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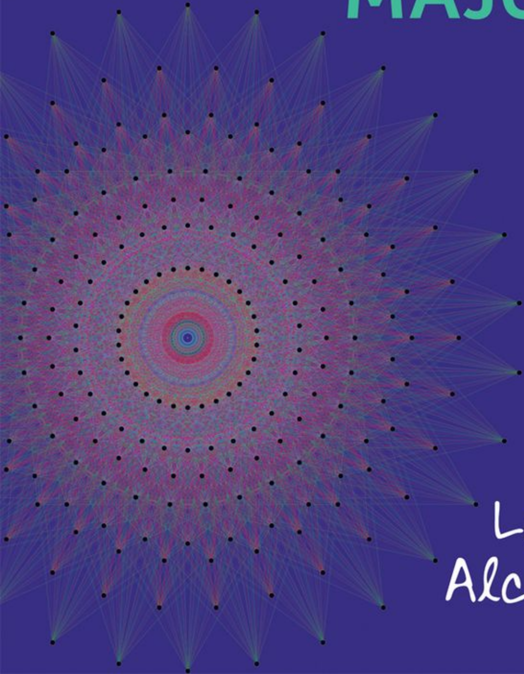
How to Study as a MATHEMATICS MAJOR



Lara Alcock

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PREFACE

Every year, thousands of students declare mathematics as their major. Many of these students are extremely intelligent and hard-working. However, even the best struggle with the demands of making the transition to advanced mathematics. Some struggles are down to the demands of increasingly independent study. Others, however, are more fundamental: the mathematics shifts in focus from calculation to proof, and students are thus expected to interact with it in different ways. These changes need not be mysterious—mathematics education research has revealed many insights into the adjustments that are necessary—but they are not obvious and they do need explaining.

This book aims to offer such explanation for a student audience, and it differs from those already aimed at similar audiences. It is not a popular mathematics book; it is less focused on mathematical curiosities or applications, and more focused on how to engage with academic content. It is not a generic study skills guide; it is focused on the challenges of coping with formal, abstract undergraduate mathematics. Most importantly, it is not a textbook. Many "transition" or "bridging" or "foundations" textbooks exist already and, while these do a good job of introducing new mathematical content and providing exercises for the reader, my view is that they still assume too much knowledge regarding the workings and values of abstract mathematics; a student who expects mathematics to come in the form of procedures to copy will not know how to interact with material presented via definitions, theorems, and proofs. Indeed, research shows that such a student will likely ignore much of the explanatory text and focus disproportionately on the obviously symbolic parts and the exercises. This book aims to head off such problems by starting where the student is; it acknowledges existing skills, points out common experiences and expectations, and re-orients students so that they know what to look for in texts and lectures on abstract mathematics. It could thus be considered a universal prelude to upper-level textbooks in general and to standard transition textbooks in particular.

Because this book is aimed at students, it is written in the style of a friendly, readable (though challenging and thought-provoking) self-help book. This means that mathematicians and other mathematics teachers will find the style considerably more narrative and conversational than is usual in mathematics books. In particular, they might find that some technical details that they would emphasize are glossed over when concepts are first introduced. I made a deliberate decision to take this approach, in order to avoid getting bogged down in detail at an early stage and to keep the focus on the large-scale changes that are needed for successful interpretation of upper-level mathematics. Technical matters such as precise specification of set membership, of function domains, and so on, are pointed out in footnotes and/or separated out for detailed discussion in the later chapters of Part 1.

To lead students to further consideration of such points, and to avoid replicating material that is laid down well elsewhere, I have included a further reading section at the end of each chapter. These lists of readings aim to be directive rather than exhaustive, and I hope that any student who is interested in mathematics will read widely from such material and thus benefit from the insights offered by a variety of experts.

This book would not have been possible without the investigations reported by the many researchers whose works appear in the references. My sincere thanks also to Keith Mansfield, Clare Charles, and Viki Mortimer at Oxford University Press, to the reviewers of the original book proposal, and to the following colleagues, friends, and students who were kind enough to give detailed and thoughtful feedback on earlier versions of this work: Nina Attridge, Thomas Bartsch, Gavin Brown, Lucy Cragg, Anthony Croft, Ant Edwards, Rob Howe, Matthew Inglis, Ian Jones, Anthony Kay, Nathalie Matthews, David Sirl, and Jack Tabeart. Thanks in particular to Matthew, who knew that I was intending to write and who gave me related books for my birthday in a successful attempt to get me started.

