



Ian Stewart

# SYMMETRY

A Very Short Introduction

OXFORD

## Symmetry: A Very Short Introduction

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# Introduction

Symmetry is an immensely important concept. A fascination with symmetric forms seems to be an innate feature of human perception, and for millennia it has influenced art and natural philosophy. More recently, symmetry has become indispensable in mathematics and science, where its applications range from atomic physics to zoology. Einstein's principle that the laws of Nature should be the same at all locations and all times, which forms the basis for fundamental physics, requires those laws to possess corresponding symmetries. But for thousands of years, the concept of symmetry was just an informal description of regularities of shape and structure. The main example was bilateral or mirror-image symmetry—for example, human bodies and faces look almost the same as their reflections. Occasionally the term was also used in connection with rotational symmetry, such as the fivefold symmetry of a starfish or the sixfold symmetry of a snowflake. The main focus was on symmetry as a geometric property of shapes, but sometimes the word was invoked in a metaphorical sense: for example, that in social disputes, both sides should be treated in the same way. The deeper implications of symmetry could not be discovered until the concept was made precise. Then mathematicians and scientists would have a solid base from which to investigate how symmetry affects the world we live in.

Today's formal concept of symmetry did not come from art or sociology. It did not come from geometry, either. Its primary source was algebra, and it emerged from a study of the solution of algebraic equations. An algebraic formula has symmetry if some of its variables can be interchanged without altering its value. In the 1800s several mathematicians, notably Niels Henrik Abel and Évariste Galois, were attempting to understand the general equation of the fifth degree. They proved, in two related but different ways, that this equation cannot be solved by any formula of the traditional kind ('radicals'). Both analysed the relation between such a solution and symmetric functions of the roots of the equation. What emerged was a new algebraic concept: a group of permutations.

After an initial hiatus while mathematicians got used to this new idea, it soon became apparent that structures remarkably similar to groups of permutations occurred naturally in many different areas of mathematics, not just algebra. Among these areas were complex function theory and knot theory. General and more abstract definitions of a group appeared, and a new subject was born: group theory. At first most work in this area was algebraic, but Felix Klein pointed out a deep connection between the concepts that made sense in any specific type of geometry and the group of transformations upon which that geometry was based. This connection allowed theorems to be transferred from one area of geometry to another, and unified what at the time was an increasingly disparate collection of geometries—Euclidean, spherical, projective, elliptic, hyperbolic, affine, inversive, and topological.

At much the same time, crystallographers realized that group theory could be used to classify the different types of crystal, by considering the symmetries of the crystal's atomic lattice. Chemists began to understand how the symmetries of molecules affected their physical behaviour. General theorems linked

symmetries of mechanical systems to the great classical conserved quantities, such as energy and angular momentum.

Symmetry is a highly visual topic with many applications, such as animal markings, locomotion, waves, the shape of the Earth, and the form of galaxies. It is fundamental to both of the core theories of physics, relativity and quantum theory, and provides a starting point for the ongoing search for a unified theory that subsumes them both. This makes the topic ideal for a Very Short Introduction. My aim is to discuss the historical origins of symmetry, some of its key mathematical features, its relevance to patterns in the natural world, including living organisms, and its applications to pattern formation and fundamental physics.

The story begins with simple examples of symmetry related to everyday life. These lead to the great breakthrough: the realization that objects do not have symmetry: they have *symmetries*. These are transformations that leave the object unchanged. This concept then extends to symmetries of more abstract entities, such as mathematical equations and algebraic structures, leading to the general notion of a group. Some of the basic theorems of the subject are then stated and motivated, without proofs.

Next, we describe some of the many different types of symmetry—translations, rotations, reflections, permutations, and so on. In combination, these transformations lead to many symmetric structures that are vital in both mathematics and science: cyclic and dihedral symmetry, frieze patterns, lattices, wallpaper patterns, regular solids, and crystallographic groups. For light relief, we discuss how group theory can be applied to some familiar games and puzzles: the Fifteen Puzzle, the Rubik cube, and sudoku.

Equipped with a refined understanding of symmetry, we examine how Nature's patterns, especially familiar ones from everyday life,

can be described and explained through symmetry. Examples include crystals, water waves, sand dunes, the shape of the Earth, spiral galaxies, animal markings, seashells, animal movement, and the spiral *Nautilus* shell. These examples motivate the concept of symmetry breaking, which is a general pattern-forming mechanism.

Delving deeper, we examine the profound impact that symmetry has had on the basic equations of mathematical physics. Symmetries of mechanical equations, now conceptualized as Lie groups, are closely related to fundamental conservation laws via Noether's Theorem. An important class, the simple Lie groups, can be classified completely. Lie groups appear in relativity and quantum mechanics, providing an entry route for the search for unified field theories—so-called Theories of Everything—such as string theory.

Symmetry

The groups that are vital to mathematical physics feed back into the mathematical foundations of symmetry in a surprising manner. Their study forms a key part of one of the great triumphs of 20th-century mathematics: the awe-inspiring classification of all finite simple groups. These turn out to be the alternating groups; finite analogues of simple Lie groups, in which the real or complex numbers are replaced by finite fields, plus some cunning variations; and twenty-six puzzling 'sporadic' groups, culminating in a truly remarkable and utterly enormous group called the monster.