing their ideas. Some were totally indifferent to the practical applications, possibly because these were made centuries later. The idealistic mathematical musings of Pythagoras and Plato have led to far more significant contributions than the purposeful act of the warehouse clerks whose introduction of the symbols + and − convinced one writer that 'a turning point in the history of mathematics arose from the common social heritage. . .' It is no doubt true that almost every great man occupies himself with the problems of his age, and that prevailing beliefs condition and limit his thinking. Had Newton been born two hundred years earlier he would most likely have been a masterful theologian. Great thinkers yield to the intellectual fashions of their times as women do to fashions in dress. Even those creative geniuses to whom mathematics was purely an avocation pursued the problems that were agitating the professional mathematicians and scientists. Nevertheless, these 'amateurs' and mathematicians generally have not been concerned primarily with the utility of their work.

Practical, scientific, aesthetic, and philosophical interests have all shaped mathematics. It would be impossible to separate the contributions and influences of any one of these forces and weigh it against the others, much less assert sweeping claims to its relative importance. On the one hand, pure thought, the response to aesthetic and philosophical interests, has decisively fashioned the character of mathematics and made such unexcelled contributions as Greek geometry and modern non-Euclidean geometry. On the other hand, mathematicians reach their pinnacles of pure thought not by lifting themselves by their bootstraps but by the power of social forces. Were these forces not permitted to revitalize mathematicians, they would soon exhaust themselves; thereafter they could merely sustain their subject in an isolation which might be splendid for a short time but which would soon spell intellectual collapse.

Another important characteristic of mathematics is its symbolic language. Just as music uses symbolism for the representation and communication of sounds, so mathematics expresses quantitative relations and spatial forms symbolically. Unlike the usual language of discourse, which is the product of custom, as well as of social and political movements, the language of mathematics is carefully, purposefully, and often ingeniously designed. By virtue of its compactness, it permits the mind to carry and work with ideas which, expressed in ordinary language, would be unwieldy. This compactness makes for efficiency of thought. Jerome K. Jerome's need to resort to algebraic symbolism, though for non-mathematical purposes, reveals clearly enough the usefulness and clarity inherent in this device:

When a twelfth-century youth fell in love he did not take three paces backward, gaze into her eyes, and tell her she was too beautiful to live. He said he would step outside and see about it. And if, when he got out, he met a man and broke his head—the other man's head, I mean—then that proved that his—the first fellow's—girl was a pretty girl. But if the other fellow broke *his* head—not his own, you know, but the other fellow's

—the other fellow to the second fellow, that is, because of course the other fellow would only be the other fellow to him, not the first fellow who—well, if he broke his head, then *his* girl—not the other fellow's, but the fellow who *was* the— Look here, if A broke B's head, then A's girl was a pretty girl; but if B broke A's head, then A's girl wasn't a pretty girl, but B's girl was.

While clever symbolism enables the mind to carry complicated ideas with ease, it also makes it harder for the layman to follow or understand a mathematical discussion.

The symbolism used in mathematical language is essential to distinguish meanings often confused in ordinary speech. For example, the word *is* is used in English in many different senses. In the sentence *He is here,* it indicates a physical location. In the sentence *An angel is white,* it indicates a property of angels that has nothing to do with location or physical existence. In the sentence *The man is running,* the word gives the tense of the verb. In the sentence *Two and two are four,* the form of *is* used denotes numerical equality. In the sentence *Men are the two-legged thinking mammals,* the form of *is* involved asserts the identity of two groups. Of course, for the purposes of ordinary discourse it is superfluous to introduce different words for all these meanings of *is.* No mistakes are made on account of these ambiguities. But the exactions of mathematics, as well as of the sciences and philosophy, compel workers in these fields to be more careful.

Mathematical language is precise, so precise that it is often confusing to people unaccustomed to its forms. If a mathematician should say, 'I did not see one person today,' he would mean that he either saw none or saw many. The layman would mean simply that he saw none. This precision of mathematics appears as pedantry or stiltedness to one who does not yet appreciate that it is essential to exact thinking, for exact thinking and exact language go hand in hand.