Finally, this book is dedicated to my teacher George Sutcliff, who allowed me to find out how well I could think.

CONTENTS

SYMBOLS

Symbol	Meaning	Section
$\mathbb{N}$	the set of all natural numbers	2.3
$\mathbb{Z}$	the set of all integers	2.3
$\mathbb{Q}$	the set of all rational numbers	2.3
$\mathbb{R}$	the set of all real numbers	2.3
$\mathbb{C}$	the set of all complex numbers	2.3
$\in$	is an element of	2.3
$\subseteq$	is a subset of	2.3
ϕ	phi (Greek letter often used for a transformation)	2.4
$\mathbb{R}^4$	the set of all 4-component vectors	2.4
$f : \mathbb{R} \to \mathbb{R}$	function f from $\mathbb{R}$ to $\mathbb{R}$	3.6
$\phi : U \to V$	transformation ϕ from the set U to the set V	4.4
$\Rightarrow$	implies	4.5
$\Leftrightarrow$	is equivalent to (or "if and only if")	4.5
$\forall$	for all	4.7
$\exists$	there exists	4.7
$\notin$	is not an element of	6.3
Σ	sigma (Greek letter used for a sum)	6.4
$\emptyset$	the empty set	8.4
$[a, b]$	closed interval	8.5
(a, b)	open interval	8.5
$\{a, b, c\}$	set containing the elements a, b and c	8.5
$\{x \in \mathbb{R} \| x^2 < 2\}$	set of all real x such that $x^2 < 2$	8.5

INTRODUCTION

This short introduction explains the aim and structure of this book, and suggests that different groups of readers might like to approach the chapters in different orders. For those who have not yet begun their undergraduate studies, it also explains some useful vocabulary.

AIM OF THE BOOK

This book is about how to make the most of a mathematics major. It is about the nature of undergraduate mathematics, about the ways in which professors expect students to think about it, and about how to keep on top of studying while enjoying undergraduate life. It is written for those who intend to study for a mathematics major, and for those who have already started.

If you are among the first group, you are probably in one of two positions. You might be a bit nervous about the whole business. Perhaps you have done well in mathematics so far, but you think that your success is mostly down to hard work. Perhaps you believe that others have some innate mathematical talent that you lack, and that in upper-level courses you will be in classes full of geniuses and will end up being found out as a fraud. As a mathematics professor, I meet quite a lot of students like this. Some of them always doubt themselves, and they get their degrees but they don't really enjoy their studies. Others do come to realize that their thinking is as good as that of anyone else. They develop more faith in themselves, they succeed, and they enjoy the whole process of learning. If you are a bit nervous, I hope that this book will help you to feel prepared, to make good progress, and to end up in this latter group.

You might, on the other hand, be confident that you are going to succeed. That's how I felt when I began my undergraduate studies in the UK. I'd always been the best student in my mathematics classes, I had no problems with the extra mathematics I took in high school, and I was pretty sure I wanted to be a mathematician. But when I arrived at college and began taking the equivalent of upper-level mathematics courses, I was forced to adjust my expectations. For a while I thought that I would only barely merit a degree and that I should seriously downgrade my career aspirations. Then, in a very gratifying turnaround, I got the hang of advanced mathematics and was eventually awarded what is known as a "first class" degree. This was largely due to a few key insights that I gained from some excellent teachers. In fact, these insights prompted me to decide that studying how people think about mathematics is even more interesting than studying mathematics, so I went on to do a PhD in Mathematics Education. These days I give lectures on undergraduate mathematics, and spend the rest of my time conducting research studies to investigate how people learn and think about it.

One simple but important thing I have learned is that, whatever their feelings on declaring mathematics as their major, most students have a lot to learn about how to study it effectively. Even those who end up doing very well are usually somewhat inefficient to start with. That's why I'm writing this book: to give you a leg-up so that your academic life is easier and more enjoyable than it would otherwise be.

However, this book is not about some magical easy way to complete a mathematics major without really trying. On the contrary, a lot of hard work will be required. But this is something to embrace. A mathematics major *should* be challenging—if it were easy, everyone would have one. And, if you've got this far in your studies, you must have experienced the satisfaction of mastering something that you initially found difficult. The book is, however, about how to make sure that you're paying attention to the right things, so that you can avoid unnecessary confusion and so that your hard work will pay off.

