From KANT *to* HILBERT

A Source Book in the Foundations of Mathematics

VOLUME I

William Ewald

George Berkeley

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From Kant to Hilbert:
A Source Book in the Foundations of Mathematics

Volume I

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From Kant to Hilbert: A Source Book in the Foundations of Mathematics

Volume I

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CLARENDON PRESS · OXFORD

Oxford University Press, Walton Street, Oxford, OX2 6DP

Oxford New York
Athens Auckland Bangkok Bombay
Calcutta Cape Town Dar es Salaam Delhi
Florence Hong Kong Istanbul Karachi
Kuala Lumpur Madras Madrid Melbourne
Mexico City Nairobi Paris Singapore
Taipei Tokyo Toronto

and associated companies in
Berlin Ibadan

Oxford is a trade mark of Oxford University Press

Published in the United States
by Oxford University Press Inc., New York

First published 1996
First published in paperback 1999

A catalogue record for this book is available from the British Library

Library of Congress Cataloging in Publication Data
From Kant to Hilbert: readings in the foundations of mathematics/William Ewald.
Includes bibliographical references and indexes

ISBN 0 19 850535 3 (Vol. 1)
ISBN 0 19 850536 1 (Vol. 2)
ISBN 0 19 850537 X (set)

1. Mathematics-Philosophy. I. Ewald, William Bragg, III 1954–
QA8.6.F77 1996 510′.1–dc20 96–1586

Printed in Great Britain by
Bookcraft (Bath) Ltd
Midsomer Norton, Avon.

To my parents

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Philosophy probably will always have its mysteries. But these are to be avoided in geometry: and we ought to guard against abating from its strictness and evidence the rather, that an absurd philosophy is the natural product of a vitiated geometry.

—*MacLaurin 1742*, p. 47

CONTENTS

Volume I

Volume II

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Introduction

The most important advances in the foundations of mathematics have occurred during three periods: classical antiquity, the seventeenth century, and the modern period. In these periods, the great philosophers are also the great mathematicians, and the great problems of mathematics are great problems of philosophy.

The first period gave us Aristotle and Euclid, Plato and Pythagoras, Zeno and Archimedes; irrational numbers, paradoxes of motion, mathematical proofs, axioms of geometry, syllogistic logic—indeed, it furnishes the groundwork for all that was to follow. This period has been intensively studied, and its contours are well known. The principal texts have been translated; commentaries have been written; works such as Heath's *History of Greek mathematics* (*Heath 1921*) provide a comprehensive survey of the main developments.

The second period gave us Galileo and Descartes, Newton and Leibniz; infinitesimals, analytic geometry, the differential and integral calculus, mathematical physics, and a plethora of mathematically inspired systems of metaphysics. This period has been less thoroughly explored. The writings are voluminous, the language is archaic, and the mathematics is complex and unfamiliar. Even the writings of Leibniz are still being published; the writings of lesser figures are often buried in obscure archives. Despite the existence of some superb monographs, the seventeenth century remains the least understood and least accessible of the three periods.

The modern period can usefully be divided into two sub-periods, which overlap both chronologically and in subject-matter. The first period commences with Kant and lasts until Hilbert; the second commences with Frege and continues to the present day. For brevity, the two periods may be referred to as the *nineteenth century* and the *twentieth century*, provided these terms are taken with a pinch of salt.

The present collection is devoted to the first of these two sub-periods. To explain why the span from Kant to Hilbert forms a natural whole, and in what respects it differs from its successor period, it will be helpful to recall the changes that took place in the foundations of mathematics in the last decades of the nineteenth century.

The publication of Frege's *Begriffschrift* in 1879 is often said to mark the birth of modern logic; and with good reason. Although others—notably De Morgan and Boole—had done important work in logic, their achievements remained on the periphery both of mathematics and of philosophy. But starting in 1879 the logical foundations of mathematics experienced something of a gold-rush. The story is well known, not least because of Jean van Heijenoort's pioneering *Source book in mathematical logic* (*van Heijenoort 1967*).[a] Frege introduces quantification theory, analyses propositions into function and argument rather than subject and predicate, and describes powerful new techniques for studying the foundations of logic and arithmetic. Charles Peirce independently makes comparable advances in logic and the algebra of relations, and his ideas are soon spread to Europe by Schröder. Dedekind produces his set-theoretic analysis of the natural numbers; Cantor discovers non-denumerable multiplicities and develops his theory of transfinite arithmetic. Peano describes a new notation for symbolic logic and uses it to present axiomatic number-theory. Russell discovers the set-theoretic paradoxes, publishes *The principles of mathematics*, and begins his collaboration with Whitehead. Hilbert searches for a consistency proof for arithmetic; Zermelo announces his well-ordering theorem. By 1904, twenty-five years after the *Begriffschrift*, mathematical logic had come into its own, and the new technical discipline of *the foundations of mathematics*, embracing logic, set-theory, formal axiomatics, and the foundations of arithmetic, had been born.

These accomplishments cannot be solely ascribed to Frege's influence (which at the time was slight); nor, indeed, was his discovery of the quantifiers as great an intellectual leap as Cantor's creation of transfinite arithmetic or Dedekind's work on the foundations of number-theory. If mathematical achievement alone were at issue, *Dedekind 1872* or *Cantor 1874* would have as great a claim to inaugurating the new era as *Frege 1879*; certainly the ideas of Cantor and Dedekind have loomed larger in twentieth-century mathematics.

But Frege is important for other reasons as well. In addition to producing the *Begriffschrift*, he wrote about logic and mathematics in a philosophical style that was to become characteristic of the twentieth century. To open his works is to encounter the familiar topics that have had an influence far beyond the philosophy of mathematics: sense and reference, truth and meaning, logical constants, formal languages, proper names, the deficiencies of psychologism, the relation of language to thought—all this at a time when philosophy was dominated by neo-Kantian epistemology. Among his contemporaries, only Peirce wrote on similar topics with similar penetration; and Peirce's philosophical influence was far less then Frege's.

In addition, Frege's interest in the foundations of mathematics is more narrowly focused than that of Cantor and Dedekind. He sees vividly the intrinsic interest, both philosophical and mathematical, of the new logic, and he pursues his foundational investigations primarily for their own sake rather than for their

[a] For an appreciation of van Heijenoort's contribution to the history of logic, see the dedicatory remarks in *Paris Logic Group 1987*, pp. 1–8. The *Source book*, the fruit of a decade's work and a lifetime's thought, is a milestone in its own right; the sagacious choice of documents, the craftsmanship of the introductory notes, and the skill of the translations can hardly be bettered. Special mention should also be made of van Heijenoort's translator, Stefan Bauer-Mengelberg, whose contribution to the foundations of mathematics seems never to have been adequately appreciated. Of the 41 selections in the *Source book*, 31 are translations. Of the 31 translations, Bauer-Mengelberg is responsible, in whole or in part, for 25; and these 25 were, for the most part, the longest and most difficult.

possible applications to other branches of mathematics. Cantor and Dedekind, in contrast, pursue the foundations of mathematics for the light it can shed on problems in real analysis and algebraic number-theory. In this respect, they belong to the nineteenth century, while Frege belongs to the twentieth.

Indeed, it is important at this point to observe that the phrase *the foundations of mathematics* can be used in two distinct senses. In one sense, it is a technical discipline within mathematics—'mathematical logic', if logic is understood to include such matters as set-theory and the foundations of arithmetic. In the other sense, the foundations of mathematics are the concepts, techniques, and structures that are central to mathematical practice—the elements of which mathematics is composed, rather than the groundwork on which it rests. Thus, the concepts of *saturated models* and *primitive recursive functions* are foundational in the logical sense, but not the mathematical; *groups* and *commutative rings* are foundational in the mathematical sense, but not the logical; *sets* and *functions* are foundational in both senses; highly specialized concepts of, say, point-set topology are foundational in neither.

Broadly speaking, there are two important criteria that distinguish the foundations of mathematics of the nineteenth century from those of the twentieth. First, the twentieth century has tended to pursue foundations in the logical sense, while the nineteenth tended to pursue them in the mathematical. Second, the twentieth century approaches the philosophy of mathematics largely *via* logic and the philosophy of language, while the nineteenth century approached it *via* epistemology. Of course there are overlappings and continuities, and these distinctions have blurry edges; but they bring out the difference in flavour between the writings of, say, Helmholtz and those of Carnap.

Frege is an important transitional figure, not only because he introduced many of the philosophical ideas that were to dominate the twentieth century, but also because he represents a narrowing of subject-matter. Apart from some minor contributions to group-theory, Frege's technical research was confined to logical foundations: a pattern that was to be followed by many of the great mathematical logicians of the twentieth century—Russell, Gödel, Herbrand, Turing, Tarski, and their successors. And the twentieth-century philosophy of mathematics has been similarly preoccupied with issues in mathematical logic: the axiom of choice, the logical analysis of number, Church's thesis, the iterative conception of set, Hilbert's programme, the logical paradoxes, the incompleteness theorems, definitions of truth.