Mathematical style aims at brevity and formal perfection. It sometimes succeeds too well and sacrifices the clarity its precision seeks to guarantee. Let us suppose we wish to express in general terms the fact illustrated in figure 1. We might be tempted to say: 'We have a right triangle. If we construct two squares each having an arm of the triangle as a side and if we construct a square having the hypotenuse of the triangle for its side, then the area of the third square is equal to the sum of the areas of the first two.' But no mathematician would deign to express himself that way. He prefers: 'The sum of the squares on the arms of a right triangle equals the square on the hypotenuse.' This economy of words makes for deftness of presentation, and mathematical writing is remarkable because it does encompass much in few words. Yet there are times when any reader of mathematical literature finds his patience sorely tried by what he would call a miserliness with ink and paper.

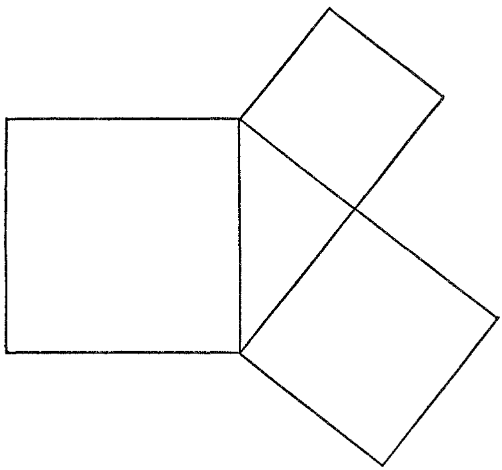

Figure 1. The Pythagorean theorem

Mathematics is more than a method, an art, and a language. It is a body of knowledge with content that serves the physical and social scientist, the philosopher, the logician, and the artist; content that influences the doctrines of statesmen and theologians; content that satisfies the curiosity of the man who surveys the heavens and the man who muses on the sweetness of musical sounds; and content that has undeniably, if sometimes imperceptibly, shaped the course of modern history.

Mathematics is a body of knowledge. But it contains no truths. The contrary belief, namely, that mathematics is an unassailable collection of truths, that it is like a final revelation from God such as religionists believe the Bible to be, is a popular fallacy most difficult to dislodge. Up to the year 1850, even mathematicians sub scribed to this fallacy. Fortunately, some events of the nineteenth century, which we propose to discuss later, showed the mathematicians the error of their ways. Not only is there no truth in the subject, but theorems in some branches contradict theorems in others. For example, some of the theorems established in geometries created during the last century contradict those proved by Euclid in his development of geometry. Though devoid of truth, mathematics has given man miraculous power over nature. The resolution of this greatest paradox in human thought will be one of our major concerns.

Because the twentieth century must distinguish mathematical knowledge from truths it must also distinguish between mathematics and science, for science does seek truths about the physical world. Mathematics has indeed been a beacon light to the sciences and has continually helped them in reaching the position they occupy in our present civilization. It is even correct to assert that modern science triumphs by virtue of mathematics. Yet we shall see that the two fields are distinct.

In its broadest aspect mathematics is a spirit, the spirit of rationality. It is this spirit that challenges, stimulates, invigorates, and drives human minds to exercise themselves to the fullest. It is this spirit that seeks to influence decisively the physical, moral, and social life of man, that seeks to answer the problems posed by our very existence, that strives to understand and control nature, and that exerts itself to explore and establish the deepest and utmost implications of knowledge already obtained. To a large extent our concern in this book will be with the operation of this spirit.

One more characteristic of mathematics is most pertinent to our story. Mathematics is a living plant which has flourished and languished with the rise and fall of civilizations. Created in some prehistoric period it struggled for existence through centuries of prehistory and further centuries of recorded history. It finally secured a firm grip on life in the highly congenial soil of Greece and waxed strong for a brief period. In this period it produced one perfect flower, Euclidean geometry. The buds of other flowers opened slightly and with close inspection the outlines of trigonometry and algebra could be discerned; but these flowers withered with the decline of Greek civilization, and the plant remained dormant for one thousand years.