STRUCTURE OF THE BOOK

This book is split into two parts. Part 1 is about mathematical content and Part 2 is about the process of learning.

Part 1 could be called "Things that your mathematics professor might not think to tell you." It describes the structure of advanced mathematics, discusses how it differs from earlier mathematics, and offers advice about things you could do to understand it. I've put this part first because it is probably what is expected by students who read this before declaring their major. Note that I do not aim to teach the mathematics—that is the job of your professors and instructors. So you will not find that the book contains a lot of mathematical content. What it contains instead is information on how to *interact* with the content. It thus includes detailed illustrations, but not exercises. If you want exercises, there are many good books you can work with, and I list some of these in the Further Reading section at the end of each chapter.

Part 2 is about how to get the most out of your lectures, and about how to organize yourself so that you can keep up with the mathematics and therefore enjoy it. I've put this part second because I expect that many students who are relatively new to college studies will be thinking, "Pah! I don't need information on study skills! I have done well in dozens of exams already and clearly I am a good student." If that is what you're thinking, good for you. But maybe Part 1 will convince you that, because the nature of the mathematics changes as you move into upper-level courses, some tweaks to your approach might be useful. Indeed, some people might be reading this book precisely because they know that they are not organizing their studies very well. Those in this position might want to read Part 2 first. And anyone who is so far behind that they find themselves in a state of panic should turn straight to Chapter 12 and start there.

To fit in with the experience of all readers, I decided to write as though I'm addressing someone who is on the point of declaring their major, and to present material that should be interesting but challenging to someone in that position. This means that you might encounter new concepts while you are reading, and you'll certainly have to think hard. I have done my best to explain everything clearly and, as I mentioned above, you should be willing to be challenged as an undergraduate student. But you might find it useful to come back to some of the ideas later, when you have more experience to draw on. I hope that this book will be useful throughout your mathematics major.

USEFUL VOCABULARY

I have tried to make each chapter fairly self-contained, so that you can jump in anywhere (though that was harder with Part 1, so I do recommend that you read most of that in order). I have also tried to introduce technical terms when they are needed, and have provided a list, prior to this introduction, summarizing the mathematical notation used in the book. However, you will find it useful to be familiar with the following vocabulary for some practical aspects of undergraduate studies.

Mathematics major: In their first two years, intending *mathematics majors* usually take a sequence of Calculus courses and some Linear Algebra. They are usually required to declare a major in their second year. Some colleges require new majors to take a "transition" or "bridging" course to prepare them for upper-level mathematics by introducing ideas about proof. Upper-level work usually includes Real Analysis and Abstract Algebra, but most institutions offer a variety of courses in both pure and applied mathematics, and students usually have some choice about which ones to take. This book is designed to be useful for all mathematics majors, although it focuses on the transition to upper-level courses and on pure mathematics.

Sections: Students electing to take a course will have to register for a particular *section*. For lower-level courses that are taken by hundreds of students every semester, there might be many sections to choose from. For upper-level courses on more specialist topics, there might be just one or two. Different sections will cover the same material but will be taught at different times and places and (often) by different professors. In institutions that operate honors programs, there might be specific honors sections; students wishing to register for these might need some form of special permission.

Recitations and problems sessions: Lots of courses have associated *recitations* or *problems sessions*; sometimes a large lecture class might be split into multiple smaller classes for recitations. These sessions focus on working through problems, and they might be run by the course professor or by an alternative instructor such as a postgraduate teaching assistant.

Problem sets: Mathematics professors distribute sets of problems for their courses. These might be lists of exercises from a course textbook, or they might be on separate printed sheets produced by the professor. Problem sets are usually distrubuted weekly, or perhaps when a topic or chapter begins or ends. I'll refer to them as *problem sets*, but they are also variously called *example sheets*, *exercises*, or just *homework*.

Coursework assignments: Mathematics professors also set work that is to be submitted for credit; that is, to contribute to the grades that students are eventually awarded. This work takes many forms, including selected problems from problem sets, separate written assignments, and tests taken during class time or on a computer. I'll refer to these things collectively as *coursework assignments*.