The picture in the nineteenth century is very different. Gauss and Bolzano, Riemann, Dedekind, Poincaré are remembered principally for their contributions to the mainstream of mathematics: their outlook is much broader than that of their twentieth-century colleagues. They drew their inspiration from the whole of mathematics—from algebra and geometry, analysis and number-theory, as well as from logic and set-theory. Each of these disciplines produced a crop of discoveries so new and unexpected, so rich in philosophical consequences, that even the mathematicians who made them could scarcely believe their eyes. Cantor, informing Dedekind in 1877 that the points of the unit interval can be correlated one-to-one with the points of a Euclidean space of any dimension, was driven to express his feelings in French: *Je le vois, mais je ne le crois pas*. The comment might stand as a motto for the century.

Throughout the century, the results come thick and fast. Bolzano in 1817 gives his 'purely analytic proof' of the intermediate value theorem, and starts on the path towards a purely arithmetical theory of the real numbers; he also publishes his study of the infinite. Gauss hits upon the idea of non-Euclidean geometry, and paves the way for Riemann by his investigations of curved manifolds; in number-theory, he defends the legitimacy of the complex integers, and describes their geometric representation in the plane.

Now follow a spate of developments in algebra. William Rowan Hamilton widens the number-concept beyond Gauss, introducing *quaternions*—new objects that fail to obey the commutative law of multiplication. Then Hermann Grassmann, in his *Ausdehnungslehre* of 1844, describes vector algebras even more general than those of Hamilton. These results and the investigations of Peacock, Gregory, De Morgan, and Boole put algebra on a new footing. No longer is it assumed that algebraic equations must behave like the familiar operations of elementary arithmetic: the rules obeyed by algebraic equations are to be distinguished from the properties of the objects studied. Boole applies the new algebraic techniques to the analysis of logic and probability; traditional Aristotelian logic is turned into a formal calculus. New algebras with strange new properties crop up everywhere, in almost tropical abundance: octonians, Clifford algebras, linear associative algebras, matrices, vector spaces. Sylvester, calling himself the 'new Adam', provides exotic names for the flora and fauna of the mathematical world—combinants, reciprocants, concomitants, discriminants, zetaic multipliers, plagiographs, skew pantographs, allotrious factors. By 1860, Benjamin Peirce is classifying hundreds of different algebras; the familiar integers now occupy only a small corner of the mathematical zoo. His son, Charles Peirce, studies the algebra of relative terms and the logic of quantification; this work, *via* Schröder, led eventually to the logical investigations of Löwenheim and Skolem.

Developments in geometry are similarly revolutionary. In Göttingen in 1854 Riemann delivers his address on the hypotheses which lie at the foundation of geometry; when this memoir is published in 1868, Clifford suggests almost at once that the new geometry may have physical applications. Helmholtz and Klein rush through the doors opened by Riemann, and develop non-Euclidean geometry into a mature mathematical discipline. Hilbert observes that 'It must be possible to replace in all geometric statements the words *point*, *line*, *plane*, by *table*, *chair*, *mug*'; from this shrewd observation spring not only his investigations into the axioms of geometry, but also, in time, proof theory and model theory.

In analysis, Weierstrass, lecturing from 1859 in Berlin, pursues his project of putting the calculus on a strictly arithmetical foundation. His student Cantor, working on questions raised by Riemann in trigonometric series, discovers the existence of non-denumerable sets of real numbers; he is led to far-reaching studies of the structure of the continuum. Dedekind, too, investigates the set-theoretic foundations of the real numbers and the integers; these investigations are intimately bound up with his creation of modern algebraic number theory. At about the same time, Frege discovers the theory of quantification, and begins his study of the logical foundations of arithmetic. Peano in 1890 discovers space-filling curves, and thereby delivers a blow to settled geometric intuitions.

Not everybody approves of the new trend towards infinitary, set-theoretic mathematics. Kronecker, urging an algorithmic conception of number theory, criticizes Weierstrass, Cantor, Dedekind, and their arithmetical continuum; so do Klein and Poincaré, but from very different points of view. Hilbert, fresh from his investigations of the foundations of geometry, takes up the cause of Cantor's transfinite numbers, and begins to extend his axiomatic method to logic and the foundations of arithmetic.

These developments are not tidy, and the central ideas cut across traditional barriers between subjects. Thus a fundamental issue in the philosophy of algebra—the distinction between manipulating meaningless *symbols* and manipulating *numbers*—is treated by Berkeley in real analysis, by Lambert in geometry, by Gauss in number-theory, by Hamilton, Gregory, and De Morgan in algebra, by Boole in logic, by Pasch, Fano, and Hilbert in geometry, and again by Hilbert in his proof theory.

This complicated network of interconnections constitutes one of the chief reasons for studying the foundations of mathematics of the nineteenth century. More than any other period, the nineteenth century enbraces the entirety of mathematical research, and can serve as a corrective to the narrowness of gaze of twentieth-century philosophy of mathematics. Logic remains an important part of the story; but it here appears in its natural connections to the other branches of mathematics.

Any attempt to assign precise chronological boundaries to this period must be arbitrary. Berkeley's philosophical writings, especially his *Analyst* of 1734, raise many of the epistemological problems that were to trouble nineteenth-century mathematicians. Although Berkeley is not usually thought of as a philosopher of mathematics, he was ahead of his time in his discussions of the foundations of the calculus, geometry, and arithmetic. But Berkeley had almost no influence upon the mathematical thinkers of the nineteenth century: he was remembered principally as an ingenious paradoxer who had denied the existence of matter, and his penetrating criticisms of Newton and Leibniz were forgotten. (Indeed, until the present work Berkeley's *Analyst* has never been reprinted except in complete editions of his writings.)

Kant, in contrast, exerted a powerful influence over the entire period, and set the stage for many of the debates that were to follow. Bolzano, Hamilton, Riemann, Peirce, Helmholtz, Frege, Cantor, Poincaré, Hilbert, and Brouwer all explicitly oriented themselves with respect to his work, whether in agreement or in opposition. His *Critique of pure reason*, the first edition of which was published in 1781, is the most conspicuous feature on the philosophical landscape, and may be taken tc mark the beginning of our period.

As for the end, Hilbert stands out as the last great mainstream mathematician to pursue the foundations of mathematics in the nineteenth-century style. His Göttingen lectures from the 1920s take a panoramic view of mathematics as a whole; they embrace mathematical physics and the foundations of geometry as well as set-theory and the philosophy of the infinite. Hilbert treats logic as being of interest in its own right, but also for the contribution it can make to the central questions of mathematics. Something of the spirit of these lectures is conveyed in his last public address, 'Logic and the knowledge of nature' (*Hilbert 1930*). This address—delivered in his and Kant's native city of Königsberg a century and a half after the publication of the *Critique of pure reason*—may be taken to mark the end of our period.

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The readings in this collection have been selected to present some of the main developments in the foundations of mathematics during the period from Kant to Hilbert.

Several classic works are here translated into English for the first time: Bolzano's *Contributions to a better-grounded presentation of mathematics*, Dedekind's correspondence with Cantor, Cantor's *Grundlagen*, Hilbert's *Axiomatic thought*. These works take their rightful place beside such warhorses as Dedekind's *Was sind und was sollen die Zahlen*? or Riemann's lecture on the foundations of geometry. Numerous subsidiary readings have also been provided to help place these central documents in historical perspective.

In general, an effort has been made to include documents that are either difficult to obtain or that have been unaccountably neglected.[b] Conversely, works that are well known and widely available have been omitted. In particular, there is no reduplication of material from *van Heijenoort 1967* or *Benacerraf and Putnam 1983*, nor from the works of Frege, Russell, or Gödel.

Both mathematical and philosophical writings have been included; most selections are a mixture. It is important to remember that most of the authors in this collection are mathematicians rather than philosophers: their contributions have to be assessed in the light of the mathematics of the age. So some technical material has been included to convey a sense of evolving mathematical styles. The aim here was to choose technical papers that would illustrate changes in style while being as accessible as possible to the non-specialist.

The selections attempt to give adequate representation to foundational work in each of the main branches of mathematics; the hope being to show the interconnections between algebra and geometry, number theory and analysis—and of all of these subjects to logic and set theory.

The selections from the British algebraists in Volume I may need a special

[b] In the latter category belong Berkeley's *Analyst*, Lambert's essay on parallel lines, Kant's early writings on geometry, and Charles Peirce's writings on logic. The neglect of Pierce is particularly difficult to explain. His accomplishments in logic were comparable to those of Frege, he wrote on a wide variety of topics central to twentieth-century philosophy, and he did so with uncommon insight; yet today he is virtually forgotten. Even *van Heijenoort 1967*, on most matters an impeccable guide, passes over his discovery of quantification theory in silence.

word of explanation, especially the long selection from Hamilton's *Lectures on quaternions*. Quaternions are no longer important to mathematics, and Hamilton's 'philosophy of pure time' was never important to philosophy. But his essay gives a fine picture of the state of algebra in 1853, and shows him grappling with one of the central foundational questions of nineteenth-century mathematics. The idea of a non-commutative algebra was almost as difficult to accept as the idea of a non-Euclidean geometry; and the idea of an algebra as a purely syntactic calculus took many decades to be absorbed. Hamilton did not achieve a final solution of this problem; but his account of his discovery of quaternions records an important milestone in the history of nineteenth-century mathematics.