Such was the state of mathematics when the plant was transported to Europe proper and once more imbedded in fertile soil. By A.D. 1600 it had regained the vigor it had possessed at the very height of the Greek period and was prepared to break forth with unprecedented brilliance. If we may describe the mathematics known before 1600 as elementary mathematics, then we may state that elementary mathematics is infinitesimal compared to what has been created since. In fact, a person possessed of the knowledge Newton had at the height of his powers would not be considered a mathematician today for, contrary to popular belief, mathematics must now be said to begin with the calculus and not to end there. In our century, the subject has attained such vast proportions that no mathematician can claim to have mastered the whole of it.

This sketch of the life of mathematics, however brief, may nevertheless indicate that its vitality has been very much dependent on the cultural life of the civilization which nourished it. In fact, mathematics has been so much a part of civilizations and cultures that many historians see mirrored in the mathematics of an age the characteristics of the other chief works of that age. Consider, for example, the classical period of Greek culture, which lasted from about 600 B.C. to 300 B.C. In emphasizing the rigorous reasoning by which they established their conclusions, the Greek mathematicians were concerned not with guaranteeing applicability to practical problems but with teaching men to reason abstractly and with preparing them to contemplate the ideal and the beautiful. It should be no surprise to learn, then, that this age has been unsurpassed in the beauty of its literature, in the supremely rational quality of its philosophy, and in the ideality of its sculpture and architecture.

It is also true that the absence of mathematical creations is indicative of the culture of a civilization. Witness the case of the Romans. In the history of mathematics the Romans appear once and then only to retard its progress. Archimedes, the greatest Greek mathematician and scientist, was killed in 211 B.C. by Roman soldiers who burst in upon him while he was studying a geometrical diagram drawn in sand. To Alfred North Whitehead,

> The death of Archimedes by the hands of a Roman soldier is symbolical of a world change of the first magnitude; the theoretical Greeks, with their love of abstract science, were superseded in leadership of the European world by the practical Romans. Lord Beaconsfield, in one of his

novels, has defined a practical man as a man who practices the errors of his forefathers. The Romans were a great race, but they were cursed with the sterility which waits upon practicality. They did not improve upon the knowledge of their forefathers, and all their advances were confined to the minor technical details of engineering. They were not dreamers enough to arrive at new points of view, which could give a more fundamental control over the forces of Nature. No Roman lost his life because he was absorbed in the contemplation of a mathematical diagram.

In fact, Cicero bragged that his countrymen, thank the gods, were not dreamers, as were the Greeks, but applied their study of mathematics to the useful.

Practical-minded Rome, which devoted its energies to administration and conquest, symbolized best perhaps by the stolid if not graceful arches under which victorious troops celebrated their homecomings, produced little that was truly creative and original. In short, Roman culture was derivative; most of the contributions made during the period of Roman supremacy came from the Greeks of Asia Minor, who were under the political domination of Rome.

These examples show us that the general character of an age is intimately related to its mathematical activity. This relationship is especially valid in our times. Without belittling the merits of our historians, economists, philosophers, writers, poets, painters, and statesmen, it is possible to say that other civilizations have produced their equals in ability and accomplishments. On the other hand, though Euclid and Archimedes were undoubtedly supreme thinkers and though our mathematicians have been able to reach farther only because, as Newton put it, they stood on the shoulders of such giants, nevertheless, it is in our age that mathematics has attained its range and extraordinary applicability. Consequently, present-day Western civilization is distinguished from any other known to history by the extent to which mathematics has influenced contemporary life and thought. Perhaps we shall come to see in the course of this book how much the present age owes to mathematics.

II

The Rule of Thumb in Mathematics

Do not imagine that mathematics is hard and crabbed and repulsive to common sense. It is merely the etherealization of common sense.