Online materials: Colleges usually have a computer environment via which students can access academic materials. In some cases this will be a *virtual learning environment* (VLE), which might be referred to by a commercial or locally-decided name (at my university, the VLE is called "Learn"). You usually have to log in to a VLE, but when you do you'll find that the system knows what courses you are registered for and provides a link to a separate page for each one. Within each course page, your professor can provide resources (lecture notes, problem solutions, links to online tests, and so on) for you to download. Alternatively, there might be no VLE as such, but course webpages maintained by your department or by individual professors. The main college pages will also have links to other services and facilities, and might allow you to make college-related financial transactions. Needless to say, someone will provide you with information about all of this.

Advisers: US-style college systems give students a lot of choice about how to organize their personal degree programs; about how to select courses to make good progress through a major (and minor) while also fulfilling other requirements. Prerequisite structures mean that this is not a trivial process, so *academic advisers* are usually available to help students make sensible decisions.

One final term worth mentioning is *independent learning*. Students sometimes misinterpret this term. They know that independent learning is expected in college, but some think this means that they will have to study alone with no help or support. This is far from true, as I hope will be clear from the whole of this book. There will be challenges, and you will have to put in some individual intellectual effort. But those with a bit of initiative will find that plenty of help is available (see especially Chapter 10) and that asking good questions and thinking in productive ways can lead to rapid progress.

With that in mind, let's get started.

PART I
Mathematics

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CHAPTER 1

Calculation Procedures

This chapter addresses some issues that frequently arise when students make the transition to advanced mathematics. It discusses ways in which undergraduates can build on their existing mathematical skills; it also identifies ways in which mathematical expectations change when a student moves from lower-level to upper-level courses. It describes different approaches to learning, and argues that certain approaches are more useful than others when dealing with advanced mathematics.

1.1 Calculation and advanced mathematics

The first part of this book is about the nature of upper-level mathematics. Upper-level mathematics has much in common with lower-level mathematics, and students who have been accepted onto a mathematics major already have an array of mathematical skills that will serve them well. On the other hand, upper-level mathematics also differs from lower-level mathematics in some important respects. This means that most students need to extend and adapt their existing skills in order to continue doing well. Making such extensions and adaptations can be difficult for those who have never really reflected on the nature of their skills, so Part 1 discusses these in some detail.

One thing you have certainly learned to do is to apply mathematical procedures to calculate answers to standard questions. Some people enjoy doing this type of work. They like the satisfaction of arriving at a page of correct answers, and they like the security of knowing that if they do everything right then their answers will, indeed, be correct. Sometimes they compare mathematics favorably with other subjects in which things seem to be more a matter of opinion and "there are no right answers."

Other people dislike this aspect of mathematics. They find it dull to do lots of repetitious exercises, and they get more satisfaction from learning about *why* the various procedures work and how they fit together. I will discuss this difference further in this chapter. For now, however, note that knowing how to apply procedures is extremely important because, without fluency in calculations, it is hard to focus your attention on higher-level concepts.

When you start taking upper-level courses, professors will expect you to be fluent in using the procedures you have already learned. They will expect you to be able to accurately manipulate algebraic expressions, to solve equations, to differentiate and integrate functions, and so on. They will expect you to be able to do these things without having to stop each time to look up a rule, and they might not be patient with students who are not able to do so. This is not because they are impatient with students in general—most professors will be very happy to spend a long time talking with you about new mathematics, or responding to students who say, "I know how to do this, but I've never really understood why we do it this way." But they will not expect to have to re-teach things you have already studied. So you should brush up your knowledge prior to beginning a course, especially if, say, you've done no mathematics all summer.

Once you do begin, you'll find that some upper-level mathematics involves learning new procedures. These procedures, unsurprisingly, will be longer and more complicated than those you met in earlier work. I am not worried about your ability to apply long and complicated procedures, however, because to have got this far you must be able to do that kind of thing. Here, I want to focus on more substantive changes in the ways in which you have to interact with the procedures.