This collection cannot pretend to offer a complete picture of the nineteenth-century foundations of mathematics: the subject is too vast for such a thing to be possible. Some important documents had to be excluded because they were too long; others, because they were too technical; yet others, because of difficulties with the copyright. Many important topics receive only cursory treatment; others are not treated at all.

Introductory notes have therefore been provided which attempt to fill gaps and to supply some of the historical background; these notes also make suggestions for further reading. But the material is too voluminous and too diffuse to be happily treated by this expedient. Readers are accordingly urged to use this collection in conjunction with one of the standard histories of the period, such as *Kline 1972*, *Klein 1926–7*, *M. Cantor 1894–1908*, *Bourbaki 1969*, or *Dieudonné 1978*. The selection *Cayley 1883* gives a masterly survey of developments in nineteenth-century mathematics, and may be found a convenient starting-point for readers to whom this period is not familiar.

References appear at the end of Volume I, and an extensive bibliography, incorporating all references from both volumes, appears at the end of Volume II. References are given in the form *Dedekind 1888*; or simply *1888* if the reference to Dedekind is clear from the context. A reference such as (*Peirce 1931–58*, Vol. iii, p. 47) refers to page 47 of volume three of Charles Peirce's *Collected papers*. Benjamin Peirce and Moritz Cantor are designated as *B. Peirce* and *M. Cantor*; Charles Peirce and Georg Cantor are simply *Peirce* and *Cantor*.

No uniform rule has been adopted for dating the selections. In most cases the date is the year of first publication, i.e. the year in which the work first became widely available to the mathematical public. Usually this is the year in which the selection was first printed; sometimes, however, it is the date of a major public address. But not all dates follow this rule.[c] When dating raises

[c] Some selections were written in the author's youth and printed decades after his death—for example, the *Habilitation* addresses of Riemann and Dedekind, which were both delivered in the presence of Gauss in Göttingen in 1854. It would have been incongrous for Dedekind's address, which was first printed in the 1930s, to be assigned a later date than the eight other Dedekind selections; so it appears here as *Dedekind 1854*. On the other hand, Riemann's address was unknown until it was printed in 1868—the same year Helmholtz began independently to write about the foundations of geometry. To have dated it 1854 would have dated it correctly with respect to Dedekind, but would have given a misleading picture of Riemann's relationship to Helmholtz and to the development of non-Euclidean geometry. So it bears the date *Riemann 1868*. Another such posthumously published work is *Gauss 1929*. This selection is so dated because the exact date of composition is unknown.

special problems, full information on dates of composition, of public delivery, and of first printing is given in the Bibliography.

The selections are arranged by author, and, with a few exceptions, are ordered chronologically.[d]

Footnote references have been given in the form in which they originally appeared. Sometimes it is useful to know which edition, translation, or reprinting an author relied upon (or to know that this information is not conveyed by the author's references); for this reason, there seemed little merit in attempting to impose a uniform system of citation. Detailed information on the mathematical works cited can be found in the Bibliography; citations of non-mathematical works (for example Cantor's references to various obscure histories of scholasticism) have not been included in the Bibliography. Asterisks and daggers have often been replaced by numerals; editorial footnotes are lettered rather than numbered. The original numbering of footnotes has, so far as possible, not been disturbed.

These volumes contain 89 selections; of these, 56 are translations; of the translations, 36 appear here for the first time. All new translations are by the editor, except for the translations of *Bolzano 1804*, *1810*, and *1817a* (which are by Stephen Russ), of *Dedekind 1876–7* (which is by David Reed), and of *Zermelo 1930* (which is by Michael Hallett). Hallett also wrote the introductory note to Zermelo; all other notes are by the editor. The editor has also made revisions to the previously existing translations, either to correct errors or to bring the terminology into line with the other translations in this collection; *Dedekind 1872* and *1888* have been heavily revised in this way, as have several of the selections from Poincaré (which were previously only partially translated).

When a translated word or phrase seemed to merit special attention, it has been given in the original language in double square brackets: for example '⟦Mannigfaltigkeit⟧'. Such bracketed words are printed in italics only if they so appeared in the original.

Editorial insertions are always given in double square brackets. Single square brackets are sometimes used for other purposes (for example to indicate deletions or additions between various editions). These uses are always explained in the accompanying note.

German authors sometimes emphasize words by increasing the spacing between letters; these emphases have always been rendered here with italics. It is also a German practice to print proper names in small capitals; this practice has not been preserved in the translations.

In his translations the editor has tried to cleave to the terminology of Stefan Bauer-Mengelberg's translations in *van Heijenoort 1967*; in particular, he has

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[d] Gauss appears after Bolzano because *Bolzano 1804* continues the discussion of themes raised by Lambert, while *Gauss 1831* leads naturally into the selections from the British algebraists. Hamilton appears after De Morgan and Gregory because his long *Preface to the lectures on quaternions*—the second Hamilton selection—is best read after their writings. Kronecker appears after his juniors Dedekind and Cantor because he is writing in response to their work.

adopted Bauer-Mengelberg's translation of Hilbert's adjective '*inhaltlich*' by the neologism 'contentual'. The editor has also striven not to break up paragraphs or sentences (although in a very few cases some exceptionally long and convoluted German sentences had to be split into two English sentences). Technical vocabulary has deliberately been translated literally. For example, the German word *Fundamentalreihe* has been translated as 'fundamental sequence' rather than as the more familiar 'Cauchy sequence'. 'Fundamental sequence' was good English mathematical usage a hundred years ago, and *Cauchyreihe* is good German mathematical usage today; but to have translated *Fundamentalreihe* as 'Cauchy sequence' would have been anachronistic.

Citations of the works in this collection present a special problem. Many of the works originally appeared in journals or books that are now difficult to obtain. Some have been reprinted in various collected works; others have been printed in more than one forum. Some selections have been translated into other European languages; others have undergone several revised editions. Page numbers in one version do not necessarily provide a clue to page numbers in another. The situation is already immensely confusing; and so it seemed best to attempt to design a context-free system of citation, rather than to add yet another layer of confusion. (This is particularly important as a courtesy to scholars working with the original texts.) The system varies, as it must, from document to document. Sometimes the original pagination is preferred; sometimes section numbers; sometimes paragraph numbers. The preferred form for citations is given in the introductory note at the beginning of each selection.

Readers are urged to regard the translations merely as an introduction to the original texts. Certainly any scholar wishing to work in this area must expect to delve deeply into the primary sources, and to read them in the original languages. This is not just because no translation is perfect (although that is true enough). But, as a glance at the bibliography will show, very little has in fact been translated. For every article in this collection, there exist many related background works, often entire volumes, that have never been translated; and these works must be consulted if the original selection is to be properly understood in its historical setting. At best, a compilation of this kind can serve as a general introduction to the subject; it is no substitute for a library, or for a thorough knowledge of the original documents.[e]

[e] This may be an appropriate spot to note some of the important topics that had to be omitted from these two volumes. I should have like to have included some readings from Dirichlet, Riemann, Heaviside, and others on the development of the concept of *function*. Hermann Grassmann's *Ausdehnungslehre* is badly slighted; but this important and influential book does not lend itself to being read in excerpts. Poincaré's work in topology deserves to be represented, as do his late articles on set theory. More extensive selections from the French analysts (notably Borel, Lebesgue, and Baire) would have shed valuable light on the reception of Cantor's ideas; so would appropriate selections from the writings of Schoenflies, Hausdorff, and the early Brouwer. The topic of implicit definitions (raised in *Gergonne 1818* and treated also by Pasch, Fano, Veronese, Peano, and Hilbert) could have formed an instructive counterpoint to the selections on the foundations of algebra, and made it clear that the idea of an uninterpreted formal calculus has roots in geometry as well as in algebra. The topic of constructivism is here treated principally *via* the medium

Acknowledgments. The idea for this collection came from a set of eight seminars given by Daniel Isaacson in Oxford in Hilary Term of 1984. The seminars treated many of the authors represented here, and in particular made a strong case for the superiority of Dedekind's analysis of the natural numbers to Frege's. I am grateful to Isaacson for many suggestions about the readings to be included, and for his constant encouragement; the book would look very different without his influence.

Another participant in the Isaacson seminar, Michael Hallett, was also of great assistance. He gave me the benefit of his detailed knowledge of the Göttingen archives, and guided me towards the *Nachlässe* of Hilbert and Dedekind, as well as towards the Cantor–Hilbert correspondence. He pointed out many documents that needed to be translated, and consented to translate the selection from Zermelo himself. In preparing the introductory notes to the selections from Cantor, I drew heavily on his *Cantorian set-theory and limitation of size*. This book, despite its forbidding title, is one of the most readable and illuminating contributions to the philosophy of mathematics in many years; it is particularly noteworthy for the deft way in which it blends philosophy, mathematics, and history. (In this respect, Hallett's book is very much in the spirit of Isaacson's seminar.) I learned much from studying his work, and hope that his style of scholarship will find many imitators.