LORD KELVIN

The cradle of mankind, as well as of Western culture, was the Near East. While the more restless abandoned this birthplace to roam the plains of Europe, their kinsmen remained behind to found civilizations and cultures. Many centuries later the wise men of the East had to assume the task of educating their still untutored relatives. Of the knowledge which these sages imparted to Western man the elements of mathematics were an integral part. Hence, to trace the impress of mathematics on modern culture, we must turn to the major Near Eastern civilizations.

We should mention in passing that simple mathematical steps were made in primitive civilizations. Such steps were no doubt prompted by purely practical needs. The barter of necessities, which takes place in even the most primitive types of human society, requires some counting.

Since the process of counting is facilitated by the use of the fingers and toes, it is not surprising that primitive man, like a child, used his fingers and toes as a tally to check off the things he counted. Traces of this ancient way of counting are imbedded in our own language, the word *digit* meaning not only the numbers 1, 2, 3 . . . but a finger or a toe as well. The use of the fingers undoubtedly accounts for the adoption of our system of counting in tens, hundreds (tens of tens), thousands (tens of hundreds), and so forth.

Even primitive civilizations developed special symbols for numbers. In this way, these civilizations showed cognizance of the fact that three sheep, three apples, and three arrows have much in common, namely the quantity *three*. This appreciation of number as an abstract idea, abstract in the sense that it does not have to relate to particular physical objects, was one of the major advances in the history of thought. Each of us in his own schooling goes through a similar intellectual process of divorcing numbers from physical objects.

Primitive civilizations also invented the four elementary operations of arithmetic, that is, addition, subtraction, multiplication, and division. That these operations did not come readily to man can be learned even from a study of contemporary backward peoples. When sheep owners of many primitive tribes sell several animals, they will not take a lump sum for the lot but must be paid for each one separately. The alternative of multiplying the number of sheep by the price per sheep confuses them and leaves them suspecting that they have been cheated.

There is little question that geometry, like the number system, was fostered in primitive civilizations to satisfy man's needs. Fundamental geometric concepts came from observation of figures formed by physical objects. It is likely that the concept of angle, for example, first came from observation of the angles formed at the elbows and knees. In many languages, including modern German, the word for the side of an angle is the word for leg. We ourselves speak of the arms of a right triangle.

The major Near Eastern civilizations from which our culture and our mathematics sprang were the Egyptian and the Babylonian. In the earliest records of these civilizations we find well-developed number systems, some algebra, and very simple geometry. For the numbers from 1 to 9 the Egyptians used simple strokes thus: I, II, III, etc. For 10 they introduced the special symbol ∩, and there were special symbols for 100, 1000, and other large numbers. For intermediate numbers they combined these symbols in a very natural manner. Thus 21 was written ∩∩I

The Babylonian method of writing quantity deserves more attention. For 1 they wrote 𒁹; 2 was represented by 𒁹𒁹; 4, by 𒁹𒁹𒁹𒁹; and so forth up to nine. The symbol 𒌋 was used for 10. Thus 33 was 𒌋𒌋𒌋𒁹𒁹𒁹. The number 𒁹𒌋𒌋𒁹 is especially significant. Here the first 𒁹 meant not 1 but 60, and the whole group represented 60 + 10 + 10 + 1 or 81. Thus the same symbol represented different values depending on its position in the number. The principle invloved here is that of place value and is precisely the one we use today. In the number 569, the 9 represents 9 units but the 6 means 6 times 10 and the 5 means 5 times 100 or 5 times 10^2. In other words, the position of a digit in the number determines the value it represents, and this value is a multiple of 10, of the square of 10, or of the cube of 10, and so on, depending on the position of the digit. The number *ten* is called the base of our system.

Because the Babylonians introduced place value in connection with the base sixty, the Greeks and Europeans used this system in all mathematical and astronomical calculations until the sixteenth century and it still survives in the division of angles and hours into 60 minutes and 60 seconds. Base ten was developed by the Hindus and introduced into Europe during the late Middle Ages.