1.2 Decisions about and within procedures

The first substantive difference is that you will have more responsibility for deciding which procedure to apply. Of course, you have learned to do this to some extent already. For instance, you have learned how to multiply out brackets and write things like:

$$(x + 2)(x - 5) = x^2 - 3x - 10.$$

But hopefully you have also learned that it is *not* sensible to multiply out when trying to simplify a fraction like this:

$$\frac{x^2(x + 2)(x - 5)}{x^2 + 2x}.$$

For the fraction, simplification is easier if we keep the factors "visible." Nonetheless, many people automatically multiply out, probably because multiplying out was one of the first things they learned to do when studying algebra. They would, however, become more effective mathematicians if they learned to stop and think first about what would allow them to make the most progress. If this doesn't apply to you for this particular type of problem, does it apply in others? Have you ever done a long calculation and then realized that you didn't need to? Could you have avoided it if you'd stopped to think first? Part of deciding which procedure to apply is giving yourself a moment to think about it before you leap in and do the first thing that comes to mind.

This might not sound like a big deal, but think for a moment about how often you *don't* have to make a choice about what procedure to apply. Often, questions in books or on tests tell you exactly what to do. They say things like, "Use the product rule to differentiate this function." Even when a question doesn't tell you outright, it is sometimes obvious from the context. In high school, if your teacher spent a lesson showing you how to apply double angle formulas, then gave you a set of questions to do, it was probably safe for you to assume that these would involve double angle formulas. This helped you out, but it means that much of the time you didn't have to decide what procedure to apply. In the wider world, and in advanced mathematics, making decisions is more highly valued and more often expected. This means that questions presented to you on problem sheets or in exams will usually just say "solve this problem" rather than "solve this problem using this procedure."

Another part of deciding which procedure to apply is being able to distinguish between cases that look similar but are best approached in different ways. For example, consider integration, and more specifically integration by parts. You may know that this is used when we want to integrate a product of two functions, one of which gets simpler when we differentiate it, and the other of which does not get any more complicated when we integrate it. For instance, in $\int xe^x dx$, x gets simpler if we differentiate it, and e^x gets no more complicated if we integrate it. You might also know, however, that sometimes mathematical situations look superficially similar, but are best tackled using different procedures. In the integration case, integration by substitution might sometimes be more appropriate. For instance, in $\int xe^{x^2} dx$, we would probably want to use integration by substitution instead of by parts. Can you see why?

Integration by substitution is a good case for another point I want to make, this time about making decisions *within* procedures. It might be that you read the end of the last paragraph and thought, "But what substitution should I use?" Perhaps your teachers or books always told you what to use, but I would argue that they shouldn't necessarily have to. After a while, you should notice that certain substitutions are useful in certain cases. If you pay attention to the structures of these cases then, even if you wouldn't be sure that you could pick a good substitution for a new case, you should have an idea of some sensible things to try. If you haven't deliberately thought about this before, I suggest you do so now. Get out some questions on integration by substitution and, without actually doing the problems, look at the suggested substitutions. Can you anticipate why those substitutions will work? Can you then anticipate what would work in similar cases? I'll come back to this illustration later in this chapter.

So how can a student improve their ability to make decisions about and within procedures? I have two suggestions. The first is to try doing exercises from a source where the procedure to be applied is not obvious. A good place to look is in end-of-chapter exercises, which tend to cover more material. The second suggestion is to turn ordinary exercises into opportunities for reflection. When you finish an exercise, instead of just moving on to the next one, stop and think about these questions:

1. Why did that procedure work?
2. What could be changed in the question so that it would still work?
3. What could be changed in the question so that it would *not* work?
4. Could I modify the procedure so that it would work for some of these cases?

All of these questions should help you build up flexibility in applying what you know.

1.3 Learning from few (or no) examples

When you learned a new procedure in high school, your teacher probably introduced it by showing you several worked examples. These examples probably varied a bit, so that the first ones were easier and the later ones were harder. Your teacher probably then set you some exercises so you could practice for yourself. You probably did these exercises with the worked examples to hand, applying the method to the new cases you were given.

In upper-level courses it's less common to have several worked examples to hand when you begin trying a problem. You might have just one or two. These will not encompass all the possible variation in applying the procedure, so you will be more responsible for working out whether you can follow it exactly, or whether you need to adjust it because you're working with a slightly different situation.