While editing this book, I was supported by four institutions: The Queen's College, Oxford; the Philosophisches Seminar of the Georg-August Universität, Göttingen; the Institute for Advanced Study, Princeton; and the Istituto Universitario Europeo, Florence. I am grateful to all four institutions for their support; to the Alexander von Humboldt Stiftung for a fellowship which made my stay in Göttingen possible; and to the Research Foundation of the University of Pennsylvania for assistance with the costs of preparing the manuscript for publication.

I also owe thanks to Stephen Russ for allowing me to publish his translations of Bolzano; to David Reed for translating *Dedekind 1876–7*; and to Angus Bowie for his help with the classical quotations. Ralf Haubrich provided useful advice on Dedekind, as did Paolo Mancosu and Walter van Stigt on Hilbert and Brouwer respectively.

No doubt there are many ways in which this work could be improved; and no doubt it contains many errors. I hope one day to produce a revised and expanded edition. Comments and suggestions will be received with gratitude; they can be communicated either to the Mathematics Editor at Oxford University Press, or to wewald@oyez.law.upenn.edu.

Philadelphia
September 1992

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of number theory; but it is also important in geometry and early topology. The writings of Weierstrass and his school; Hilbert's unpublished lectures on the foundations of mathematics; Riemann's philosophical speculations; Weyl on the continuum; the chief methodological writings of the mathematical physicists—all of these important topics had to be omitted if the topics represented here were to receive adequate treatment. Anybody who rummages in the titles listed in the Bibliography will be able to extend this list of omissions almost indefinitely.

1
George Berkeley (1685–1753)

Of all treatises written on the subject in the eighteenth century, Berkeley's *Analyst* was the most sustained and penetrating critique of the methodology of the infinitesimal calculus. Despite its early date (1734) and the fact that it was largely ignored by mathematicians, this work foreshadows the foundational research of the nineteenth century, and provides a link between the mathematical preoccupations of the seventeenth and eighteenth centuries and those of the nineteenth.

Broadly speaking, the mathematicians of the seventeenth and eighteenth centuries—and in particular Newton, Leibniz, and Euler—had been more concerned with exploiting and extending the techniques of the differential and integral calculus than with tidying its foundations. The mathematicians of the early nineteenth century, in contrast, bent their efforts towards placing the calculus on an unobjectionable footing—towards chasing away the obscurities that surrounded such notions as *infinitesimal*, *limit*, and *differential*. This project was pursued during the first two-thirds of the nineteenth century by such mathematicians as Gauss, Bolzano, Cauchy, Abel, Fourier, Riemann, and Weierstrass. Their investigations into the foundations of real analysis in turn inspired the later studies by Dedekind, Cantor, Frege, Peano, Peirce, Russell, and Hilbert of set-theory, logic, and the foundations of arithmetic.

Berkeley's *Analyst* prefigures this entire development, and his philosophically-motivated criticisms of the Newtonian mathematics of the eighteenth century raise many issues that will loom large in the selections that follow. These issues may be loosely grouped under four headings.

1. His critique of *infinitesimals* raises not only the issue of justifying the central concepts of the calculus (a principal concern in *MacLaurin 1742*, *D'Alembert 1765a,b*, and *Bolzano 1817a*) but also the more general issue of the legitimacy of the *actual infinite* in mathematics. (This issue is already present in 'Of infinites' *Berkeley 1901* [1707]).) Berkeley's objections to the actual infinite will be encountered again, wearing different masks, in many of the selections that follow: for instance, in the selections from Kronecker, Cantor, Hilbert, Brouwer, and Poincaré.

2. Berkeley, as befits the empiricist philosopher who wrote *An essay towards a new theory of vision* (1709), makes acute comments about the relationship between geometry, human visual perception, and the foundations of the calculus. These themes will reappear in selections from Bolzano; from Riemann and Clifford; and from Helmholtz and Klein.

3. In numerous places Berkeley discusses the problem of the *reference* of

mathematical expressions; declaring, for instance, in *Of infinites*, that ' 'Tis plain to me we ought to use no sign without an idea answering to it'; one of his cardinal criticisms of Newton is that infinitesimals can have no empirical reference. This topic of the reference of mathematical expressions (a topic on which Berkeley occasionally shifted his position) was to be central to the development of algebra during the nineteenth century, and we shall encounter it again in the selections from Lambert, Gauss, Gregory, De Morgan, Hamilton, Boole, Dedekind, and Hilbert.

4. All these mathematical issues—in analysis, geometry, and algebra—are intertwined in Berkeley's thought with more general concerns about *mathematical truth, the rigour* of demonstrations, the *applicability* of mathematics to the empirical world, and the *scope* and *limits* of mathematical knowledge. These topics were not central to the mathematical thought of Berkeley's immediate predecessors and successors, but were to dominate the mathematical research of the nineteenth century; it is remarkable that they should have been explored so early and in such depth in an essentially philosophical treatise. Philosophers more often react to developments in mathematics than anticipate them, and in modern times perhaps only Descartes, Leibniz, and Kant can be said to have equalled Berkeley's insight into the foundations of mathematics .

The connection between Berkeley's mathematical interests and the leading strands of his philosophy was not adventitious: his education at Trinity College, Dublin was in mathematics and logic as well as in philosophy and the classics, and the foundations of mathematics was to be a lifelong preoccupation. His investigations into the philosophy of mathematics commenced with his first published work (in effect, his bachelor's thesis), the *Arithmetica et miscellanea mathematica* (*1707*; probably written 1704); they were continued in his early, unpublished *Philosophical commentaries* (*1707–8*) and *Of infinites* (*1901* [1707])—especially in *Notebook B* of the *Commentaries*, where remarks on optics and on the nature of the soul mingle with observations on algebra, geometry, and the infinitesimal calculus; they play a major though subsidiary role in his *Principles of human knowledge*, *De motu*, and *Alciphron*; and they are once more the centre of attention in his last major philosophical enterprise, the critique of Newton and the Newtonians in *The analyst*.

Berkeley would never have acknowledged a sharp distinction between his studies in metaphysics and his studies in 'natural philosophy', and it is important to remember, even in reading his works on epistemology and immaterialism, that his thought was as heavily affected by Newton and Boyle as by Locke, Bacon, and Malebranche. Indeed, Berkeley's writings show the futility of attempting to draw a sharp boundary between philosophy and mathematics. We have already observed that his philosphical reflections led him to deep criticisms of current mathematical practice. But the influence goes in the other direction as well, and his study of mathematics and the physical sciences affected his general metaphysical position—so much so that the two are often difficult to disentangle. His discussions of infinitesimals are interwoven with arguments about *minima sensibilia* and with his doctrine that, for sensible objects, *esse* is *percipi*; his discussion in the *Principles of human knowledge* of algebra and of geometric reasoning is bound up with his view of language and his rejection of abstract ideas; his critique of the foundations of the calculus is motivated by many of the same epistemological considerations that underlie his critique of the idea of material substance. It is these connections (rather than his powerful but already-known logical criticisms of Newton's reasoning in the *Principia*) that give his philosophy of mathematics its depth and its strength and its present interest.

Despite the penetration of his writings—their wealth of implications both for mathematics and for philosophy—Berkeley had little actual influence on the development of mathematics. He was nobody's inspiration, and indeed to this day is rarely thought of as a philosopher of mathematics at all. The nineteenth century remembered him principally as an ingenious paradoxer, the precursor of Hume who had denied the existence of material substance; his writings on mathematics have largely been forgotten. (Two exceptions are reproduced below: *MacLaurin 1742* and the correspondence between William Rowan Hamilton and Augustus De Morgan.) But even if Berkeley did not himself initiate or influence the great period of nineteenth-century foundational research, he nevertheless glimpsed many of its central themes; and his writings are therefore an appropriate starting-point for these volumes.

The selections below represent the main lines of his thought; but there is a great deal of supplementary material in *Berkeley 1948–57*, especially in Volumes 1 and 4. *Stammler 1922* and *Jesseph 1993* are general studies of Berkeley's philosophy of mathematics; *Boyer 1939*, *Baron 1969*, *Edwards 1979*, and *Grattan-Guinness 1980* give accounts of the mathematical background.

A. *FROM THE* PHILOSOPHICAL COMMENTARIES (*BERKELEY 1707–8*)

The *Philosophical commentaries* (so named by Berkeley's editors in *Berkeley 1948–57*) consist of two notebooks, now in the British Library, each containing some four hundred working notes on philosophical topics. They were written in 1707–8 (roughly the time of Berkeley's election as a Fellow of Trinity College, Dublin) and were first printed in *Berkeley 1871*. The notebooks were physically bound together in the wrong order, so that most of the entries in Notebook B were in fact written before those in Notebook A. The notebooks show him groping towards the doctrines of *An essay towards a new theory of vision* (1709) and *A treatise concerning the principles of human knowledge* (1710). Many of the notes express views that Berkeley later discarded, while others may reflect his reading rather than his own thoughts. For this reason they should be cited with caution.