The principle of place value is so important that it merits a bit of discussion. Taken in conjunction with base ten, ten symbols suffice to represent any quantity no matter how large. The representation is systematic and compact compared to other methods such as the Egyptian. Even more important is the fact that the principle permitted the development of our modern efficient methods of computation.

We should notice too that it is not necessary to use ten as a base. Any whole number would do as well in principle. Suppose, for example, that a person were to use five. He would then need just five symbols, say 1, 2, 3, 4, and 0. To indicate the quantity five he would write 10, the 1 this time meaning 1 times five, just as the 1 in the familiar 10 means 1 times ten. To write six in the base five he would write 11. Seven would be 12. Eleven would be 21. Twenty-five would be 100 or 1 times $5^2 + 0$ times $5 + 0$ units. To use the base five systematically he would of course have to learn the relevant addition and multiplication tables. Thus $3 + 4$ would be 12; $13 + 14$, the two numbers being in base five and representing eight and nine, respectively, would be 32; and so on. The question, what is the best base, has been seriously considered and there are good reasons to favor twelve. Custom rules in favor of ten, however, as far as ordinary uses of numbers are concerned.

To use the principle of place value to best advantage a zero is required, for there must be some way to distinguish 503, say, from 53. The Babylonians used a special symbol to separate the 5 and 3 in the former case but failed to recognize that this symbol could also be treated as a number; that is, they failed to see that zero indicates quantity and can be added, subtracted, and used generally like other numbers. The number *zero* must be carefully distinguished from the concept of nothing. A student's grade in a mathematics course is nothing if he never took that course. If, on the other hand, he did take the course and his work was judged worthless, his grade would be zero.

For early civilizations computation with fractions was not a simple matter. The Babylonians lacked adequate notation. Thus <<< meant $^{30}/_{60}$, as well as 30; the correct value had to be understood from the context. The Egyptians found it necessary to reduce a fraction to a sum of fractions in each of which the numerator was unity. Thus they would express $5/8$ as $1/2 + 1/8$ before computing with it. Though modern methods of handling fractions are much more efficient they still give trouble to many adults.

The ancient civilizations of Babylonia and Egypt carried their arithmetic beyond the use of integers and fractions. We know they were able to solve some problems involving unknown quantities, although by methods cruder and less general than we learn in our secondary schools. Babylonia is, in fact, considered to be the source of some of Euclid's knowledge of algebra.

Whereas the Babylonians developed a superior arithmetic and algebra, the Egyptians are generally considered to have surpassed them in geometry. There is much speculation about why this was so. One reason offered by historians is that the Egyptians never developed convenient methods of working with numbers, particularly fractions, and consequently were prevented from going further in the field of algebra. Instead they emphasized geometry. Another view is that geometry is a 'gift of the Nile.' Herodotus relates that in the fourteenth century B.C. King Sesostris had so divided the land among all Egyptians that each received a rectangle of the same size and was taxed accordingly. If a man lost any of his land by the annual overflow of the Nile, he had to report the loss to the Pharaoh who would then send an overseer to measure the loss and make a proportionate abatement of the tax. Thus from the soil of Egypt the science of geometry—*geo* meaning earth, *metron* meaning measure—arose and flourished. Herodotus may have correctly selected the reason for the emphasis on geometry in Egypt but seems to have overlooked its existence for millenniums preceding the fourteenth century B.C.

Egyptian and Babylonian geometry was of the rule-of-thumb or practical variety. Straight lines meant no more than stretched pieces of cord; the Greek word 'hypotenuse,' in fact, means stretched against,' presumably against the two arms of the right angle. A plane was merely the surface of a piece of flat land. Their formulas for volume of granaries and areas of land were arrived at by trial and error. As a consequence, many of these formulas were definitely faulty. For example, an Egyptian formula for the area of a circle was 3.16 times the square of the radius. This is not correct though close enough for the uses the Egyptians made of it.