For a straightforward example in which an adjustment is necessary, consider school students who have learned to solve quadratic equations like $x^2 - 5x + 6 = 0$ by factorizing then setting the factors equal to 0. They might write something like this:

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ (x-2)(x-3) &= 0 \\ x - 2 = 0 \text{ or } x - 3 &= 0 \\ &\qquad\qquad x = 2 \text{ or } x = 3. \end{aligned}$$

Now suppose such a student is asked to solve the equation $x^2 - 5x + 6 = 8$, and writes this:

$$\begin{aligned} x^2 - 5x + 6 &= 8 \\ (x-2)(x-3) &= 8 \\ x - 2 = 8 \text{ or } x - 3 &= 8 \\ &\qquad\qquad x = 10 \text{ or } x = 11. \end{aligned}$$

What exactly has gone wrong here? Make sure you can answer this—spotting the errors in logical arguments is important. Can you explain what the error is and why it is an error? Can you see, nonetheless, why someone might make this mistake? Notice that it isn't crazy—the procedure looks on the surface like it might work because the equation seems to be of the same kind. It *doesn't* work in this case because, while it is true that if $ab = 0$ then we must have $a = 0$ or $b = 0$, it is *not* true that if $ab = 8$ then we must have $a = 8$ or $b = 8$. A sensible modification of the procedure would have been to subtract 8 from both sides first, and work with an equation in the standard form.

This is a simple illustration and, when you learned to do this kind of thing, your teachers probably didn't expect you to use just one worked example to work out how to deal with related but non-identical cases. Indeed, in lower-level mathematics, students are usually only shown cases in which everything works—comparatively little time is devoted to recognizing examples for which standard procedures do not apply. So you might not have had much practice at the critical thinking needed to spot the limitations of procedures, and you should be prepared to develop this skill as you work toward your major.

In fact, in upper-level mathematics, you will sometimes be asked to apply a procedure without having seen *any worked examples at all.* This might sound impossible, but it isn't, because useful information sometimes comes in other forms. In particular, applying procedures often involves substituting things into formulas. For instance, the product rule for differentiation involves a formula, and is often expressed as follows (if the formula you use does not look exactly like this, can you see how it corresponds to the one you're familiar with?):

$$\text{If } f = uv \text{ then } \frac{\mathrm{d}f}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

To apply the product rule, we decide what we want u and v to be, then work out all the other things we need and substitute them into the formula. When you first saw this, your teacher probably went through a few examples, showing you how to do this. Was that really necessary, though? If you learned something similar now, would you need someone to walk you through it step by step, or could you just make all the appropriate substitutions and follow it through yourself?

For another illustration, consider the definition of the derivative. You may have seen this when you were first introduced to differentiation. You'll have to engage with it in detail during your major, so we'll use it here to show how you might apply a general formula without needing a worked example. Here it is:

Definition: $\dfrac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, provided this limit exists.

The question of why this is a reasonable definition is important, as is the question of why we need to say "provided this limit exists." However, we're not focusing on those questions here (look out for the answers in a course called Advanced Calculus or Analysis). For now, suppose we've been given this definition and asked to use it to find the derivative of the function f given by $f(x) = x^3$ (I know you already know what the answer is, but humor me for a moment). What would we do? Well, we want to find $\mathrm{d}f/\mathrm{d}x$, so the formula can be used exactly as it is—we don't need to do any rearranging. We have a formula for f, so we can substitute that in:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}.$$

Then we can simplify the resulting expression:

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h \to 0} 3x^2 + 3xh + h^2.
\end{aligned}$$

Finally, we can observe that as h tends to 0, $3x^2$ stays as it is, but $3xh$ tends to 0 and so does h^2. So the limit is equal to $3x^2$. Hence, by substituting into the definition, we have established that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 3x^2.$$

This is what we were expecting.

Recognizing that you can apply general formulas directly to examples, on your own, should make you feel mathematically empowered. It might still be reassuring to have someone walk you through some worked examples, but in many cases you don't need that level of support any more.

1.4 Generating your own exercises

Sometimes, as I said, university professors will give you only a small number of worked examples. Similarly, sometimes they will set you only a small number of exercises. They might, for instance, demonstrate how we can prove that the function f given by $f(x) = 2x$ is continuous,[1] and ask you to write a similar proof for $f(x) = 3x$. They might not ask for anything more, but their intention will be that you can then see how to write a similar proof for $f(x) = 4x$ and for $f(x) = 265x$, and so on. Even if you can, you might be well-advised to write out a proof for a few cases anyway, just for practice. In high school, your teacher probably took responsibility for deciding how much practice you should do, but a professor is more likely to set just one exercise and leave it to you to judge whether you'd benefit from inventing similar ones.