The individual entries often have symbols beside them in the margin; the precise significance of these symbols is uncertain. The symbol + probably indicates an entry about which Berkeley later changed his mind; the symbol **x** apparently indicates entries that found their way into the *New theory of vision*. Other symbols are explained by Berkeley; for instance, the symbol **Mo.** that appears by entry 769 below indicates that the entry concerns moral philosophy. Berkeley occasionally changed or erased a marginal symbol: another reason for citing the following entries with caution. (A discussion of Berkeley's marginal symbols can be found in the editors' introduction to the *Philosophical commentaries* in *Berkeley 1948–57*, Vol. 1, or more fully in the article on this subject by A.A. Luce in *Hermathena* for 1970.)

The selection below is reprinted from *Berkeley 1948–57*, Vol. 1; emendations in single square brackets are from that edition. References to *Berkeley 1707–8* should be to the paragraph numbers, which first appeared in the edition of 1948. (In the 1948 numeration the paragraphs of the *recto* side of the notebook are numbered consecutively; entries on the facing *verso* side—which seem to reflect Berkeley's later thoughts—are assigned the number of the facing paragraph followed by a small Roman *a*. Thus note 341a is Berkeley's facing-page comment on note 341.)

FROM NOTEBOOK B:

x 341. When a small line upon Paper represents a mile the Mathematicians do not calculate the 1/10000 of the Paper line they Calculate the 1/10000 of the mile 'tis to this the[y] have regard, tis of this the[y] think if they think or have any idea at all. the inch perhaps might represent to their imaginations the mile but ye 1/10000 of the inch can not be made to represent anything it not being imaginable.

x 341a. But the 1/10000 of a mile being somewhat they think the 1/10000 of the inch is somewhat, w^{n} they think of y^{t} they imagine they think on this.

x 351. We need not strain our Imaginations to conceive such little things. Bigger may do as well for intesimals since the integer must be an infinite.

x 352. Evident y^{t} w^{ch} has an infinite number of parts must be infinite.

x 353. Qu: whether extension be resoluble into points id [sic] does not consist of.

x 354. Axiom. No reasoning about things whereof we have no idea. Therefore no reasoning about Infinitesimals.

x 354a. nor can it be objected that we reason about Numbers w^{ch} are only words & not ideas, for these Infinitesimals are words, of no use, if not suppos'd to stand for Ideas.

x 355. Much less infinitesimals of infinitesimals &c.

x 356. Axiom. No word to be used without an idea.

+ 368. I'll not admire the mathematicians. tis w^{t} any one of common sense might attain to by repeated acts. I know it by experience, I am but one of common sense, and I etc

+ 372. I see no wit in any of them but Newton. The rest are meer triflers, meer Nihilarians.

+ 375. Mathematicians have some of them good parts, the more is the pity. Had they not been Mathematicians they had been good for nothing. they were such fools they knew not how to employ their parts.

x 395. I can square the circle, &c they cannot, w^{ch} goes on the best principles.

FROM NOTEBOOK A:

x 449. If the Disputations of the Schoolmen are blam'd for intricacy triflingness & confusion, yet it must be acknowledg'd that in the main they treated of great & important subjects. If we admire the Method & acuteness of the Math: the length, the subtilty, the exactness of their Demonstrations, we must nevertheless be forced to grant that they are for the most part about trifling subjects & perhaps nothing at all.

x 633. Mem: upon all occasions to use the Utmost Modesty. to Confute the Mathematicians wth the utmost civility & respect. not to stile them Nihilarians etc:

x 750. Words (by them meaning all sort of signs) are so necessary that instead of being (w^{n} duly us'd or in their own Nature) prejudicial to the Advancement of knowlege, or an hindrance to knowlege that wthout them there could in Mathematiques themselves be no demonstration.

751. Mem: To be eternally banishing Metaphisics &c & recalling Men to Common Sense.

x 761. I am better inform'd & shall know more by telling me there are 10000 men than by shewing me them all drawn up. I shall better be able to judge of the Bargain you'd have me make w^{n} you tell me how much (i.e. the name of y^{e}) mony lies on y^{e} Table than by offering & shewing it without naming. In short I regard not the Idea the looks but the names. Hence may appear the Nature of Numbers.

x 762. Children are unacquainted with Numbers till they have made some Progress in language. This could not be if they were Ideas suggested by all the senses.

x 763. Numbers are nothing but Names, never Words.

x 764. Mem: Imaginary roots to unravel that Mystery.

x 765. Ideas of Utility are annexed to Numbers.

x 766. In Arithmetical Problems Men seek not any Idea of Number, they onely seek a Denomination. this is all can be of use to them.

x 767. Take away the signs from Arithmetic & Algebra, & pray w^{t} remains?

x 768. These are sciences purely Verbal, & entirely useless but for Practise in Societys of Men. No speculative knowledge, no comparing of Ideas in them.

Mo. 769. Sensual Pleasure is the Summum Bonum. This is the Great Principle of Morality. This once rightly understood all the Doctrines even the severest of the [Gospels] may cleerly be Demonstrated.

x 770. Qu: whether Geometry may not be properly reckon'd among the Mixt Mathematics. Arithmetic and Algebra being the only abstracted pure i.e. entirely Nominal. Geometry being an application of these to Points.

B. OF INFINITES
(*BERKELEY 1901* [1707])

The following selection is an undated manuscript in the possession of Trinity College, Dublin; it was probably written in 1707–8. The manuscript lay neglected for many years in the Molyneux archives, and was first printed in 1901. Luce and Jessop, Berkeley's editors, conjecture that it was intended as an address to the Dublin ‖Philosophical‖ Society. This Society had been founded in 1683 by William Molyneux, and revived by his son Samuel in or around 1707; most of the documents in the Molyneux archives are in fact learned documents contributed to this society. Whether Berkeley's address was in fact given is unknown. The manuscript deals with infinitesimals, the necessity of having an *idea* to correspond to each *sign*, and the distinction (at least as ancient as Aristotle) between the potential and the completed infinite. The issue of infinity was to preoccupy Berkeley in his later writings, not only because of its mathematical importance (and in particular its centrality in the mathematics of Newton and Leibniz), but also because it poses the following challenge to Berkeley's philosophy: if space is infinitely divisible, then it can be divided beyond the *minimum sensibile*; but then something can exist which cannot be sensed; and this conclusion places in jeopardy the doctrine of *esse est percipi*, which is the very cornerstone of Berkeley's idealist metaphysics and epistemology.

The text below is reprinted from *Berkeley 1948–57*, Vol. 4. References to *Berkeley 1901* [1707] should be to the paragraph numbers, which have been added in this reprinting.

‖1‖ THO' some mathematicians of this last age have made prodigious advances, and open'd divers admirable methods of investigation unknown to the ancients, yet something there is in their principles which occasions much controversy & dispute, to the great scandal of the so much celebrated evidence of Geometry. These disputes and scruples, arising from the use that is made of quantitys infinitely small in the above mentioned methods, I am bold to think they might easily be brought to an end, by the sole consideration of one passage in the incomparable Mr. Locke's treatise of *Humane Understanding*, b. 2. ch. 17, sec. 7, where that authour, handling the subject of infinity with that judgement & clearness wch is so peculiar to him, has these remarkable words:

'I guess we cause great confusion in our thoughts when we joyn infinity to any suppos'd idea of quantity the mind can be thought to have, and so discourse or reason about an infinite quantity, *viz.* an infinite space or an infinite duration. For our idea of infinity being as I think an endless growing idea, but the idea of any quantity the mind has being at that time terminated in that idea, to join infinity to it is to adjust a standing measure to a growing bulk; &, therefore, I think 'tis not an insignificant subtilty if I say we are carefully to distinguish between the idea of infinity of space and the idea of space infinite.'

[2] Now if what Mr. Locke says were, *mutatis mutandis*, apply'd to quantitys infinitely small, it would, I doubt not, deliver us from that obscurity & confusion wch perplexes otherwise very great improvements of the Modern Analysis. For he that, with Mr. Locke, shall duly weigh the distinction there is betwixt infinity of space & space infinitely great or small, & consider that we have an idea of the former, but none at all of the later, will hardly go beyond his notions to talk of parts infinitely small or *partes infinitesimae* of finite quantitys, & much less of *infinitesimae infinitesimarum,* and so on. This, nevertheless, is very common with writers of fluxions or the differential calculus, &c. They represent, upon paper, infinitesimals of several orders, as if they had ideas in their minds corresponding to those words or signs, or as if it did not include a contradiction that there should be a line infinitely small & yet another infinitely less than it. 'Tis plain to me we ought to use no sign without an idea answering it; & 'tis as plain that we have no idea of a line infinitely small, nay, 'tis evidently impossible there should be any such thing, for every line, how minute soever, is still divisible into parts less than itself; therefore there can be no such thing as a line *quavis data minor* or infinitely small.