The Egyptians and Babylonians made numerous practical applications of their mathematics. Their papyri and clay tablets show promissory notes, letters of credit, mortgages, deferred payments, and the proper apportionment of business profits. Although arithmetic and algebra were used in such commercial transactions, geometrical formulas produced the areas of fields and the amounts of grain stored in cylindrical and pyramidal granaries. In addition, the Babylonians and Egyptians were indefatigable builders. Even in this age of skyscrapers their temples and pyramids appear to us to be admirable engineering achievements. The Babylonians were also highly skilled irrigation engineers. Through cleverly dug canals, the Tigris and Euphrates Rivers, the life's blood of these people, fertilized the land and made possible in that dry, hot climate the support of thriving and populous cities such as Ur and Babylon.

But it is a mistake—no matter how often it is repeated—to believe that mathematics in Egypt and Babylonia was confined just to the solution of practical problems. This belief is as false for those times as it is for our own. Instead we find, upon closer investigation, that the exact expression of man's thoughts and emotions, whether artistic, religious, scientific, or philosophical, involved then, as today, some aspect of mathematics. In Babylonia and Egypt the association of mathematics with painting, architecture, religion, and the investigation of nature was no less intimate and vital than its use in commerce, agriculture, and construction.

Those writers who believe that mathematics possesses only utilitarian value often read into history a practical motivation for mathematical activity that logically could not have existed. Their argument runs like this: mathematics was applied to calendar reckoning and navigation; hence the creation of mathematics was motivated by these practical problems much as the need to count led to the number system. This *post hoc ergo propter hoc* type of argument has no history and very little probability in its favor. No mariner lost at sea suddenly decided that the stars were the answer to his navigation problem; nor did some Egyptian farmer, concerned about the number of days until the annual flood of the Nile, decide that he would thereafter watch the course of the sun.

Preceding the use of astronomy and of mathematics for navigation and calendar reckoning there must have been centuries during which men filled with instinctive wonder and awe of nature, men with irrepressible philosophical drives, patiently observed the movement of the sun, moon, and stars. These seers, obsessed by the mystery of nature, overcame the handicaps of lack of instruments and woefully inadequate mathematics to distill from their observations the patterns the heavenly bodies describe. These are the men who very early in the Egyptian civilization learned that the solar year, the year of the seasons, consists of about 365 days.

Their patience and persistence accomplished even more. They observed that the star Sirius appeared in the sky at sunrise on that day of the year when the annual flood of the Nile reached Cairo. This observation must have been made for many years before it was decided to chart Sirius' path in the heavens in order to predict the flood. More than that, since the calendar year of 365 days was a quarter of a day short of the true solar year, after several years the calendar no longer told when Sirius would appear in the sky at dawn. Only after 1460 years, that is, 4×365, would the calendar and the position of Sirius in the sky agree once more. This period of 1460 years, called the Sothic cycle, was also known to the Egyptian astronomers. Surely the existence of such regularities in the heavens had to be recognized before anyone could think of applying them.

Once the astronomical and mathematical studies revealed these regularities, the Babylonian and Egyptian learned to watch the face of the sky. He hunted, fished, sowed, reaped, danced, and performed religious ceremonies at the times the heavens dictated. Soon particular constellations received the names of the activities their appearance sanctioned. Sagittarius, the hunter, and Pisces, the fish, are still in the sky.

The heavens decided the time of events. But such imperious masters would brook no tardiness in compliance with their orders. The Egyptian, who made his living by tilling the soil which the Nile covered with rich silt during its annual overflow of the country, had to be well prepared for the flood. His home, equipment, and cattle had to be temporarily removed from the area, and arrangements made for sowing immediately afterward. Hence the coming of the flood had to be predicted. Not only in Egypt but in all lands it was necessary to know beforehand the time for planting and the coming of holidays and days of sacrifice.