You might also be well-advised to stop and ask yourself about the limitations of such a proof. Would it, for instance, work for negative values? Would the same steps apply immediately for $f(x) = -10x$, or would you need to make some kind of adjustment in that case? What about for $f(x) = 0x$? Indeed, could you generalize properly, and write an argument for $f(x) = cx$? Would you need to place any restrictions on c? Mathematicians consider this kind of thinking to be very natural. They probably always did it for themselves, without having to be told to. You should too.

The upshot of this chapter so far is that you will not succeed in upper-level mathematics if you always try to solve problems by finding something that looks similar and copying it. In some cases, if you copy without

[1] If you can't see why there would be anything to prove, wait for the discussion in Chapter 5.

thinking, you could end up writing nonsense because some property that holds in the worked example does not hold in your problem. In other cases, copying might just be inefficient—the first approach you choose might work perfectly well but take twice as long as a different method. In still other cases, there might not be any examples to follow at all. You might have to recognize that a particular definition or theorem can be applied, then apply it, going straight from the general statement to your specific case via appropriate substitutions.

This makes upper-level mathematics more demanding than mathematics you have encountered before. You have to attend more carefully to whether your symbolic manipulations are valid for the example that you are working with. This is not always easy, but again you can practice by asking yourself the questions at the end of Section 1.2.

1.5 Writing out calculations

If you've read the Contents pages of this book, you will have noticed that there is an entire chapter devoted to writing mathematics. I'm not going to say much about this here, but I have a few comments that are particularly relevant while we're talking about calculation procedures.

There are probably some procedures for which you can happily keep everything in mind and write down only minimal working, but others for which you tend to make errors so that it is worth doing one step at a time and writing everything down. If you have been told that you should *always* write out all of your working, this would be a good time to begin letting go of that idea. One of the great things about mathematics is that it is very compressible: we can understand complex ideas by mentally "chunking" their components. For example, when expanding brackets, you probably originally learned to write everything out, like this:

$$(x+2)(x-5) = x^2 + 2x - 5x + 2(-5) = x^2 - 3x - 10.$$

But now you probably do much of it in your head and just write this:

$$(x+2)(x-5) = x^2 - 3x - 10.$$

Doing so helps you to do lots of calculations faster, which helps you to keep your eye on whatever overall problem you are trying to solve.

At lower-levels, it may be in your interest to write out a certain amount of working in order to satisfy an examiner, but you can still think for yourself about what you actually need. At upper-levels, you will find that your professors sometimes take routine calculations for granted; they will "skip steps" and expect you to be able to fill them in for yourself. For instance, to go back to integration by substitution, I am happy to just write down:

$$\int x \cos\left(x^2\right) \mathrm{d}x = \frac{1}{2} \sin\left(x^2\right) + c.$$

I can do the integration quickly in my head because I know that I am looking for something that differentiates to $x \cos\left(x^2\right)$. I know that the answer will be something like $\sin\left(x^2\right)$ so I'd write that down first. I can then see that if I differentiate $\sin\left(x^2\right)$ (using the chain rule), I get $2x \cos\left(x^2\right)$, so I just need to divide the $\sin\left(x^2\right)$ by 2 to get what I want. If you haven't done so already, you might try using similar thinking for the integral $\int xe^{x^2} \mathrm{d}x$ as mentioned earlier in this chapter.

Either calculation can be written out fully using the substitution $u = x^2$ (if you're unsure about this, try it, and you'll find that what you write is more or less exactly the reasoning I just went through). If I were lecturing a freshman Calculus course, I might expect a student to write everything out. If I were lecturing anything other than that, however, I would expect my students to be able to do this kind of thing in their heads. I probably wouldn't mind if they just wrote down the answer; chances are they'd be doing it as part of some larger problem, and their solutions to the larger problem would be more concise if they chose not to include every last detail. It's certainly likely that if I needed this calculation in a lecture, I would just write down the answer and expect students to be able to check its correctness for themselves. If you get used to seeing mathematics as compressible, you should be fine with this.

I should stress, however, that your teachers haven't said anything wrong if they've told you to write out your working in full. Doing so certainly has benefits: it allows you to check your work for errors and to remember what you were thinking when you come back to a problem. But compression is also important, as is making judgments about what will make your overall argument clearest to a reader. We will return to these themes throughout the book.