[3] Further, it plainly follows that an infinitesimal even of the first degree is meerly *nothing*, from wt Dr. Wallis, an approv'd mathematician, writes at the 95th proposition of his *Arithmetic of Infinites*,[a] where he makes the asymptotic space included between the 2 asymptotes and the curve of an hyperbola to be in his stile a *series reciproca primanorum*, so that the first term of the series, *viz.*, the asymptote, arises from the division of 1 by 0. Since therefore, unity, *i.e.* any finite line divided by 0, gives the asymptote of an hyperbola, *i.e.* a line infinitely long, it necessarily follows that a finite line divided by an infinite gives 0 in the quotient, *i.e.* that the *pars infinitesima* of a finite line is just nothing. For by the nature of division the dividend divided by the quotient gives the divisor. Now a man speaking of lines infinitely small will hardly be suppos'd to mean nothing by them, and if he understands real finite quantitys he runs into inextricable difficultys.

[4] Let us look a little into the controversy between Mr. Nieuentiit[b] and Mr. Leibnitz. Mr. Nieuentiit allows infinitesimals of the first order to be real quantitys,

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[a] [Wallis is also mentioned by Berkeley below; see *Berkeley 1734*, §17.]

[b] [Bernard Nieuwentijdt or Nieuwentijt (1654–1718) was a Dutch philosopher and mathematician, and a defender of religion; he is mentioned again in Berkeley's *Siris*, §190. His *Analysis infinitorum, seu curvilineorum proprietates ex polygonorum natura deductae* (Amsterdam, 1695) was one of the first expositions of Leibniz's differential calculus.]

but the *differentiae differentiarum* or infinitesimals of the following orders he takes away making them just so many noughts. This is the same thing as to say the square, cube, or other power of a real positive quantity is equal to nothing; wch is manifestly absurd.

[5] Again Mr. Nieuentiit lays down this as a self evident axiom, *viz.*, that betwixt two equal quantitys there can be no difference at all, or, which is the same thing, that their difference is equal to nothing. This truth, how plain soever, Mr. Leibnitz sticks not to deny, asserting that not onely those quantitys are equal which have no difference at all, but also those whose difference is incomparably small. *Quemadmodum* (says he) *si lineae punctum alterius lineae addas quantitatem non auges.*[c] But if lines are infinitely divisible, I ask how there can be any such thing as a point? Or granting there are points, how can it be thought the same thing to add an indivisible point as to add, for instance, the *differentia* of an ordinate, in a parabola, wch is so far from being a point that it is itself divisible into an infinite number of real quantitys, whereof each can be subdivided *in infinitum*, and so on, according to Mr. Leibnitz. These are difficultys those great men have run into, by applying the idea of infinity to particles of extension exceeding small, but real and still divisible.

[6] More of this dispute may be seen in the *Acta Eruditorum* for the month of July, A.D. 1695, where, if we may believe the French authour of *Analyse des infiniment petits*, Mr. Leibnitz has sufficiently established & vindicated his principles. Tho' 'tis plain he cares not for having 'em call'd in question, and seems afraid that *nimia scrupulositate arti inveniendi obex ponatur,*[d] as if a man could be too scrupulous in Mathematics, or as if the principles of Geometry ought not to be as incontestable as the consequences drawn from them.

[7] There is an argument of Dr. Cheyne's,[e] in the 4th chapter of his *Philosophical Principles of Natural Religion* which seems to make for quantitys infinitely small. His words are as follows:

> 'The whole abstract geometry depends upon the possibility of infinitely great & small quantitys, & the truths discover'd by methods wch depend upon these suppositions are confirm'd by other methods wch have other foundations.'

[8] To wch I answer that the supposition of quantitys infinitely small is not essential to the great improvements of the Modern Analysis. For Mr. Leibnitz acknowleges his *Calculus differentialis* might be demonstrated *reductione ad*

[c] ['For example, if you add to a line a point of a second line, you do not increase its size'.]
[d] ['a barrier would be erected to the art of discovery by too great a concern for accuracy'.]
[e] [George Cheyne (1671–1743) was a London physician. In 1702 he published *A new theory of fevers*, a work which attempted to extend the methods of Newtonian celestial mechanics to the human body; Cheyne proposed a quasi-mathematical explication of fevers, based on hydraulics and on a view of the body as a system of pipes and fluids. In 1703 he published a treatise on fluxions, the *Fluxionum methodus inversa*; the work was however riddled with mathematical errors. His *Philosophical principles of natural religion*, referred to by Berkeley, appeared in 1705; it attempts to prove the existence of God from the existence of gravitation. Cheyne later renounced his earlier 'riotous' Newtonian life, moved to Bath, and devoted himself to purely medical writings, producing his *Essay on the gout* in 1720. His name is mentioned by Berkeley in *Berkeley 1707–8a*, §§367, 387, and 459.]

absurdum after the manner of the ancients; & Sir Isaac Newton in a late treatise informs us his method of Fluxions can be made out *a priori* without the supposition of quantitys infinitely small.

⟦9⟧ I can't but take notice of a passage in Mr. Raphson's[f] treatise *De Spatio Reali seu Ente Infinito*, chap. 3, p. 50, where he will have a particle infinitely small to be *quasi extensa.* But wt Mr. Raphson would be thought to mean by *pars continui quasi extensa* I cannot comprehend. I must also crave leave to observe that some modern writers of note make no scruple to talk of a sphere of an infinite radius, or an aequilateral triangle of an infinite side, which notions if thoroughly examin'd may perhaps be found not altogether free from inconsistencys.

⟦10⟧ Now I am of opinion that all disputes about infinites would cease, & the consideration of quantitys infinitely small no longer perplex Mathematicians, would they but joyn Metaphysics to their Mathematics, and condescend to learn from Mr. Locke what distinction there is betwixt infinity and infinite.

C. LETTER TO SAMUEL MOLYNEUX (*BERKELEY 1709*)

The following letter concerns the relationship between words and ideas in mathematics; Berkeley here denies that words must suggest the ideas they stand for 'at every turn'. He was shortly to revisit this subject in §§19 and 20 of the *Principles of human knowledge.*

The letter was addressed to Samuel Molyneux (1689–1728), to whom Berkeley had dedicated the *Miscellanea mathematica* in 1707. Molyneux (who was four years Berkeley's junior) later became a distinguished astronomer; he was the son of William Molyneux, an astronomer, physicist, and Member of Parliament, whose work on optics, the *Dioptrica nova* (1692), was the basis for Berkeley's *Essay towards a new theory of vision* (1709). (The elder Molyneux had also translated Descartes into English, and posed to Locke the question whether a man blind since birth, on gaining his sight, would be able to distinguish, by sight alone, between a sphere and a cube that he previously knew only by the sense of touch.)

The letter is reprinted from *Berkeley 1948–57*, Vol. 8.

[f] ⟦Joseph Raphson, FRS, wrote treatises on fluxions and on the nature of space; his *De spatio reali, seu ente infinito conamen mathematico-metaphysicum* (1697) is here referred to by Berkeley. Berkeley also mentions his name in *Berkeley 1707–8a*, §§298 and 827, and in his letter to Samuel Johnson of 24 Mar. 1730.⟧

Trin. Coll. Dec. 19. 1709

Dr. Molyneux.

You desire to know what a Geometer thinks of when he demonstrates Properties of a Curve formd by a Ray of Light as it passes through the Air. His Imagination or Memory (say You) can affoard him no Idea of it, and as for the rude Idea of this or that Curve which may be suggested to him by Fancy that has no Connexion with his Theorems & Reasonings. I answer first, That in my Opinion he thinks of the Various Density of the Atmosphere the Obliquity of the Incidence and the Nature of Refraction in Air by which the Ray is bent into a Curve, 'tis on these he meditates and from these principles he proceeds to investigate the Nature of the Curve. Secondly. It appears to Me That in Geometricall Reasonings We do not make any Discovery by contemplating the Ideas of the Lines whose Properties are investigated. For Example, In order to discover the Method of drawing Tangents to a Parabola, 'tis true a Figure is drawn on paper & so suggested to your Fancy, but no Matter whether it be of a Parabolic Line or no, the Demonstration proceeds as well tho it be an Hyperbole or the Portion of an Ellipsis, provided that I have regard to the Equation expressing the Nature of a Parabola wherein the Squares of the Ordinates are every where equall to the Rectangles under the Abscissae & Parameters, it being this Equation or the Nature of the Curve thus expressed and not the Idea of it that leads to the Solution‖.‖ Again You tell Me that if, as I think, Words do not at every Turn suggest the respective Ideas they are supposd to stand for it is purely by Chance Our Discourse hangs together, and is found after 2 or 3 hours jingling & permutation of Sounds to agree with our Thoughts. As for what I said of Algebra, You are of Opinion the Illustration will not hold good because there are no Set rules except those of the Syllogisms whereby to range & permute o‖u‖r Words like to the Algebraic Process. In Answer to all which I observe first, That if We put Our Words together any how and at Random then indeed there may be some Grounds for what You say, but if people lay their Words together with Design and according to Rule then there can be no Pretence so far as I can see for your Inference. Secondly. I cannot but dissent from what You say, of there being no Set Rules for the Ranging and Disposition of Words but only the Syllogistic, for to Me it appears That all Grammar & every part Logic contain little else than Rules for Discourse & Ratiocination by Words. And those who do not expresly set themselves to study those Arts do nevertheless learn them insensibly by Custom. I am very sleepy & can say no more but that I am

Yo‖u‖rs &c.