Prediction was not possible, however, by merely keeping count of the passing days and nights. For the calendar year of 365 days soon lost all relation to the seasons just because it was short by a quarter of a day. Prediction of a holiday or the Nile flood even a few days in advance required an accurate knowledge of the motions of the heavenly bodies and of mathematics that was possessed only by the priests. These votaries, knowing the importance of the calendar for the regulation of daily life and for provident preparation, capitalized on this knowledge to secure power over the uninformed masses. In fact, it is believed that the Egyptian priests knew the solar year, that is, the year of the seasons, to be 365¼ days in length but deliberately withheld this knowledge from the people. Knowing also when the flood was due, the priests could pretend to bring it about with their rites while making the poor farmer pay for the performance. Knowledge of mathematics and science was power then as it is today.

Though wonder about the heavens led to mathematics through its respectable relation astronomy, religious mysticism, itself an expression of wonder about life, death, wind, rain, and the panorama of nature, soon fastened on mathematics through its now disreputable relation astrology. Of course, the importance of astrology in ancient religions must not be judged by its discredited position today. In almost all these religions, the heavenly bodies, the sun especially, were gods who ruled over events on the Earth. The will and plans of these gods might be fathomed by studying their activities, their regular comings and goings, the sudden visitations of meteors, and the occasional eclipses of the sun and moon. It was as natural for the ancient priests to work out formulas for the divination of the future based on the motions of the planets and star constellations as it is for the modern scientist to study and master nature with his techniques.

Even if the heavenly bodies had not been gods, a scientifically immature people would have had good reasons to associate the positions of the sun, moon, and stars with human affairs. The dependence of crops upon the sun and upon weather in general, the mating of animals at definite seasons of the year, the periodicity in women, which even Aristotle and Galen believed to be controlled by the action of the moon, and numerous other similar associations, all lent strong credence to such a doctrine. To the Egyptians, in particular, the coming of the Nile flood on just the day that Sirius appeared in the sky at sunrise meant one thing: Sirius caused the flood.

Religious mysticism expressed itself directly *de more geometrico* in the construction and orientation of beautiful temples and pyramids. Every major Babylonian city built a *ziggurat,* a temple in the form of a tower. This was an imposing edifice erected on top of a succession of terraces, approached by broad flights of steps, and clearly visible for miles around. The Egyptian pyramids and temples are, of course, well known. The pyramids in particular were constructed with special care because they were royal tombs, and the Egyptians believed that construction according to exact mathematical prescriptions was essential for the future life of the dead. The orientation of these religious structures in relation to the heavenly bodies is well illustrated by the famous temple of Amon-Ra, the sun god, at Karnak. The building faced the setting sun at the summer solstice, and on that day the sun shone directly into the temple and illuminated the rear wall.

Nor did religious mysticism overlook the intriguing properties of numbers as a vehicle for expressing its ideas. The numbers three and seven attracted special attention. Since the universe was evidently constructed in a definite period of time, why not utilize a desirable number like seven? That it should be a matter of days seemed a good compromise between the power of God and the complexity of nature.

The science of the cabala illustrates how far religionists were willing to go to explain the mystery of the universe in terms of number. Tradition credits the Babylonian priests with the invention of this mystic and demoniac science of numbers, which the Hebrews later expanded. This pseudo-science was based on the following idea. Each letter of the alphabet was associated with a number. In fact, the Greeks and Hebrews used the letters of the alphabet as their number symbols. With each word was associated the number that was the sum of the numbers attached to the letters spelling the word. Two words having the same associated number were believed to be related, and this connection was used to make predictions. Thus a man's death might be prophesied because the numbers attached to the name of some enterprise he planned to undertake and to the word death were the same.

Man's artistic interests vied with his religious feelings to discover and utilize mathematical knowledge. While the architects studied and applied geometry to the design and construction of beautiful public buildings, temples, and royal palaces, the painters were attracted by geometrical figures as a means of expressing their conceptions of beauty. Artists of the city of Susa, in Persia, used geometrical forms six thousand years ago in a conventionalized artistic style as sophisticated as that of modern abstract art. Goats, whose fore and hind quarters were triangles and whose horns were sweeping semicircles, and storks, whose bodies and heads were drawn as large and small triangles, decorated their pottery. Geometry was not, as Herodotus claimed, the gift of the Nile alone. The artists too presented this gift to civilization.

