



Kenneth Falconer

FRACTALS

A Very Short Introduction

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Fractals: A Very Short Introduction

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OXFORD
UNIVERSITY PRESS

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Great Clarendon Street, Oxford, OX2 6DP,
United Kingdom

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide. Oxford is a registered trade mark of
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First Edition published in 2013

Impression: 1

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Published in the United States of America by Oxford University Press
198 Madison Avenue, New York, NY 10016, United States of America

British Library Cataloguing in Publication Data
Data available

ISBN 978-0-19-967598-2

Printed in Great Britain by
Ashford Colour Press Ltd, Gosport, Hampshire

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Preface

To many people, the word ‘geometry’ conjures up circles, cubes, cylinders, and other regular or smooth objects. Familiar artefacts, such as buildings, furniture, or cars, make wide use of such shapes. However, many phenomena in nature and science are anything but regular or smooth. For example, a natural landscape may include bushes, trees, rugged mountains, and clouds, which are far too intricate to be represented by classical geometric shapes.

Surprisingly, apparently complex and irregular objects can often be described in remarkably simple terms. Fractal geometry provides a framework in which a simple process, involving a basic operation repeated many times, can give rise to a highly irregular result. Fractal constructions can represent natural objects but also give rise to a vast array of other shapes, which may be of extraordinary complexity. The phrase ‘the beauty of fractals’ is often heard, a phrase which reflects the unending intricacy of fractal designs alongside the simplicity which underlies their ever-repeating form. Indeed, complex but attractive fractal pictures have become an art form in their own right, with exhibitions, competitions, and their use on designer clothing.

The aim of this book is to show, in basic terms, how fractals may be constructed, described, and analysed as geometrical objects

and how these ‘mathematical’ fractals relate to the ‘real’ fractals of nature or science. Geometry, particularly fractal geometry, is very much a visual subject, and diagrams and a visual intuition are key to its appreciation. Inevitably some mathematics is required, but this is presented alongside a visual and intuitive interpretation and hopefully will not present too great an obstacle to the dedicated reader prepared to think through the concepts involved. In a couple of places, to keep the mathematics in the main part of the chapters to the minimum, further details are deferred to an Appendix at the end of the book.

Since fractals became popular in the 1980s, their theory and applications have developed beyond recognition, with fractals pervading many areas of mathematics and science, as well as economics and social science. Fractals are central to a great deal of sophisticated research, which has at its heart the simple ideas that are sketched in this Very Short Introduction. If the reader is able to appreciate some of these ideas then the book will have succeeded in its aim.

I am most grateful to Isobel Falconer, Timothy Gowers, and Emma Ma for reading earlier drafts of this book, to Ben Falconer and Jonathan Fraser for assistance with producing some of the figures, and to Carol Carnegie, Prabhavathy Parthiban, Joy Mellor, and Latha Menon of Oxford University Press for their work in seeing this Very Short Introduction through to publication.

Kenneth Falconer
St Andrews, Scotland, 2013

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