G. BERKELEY.

D. *FROM* A TREATISE CONCERNING THE PRINCIPLES OF HUMAN KNOWLEDGE, PART ONE (*BERKELEY 1710*)

The *Principles of human knowledge, Part One*, Berkeley's major philosophical work, was first printed in 1710 while Berkeley was a Fellow of Trinity College, Dublin. As the title indicates, Berkeley planned to publish a Part Two; however, he lost the manuscript during his travels on the Continent in 1716–20, and never rewrote it. The *Principles* is so well known to students of philosophy that it needs no detailed introduction here; for a modern discussion of the place of the *Principles* in British empiricism, see *Warnock 1982* and the works therein cited. Students of Berkeley's thought and its connections to earlier manifestations of British nominalism are urged to read the perceptive review of his *Works* by C.S. Peirce (*Peirce 1871*). The passages excerpted here discuss abstract ideas and language; the nature of arithmetic and the finite integers; spatial extension and infinite divisibility. The passages have been selected not only for their intrinsic interest, but also for the light they shed on Berkeley's other writings in this collection. In particular, the central role played by Berkeley's attack on abstract ideas should not be overlooked, and the extended discussion of abstract ideas in §§6–25 of the Introduction to the *Principles* ought to be compared with the remarks on language and mathematics in the *Philosophical commentaries* (§§354–6 and 761–70), as well as with the attack upon infinitesimals in *The analyst*.

The text below consists of §§6–25 from the Introduction and of §§118–34 from Part One. The text is based on the second edition of 1734; variants from the first edition are recorded in *Berkeley 1948–57*, Vol. 2. References to *Berkeley 1710* should be to the paragraph numbers, which appeared in the original edition.

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6 In order to prepare the mind of the reader for the easier conceiving what follows, it is proper to premise somewhat, by way of introduction, concerning the nature and abuse of language. But the unravelling this matter leads me in some measure to anticipate my design, by taking notice of what seems to have had a chief part in rendering speculation intricate and perplexed, and to have occasioned innumerable errors and difficulties in almost all parts of knowledge. And that is the opinion that the mind hath a power of framing *abstract ideas* or notions of things. He who is not a perfect stranger to the writings and disputes of philosophers, must needs acknowledge that no small part of them are spent about abstract ideas. These are in a more especial manner, thought to be the object of those sciences which go by the name of *Logic* and *Metaphysics*, and of all that which passes under the notion of the most abstracted and sublime

learning, in all which one shall scarce find any question handled in such a manner, as does not suppose their existence in the mind, and that it is well acquainted with them.

7 It is agreed on all hands, that the qualities or modes of things do never really exist each of them apart by itself, and separated from all others, but are mixed, as it were, and blended together, several in the same object. But we are told, the mind being able to consider each quality singly, or abstracted from those other qualities with which it is united, does by that means frame to itself abstract ideas. For example, there is perceived by sight an object extended, coloured, and moved: this mixed or compound idea the mind resolving into its simple, constituent parts, and viewing each by itself, exclusive of the rest, does frame the abstract ideas of extension, colour, and motion. Not that it is possible for colour or motion to exist without extension: but only that the mind can frame to itself by *abstraction* the idea of colour exclusive of extension, and of motion exclusive of both colour and extension.

8 Again, the mind having observed that in the particular extensions perceived by sense, there is something common and alike in all, and some other things peculiar, as this or that figure or magnitude, which distinguish them one from another; it considers apart or singles out by itself that which is common, making thereof a most abstract idea of extension, which is neither line, surface, nor solid, nor has any figure or magnitude but is an idea entirely prescinded from all these. So likewise the mind by leaving out of the particular colours perceived by sense, that which distinguishes them one from another, and retaining that only which is common to all, makes an idea of colour in abstract which is neither red, nor blue, nor white, nor any other determinate colour. And in like manner by considering motion abstractedly not only from the body moved, but likewise from the figure it describes, and all particular directions and velocities, the abstract idea of motion is framed; which equally corresponds to all particular motions whatsoever that may be perceived by sense.

9 And as the mind frames to itself abstract ideas of qualities or modes, so does it, by the same precision or mental separation, attain abstract ideas of the more compounded beings, which include several coexistent qualities. For example, the mind having observed that Peter, James, and John, resemble each other, in certain common agreements of shape and other qualities, leaves out of the complex or compounded idea it has of Peter, James, and any other particular man, that which is peculiar to each, retaining only what is common to all; and so makes an abstract idea wherein all the particulars equally partake, abstracting entirely from and cutting off all those circumstances and differences, which might determine it to any particular existence. And after this manner it is said we come by the abstract idea of *man* or, if you please, humanity or human nature; wherein it is true, there is included colour, because there is no man but has some colour, but then it can be neither white, nor black, nor any particular colour; because there is no one particular colour wherein all men partake. So likewise there is included stature, but then it is neither tall stature nor low stature, nor yet middle stature, but something abstracted from all these. And so of the rest. Moreover, there being a great variety of other creatures that partake in some

parts, but not all, of the complex idea of *man*, the mind leaving out those parts which are peculiar to men, and retaining those only which are common to all the living creatures, frameth the idea of *animal*, which abstracts not only from all particular men, but also all birds, beasts, fishes, and insects. The constituent parts of the abstract idea of animal are body, life, sense, and spontaneous motion. By *body* is meant, body without any particular shape or figure, there being no one shape or figure common to all animals, without covering, either of hair or feathers, or scales, &c. nor yet naked: hair, feathers, scales, and nakedness being the distinguishing properties of particular animals, and for that reason left out of the *abstract idea*. Upon the same account the spontaneous motion must be neither walking, nor flying, nor creeping, it is nevertheless a motion, but what that motion is, it is not easy to conceive.

10 Whether others have this wonderful faculty of *abstracting their ideas*, they best can tell: for myself I find indeed I have a faculty of imagining, or representing to myself the ideas of those particular things I have perceived and of variously compounding and dividing them. I can imagine a man with two heads or the upper parts of a man joined to the body of a horse. I can consider the hand, the eye, the nose, each by itself abstracted or separated from the rest of the body. But then whatever hand or eye I imagine, it must have some particular shape and colour. Likewise the idea of man that I frame to myself, must be either of a white, or a black, or a tawny, a straight, or a crooked, a tall, or a low, or a middle-sized man. I cannot by any effort of thought conceive the abstract idea above described. And it is equally impossible for me to form the abstract idea of motion distinct from the body moving, and which is neither swift nor slow, curvilinear nor rectilinear; and the like may be said of all other abstract general ideas whatsoever. To be plain, I own myself able to abstract in one sense, as when I consider some particular parts or qualities separated from others, with which though they are united in some object, yet, it is possible they may really exist without them. But I deny that I can abstract one from another, or conceive separately, those qualities which it is impossible should exist so separated; or that I can frame a general notion by abstracting from particulars in the manner aforesaid. Which two last are the proper acceptations of *abstraction*. And there are grounds to think most men will acknowledge themselves to be in my case. The generality of men which are simple and illiterate never pretend to *abstract notions*. It is said they are difficult and not to be attained without pains and study. We may therefore reasonably conclude that, if such there be, they are confined only to the learned.

11 I proceed to examine what can be alleged in defence of the doctrine of abstraction, and try if I can discover what it is that inclines the men of speculation to embrace an opinion, so remote from common sense as that seems to be. There has been a late deservedly esteemed philosopher,[a] who, no doubt, has given it very much countenance by seeming to think the having abstract general ideas is what puts the widest difference in point of understanding betwixt man and beast. 'The having of general ideas (*saith he*) is that which puts

[a] ‖Locke.‖

a perfect distinction betwixt man and brutes, and is an excellency which the faculties of brutes do by no means attain unto. For it is evident we observe no footsteps in them of making use of general signs for universal ideas; from which we have reason to imagine that they have not the faculty of *abstracting* or making general ideas, since they have no use of words or any other general signs. *And a little after.* Therefore, I think, we may suppose that it is in this that the species of brutes are discriminated from men, and 'tis that proper difference wherein they are wholly separated, and which at last widens to so wide a distance. For if they have any ideas at all, and are not bare machines (as some would have them) we cannot deny them to have some reason. It seems as evident to me that they do some of them in certain instances reason as that they have sense, but it is only in particular ideas, just as they receive them from their senses. They are the best of them tied up within those narrow bounds, and have not (as I think) the faculty to enlarge them by any kind of *abstraction*.' *Essay on Hum. Underst.* B.2. C.11. Sect. 10 and 11. I readily agree with this learned author, that the faculties of brutes can by no means attain to *abstraction*. But then if this be made the distinguishing property of that sort of animals, I fear a great many of those that pass for men must be reckoned into their number. The reason that is here assigned why we have no grounds to think brutes have abstract general ideas, is that we observe in them no use of words or any other general signs; which is built on this supposition, to wit, that the making use of words, implies the having general ideas. From which it follows, that men who use language are able to abstract or generalize their ideas. That this is the sense and arguing of the author will further appear by his answering the question he in another place puts. 'Since all things that exist are only particulars, how come we by general terms?' *His answer is*, 'Words become general by being made the signs of general ideas.' *Essay on Hum. Underst.* B.3. C.3. Sect. 6. But it seems that a word becomes general by being made the sign, not of an abstract general idea but, of several particular ideas, any one of which it indifferently suggests to the mind. For example, when it is said *the change of motion is proportional to the impressed force*, or that *whatever has extension is divisible*; these propositions are to be understood of motion and extension in general, and nevertheless it will not follow that they suggest to my thoughts an idea of motion without a body moved, or any determinate direction and velocity, or that I must conceive an abstract general idea of extension, which is neither line, surface nor solid, neither great nor small, black, white, nor red, nor of any other determinate colour. It is only implied that whatever motion I consider, whether it be swift or slow, perpendicular, horizontal or oblique, or in whatever object, the axiom concerning it holds equally true. As does the other of every particular extension, it matters not whether line, surface or solid, whether of this or that magnitude or figure.