The Egyptian and Babylonian civilizations drew inspiration for mathematical activity from many human needs and interests, but they fell short of greatness both in their understanding of mathematics and in their actual contributions to the subject. They accumulated simple formulas and numerous elementary rules and techniques, all of which answered questions arising in particular situations. There was, however, no general development of a subject nor do the texts enunciate any general principles. The Ahmes papyrus, from which we derive most of our knowledge of Egyptian mathematics, merely works out specific problems; no explanations or reasons for the operation are furnished. It has been suggested that the Babylonian and Egyptian priests may have possessed general mathematical principles and may have kept that knowledge secret. This is largely speculation, supported partly by the title of the Ahmes papyrus: *Directions for Obtaining Knowledge of All Dark Things,* and partly by the general character of the Egyptian theocracy with its oral transmission of knowledge and its attempt to develop in the people a reverence for the ruling class.

The failure to build a major scientific body of knowledge or to encompass details in some broad synthesis is noticeable also in Egyptian and Babylonian astronomy. During thousands of years of observation no theory was ever developed to correlate and illuminate the observations.

Too much has been made of the mathematics used in the construction of pyramids and temples as evidence of the profundity of ancient mathematics. It is pointed out by some writers that the sides of a pyramid are almost exactly the same length and that the right angles are very close to 90°. Not mathematics, however, but care and patience were required to obtain such results. Accurate computers are not necessarily great mathematicians and neither were the pyramid builders. What is amazing about their work was the organization and engineering of such large-scale efforts.

From the modern point of view Egyptian and Babylonian mathematics was defective in another very important respect: the conclusions were established empirically. It will profit us shortly if we examine the method by which the Egyptian and Babylonian acquired his formulas.

Suppose a farmer wished to enclose 100 square feet of area as cheaply as possible and desired to have the area rectangular in shape. To keep the cost of fencing low he would want the perimeter to be as small as possible. Now he can lay out a rectangle with 100 square feet of area by using dimensions such as 50 by 2 feet, 20 by 5 feet, 8 by 12½ feet, and many other combinations. The perimeters of these various rectangles, however, are not the same despite the fact that the areas are all 100 square feet. For example, the dimensions 2 by 50 require a perimeter of 104 feet; the dimensions 5 by 20 require a perimeter of only 50 feet; and so forth. From our few calculations we can readily see that the differences in perimeter for different dimensions can be considerable.

Now the farmer is in a plight. If he knows some arithmetic he can try various dimensions which yield an area of 100 square feet and take those which yield the smallest perimeter. But since the possibilities are infinite he can never try all of them; hence he cannot determine the best choice. An alert farmer might notice that the more nearly equal the two dimensions are the smaller the perimeter required. He might suspect, then, that the square with dimensions 10 feet by 10 feet requires the smallest perimeter. But he could not be sure. His trial-and-error procedure, however, has led to a likely conclusion, namely, that of all rectangles with a given area, the square has the least perimeter.

The farmer would no doubt use this conjecture and, because arithmetic and continued experience with rectangular areas support this conclusion, it would be handed down to posterity as a reliable mathematical fact. Of course, the conclusion is by no means established and no modern mathematics student would be permitted to 'prove' it in this manner. About the best that can be said for this ancient approach to mathematical knowledge is that it substitutes patience for brilliance.

One other aspect of the mathematics of ancient times deserves our attention. The priests monopolized all learning, mathematics included, in order to use it for their own ends. Knowledge gave them power; and by restricting knowledge they reduced the likelihood that anyone would be able to challenge that power. Moreover, ignorance begets fear and people who are afraid turn to leaders who will guide and reassure them. In this way, the priests reinforced their position and were able to maintain their rule over the people. The theocracies of Babylonia and Egypt compare very unfavorably with civilizations in which there was no dominant priestly class. We shall see that the few hundred years during which the Greeks flourished and the last few hundred years of our modern era produced infinitely more knowledge and progress than the millenniums of the two ancient civilizations.