12 By observing how ideas become general, we may the better judge how words are made so. And here it is to be noted that I do not deny absolutely there are general ideas, but only that there are any *abstract general ideas*: for in the passages above quoted, wherein there is mention of general ideas, it is

always supposed that they are formed by *abstraction*, after the manner set forth in Sect. 8 and 9. Now if we will annex a meaning to our words, and speak only of what we can conceive, I believe we shall acknowledge, that an idea, which considered in itself is particular, becomes general, by being made to represent or stand for all other particular ideas of the same sort. To make this plain by an example, suppose a geometrician is demonstrating the method, of cutting a line in two equal parts. He draws, for instance, a black line of an inch in length, this which in itself is a particular line is nevertheless with regard to its signification general, since as it is there used, it represents all particular lines whatsoever; for that what is demonstrated of it, is demonstrated of all lines or, in other words, of a line in general. And is that particular line becomes general, by being made a sign, so the name *line* which taken absolutely is particular, by being a sign is made general. And as the former owes its generality, not to its being the sign of an abstract or general line, but of all particular right lines that may possibly exist, so the latter must be thought to derive its generality from the same cause, namely, the various particular lines which it indifferently denotes.

13 To give the reader a yet clearer view of the nature of abstract ideas, and the uses they are thought necessary to, I shall add one more passage out of the *Essay on Human Understanding*, which is as follows. '*Abstract ideas* are not so obvious or easy to children or the yet unexercised mind as particular ones. If they seem so to grown men, it is only because by constant and familiar use they are made so. For when we nicely reflect upon them, we shall find that general ideas are fictions and contrivances of the mind, that carry difficulty with them, and do not so easily offer themselves, as we are apt to imagine. For example, does it not require some pains and skill to form the general idea of a triangle (which is yet none of the most abstract comprehensive and difficult) for it must be neither oblique nor rectangle, neither equilateral, equicrural, nor scalenon, but *all and none* of these at once. In effect, it is something imperfect that cannot exist, an idea wherein some parts of several different and *inconsistent* ideas are put together. It is true the mind in this imperfect state has need of such ideas, and makes all the haste to them it can, for the conveniency of communication and enlargement of knowledge, to both which it is naturally very much inclined. But yet one has reason to suspect such ideas are marks of our imperfection. At least this is enough to shew that the most abstract and general ideas are not those that the mind is first and most easily acquainted with, nor such as its earliest knowledge is conversant about.' B.4. C.7. Sect. 9. If any man has the faculty of framing in his mind such an idea of a triangle as is here described, it is in vain to pretend to dispute him out of it, nor would I go about it. All I desire is, that the reader would fully and certainly inform himself whether he has such an idea or no. And this, methinks, can be no hard task for anyone to perform. What more easy than for anyone to look a little into his own thoughts, and there try whether he has, or can attain to have, an idea that shall correspond with the description that is here given of the general idea of a triangle, which is, *neither oblique, nor rectangle, equilateral, equicrural, nor scalenon, but all and none of these at once?*

14 Much is here said of the difficulty that abstract ideas carry with them, and the pains and skill requisite to the forming them. And it is on all hands agreed that there is need of great toil and labour of the mind, to emancipate our thoughts from particular objects, and raise them to those sublime speculations that are conversant about abstract ideas. From all which the natural consequence should seem to be, that so difficult a thing as the forming abstract ideas was not necessary for *communication*, which is so easy and familiar to all sorts of men. But we are told, if they seem obvious and easy to grown men, *it is only because by constant and familiar use they are made so*. Now I would fain know at what time it is, men are employed in surmounting that difficulty, and furnishing themselves with those necessary helps for discourse. It cannot be when they are grown up, for then it seems they are not conscious of any such pains-taking; it remains therefore to be the business of their childhood. And surely, the great and multiplied labour of framing abstract notions, will be found a hard task for that tender age. Is it not a hard thing to imagine, that a couple of children cannot prate together, of their sugar-plumbs and rattles and the rest of their little trinkets, till they have first tacked together numberless inconsistencies, and so framed in their minds *abstract general ideas*, and annexed them to every common name they make use of?

15 Nor do I think them a whit more needful for the *enlargement of knowledge* than for *communication*. It is I know a point much insisted on, that all knowledge and demonstration are about universal notions, to which I fully agree: but then it doth not appear to me that those notions are formed by *abstraction* in the manner premised; *universality*, so far as I can comprehend, not consisting in the absolute, positive nature or conception of anything, but in the relation it bears to the particulars signified or represented by it: by virtue whereof it is that things, names, or notions, being in their own nature particular, are rendered *universal*. Thus when I demonstrate any proposition concerning triangles, it is to be supposed that I have in view the universal idea of a triangle; which ought not to be understood as if I could frame an idea of a triangle which was neither equilateral nor scalenon nor equicrural. But only that the particular triangle I consider, whether of this or that sort it matters not, doth equally stand for and represent all rectilinear triangles whatsoever, and is in that sense *universal*. All which seems very plain and not to include any difficulty in it.

16 But here it will be demanded, how we can know any proposition to be true of all particular triangles, except we have first seen it demonstrated of the abstract idea of a triangle which equally agrees to all? For because a property may be demonstrated to agree to some one particular triangle, it will not thence follow that it equally belongs to any other triangle, which in all respects is not the same with it. For example, having demonstrated that the three angles of an isosceles rectangular triangle are equal to two right ones, I cannot therefore conclude this affection agrees to all other triangles, which have neither a right angle, nor two equal sides. It seems therefore that, to be certain this proposition is universally true, we must either make a particular demonstration for every

particular triangle, which is impossible, or once for all demonstrate it of the *abstract idea of a triangle,* in which all the particulars do indifferently partake, and by which they are all equally represented. To which I answer, that though the idea I have in view whilst I make the demonstration, be, for instance, that of an isosceles rectangular triangle, whose sides are of a determinate length, I may nevertheless be certain it extends to all other rectilinear triangles, of what sort or bigness soever. And that, because neither the right angle, nor the equality, nor determinate length of the sides, are at all concerned in the demonstration. It is true, the diagram I have in view includes all these particulars, but then there is not the least mention made of them in the proof of the proposition. It is not said, the three angles are equal to two right ones, because one of them is a right angle, or because the sides comprehending it are of the same length. Which sufficiently shews that the right angle might have been oblique, and the sides unequal, and for all that the demonstration have held good. And for this reason it is, that I conclude that to be true of any obliquangular or scalenon, which I had demonstrated of a particular right-angled, equicrural triangle; and not because I demonstrated the proposition of the abstract idea of a triangle. ⟦And here it must be acknowledged that a man may consider a figure merely as triangular, without attending to the particular qualities of the angles, or relations of the sides. So far he may abstract: but this will never prove, that he can frame an abstract general inconsistent idea of a triangle. In like manner we may consider Peter so far forth as man, or so far forth as animal, without framing the forementioned abstract idea, either of man or of animal, in as much as all that is perceived is not considered.⟧[b]

17 It were an endless, as well as an useless thing, to trace the Schoolmen, those great masters of abstraction, through all the manifold inextricable labyrinths of error and dispute, which their doctrine of abstract natures and notions seems to have led them into. What bickerings and controversies, and what a learned dust have been raised about those matters, and what mighty advantage hath been from thence derived to mankind, are things at this day too clearly known to need being insisted on. And it had been well if the ill effects of that doctrine were confined to those only who make the most avowed profession of it. When men consider the great pains, industry and parts, that have for so many ages been laid out on the cultivation and advancement of the sciences, and that notwithstanding all this, the far greater part of them remain full of darkness and uncertainty, and disputes that are like never to have an end, and even those that are thought to be supported by the most clear and cogent demonstrations, contain in them paradoxes which are perfectly irreconcilable to the understandings of men, and that taking all together, a small portion of them doth supply any real benefit to mankind, otherwise than by being an innocent diversion and amusement. I say, the consideration of all this is apt to throw them into a despondency, and perfect contempt of all study. But this may

[b]⟦Last three sentences of §16 added in second edition.⟧

perhaps cease, upon a view of the false principles that have obtained in the world, amongst all which there is none, methinks, hath a more influence over the thoughts of speculative men, than this of abstract general ideas.

18 I come now to consider the source of this prevailing notion, and that seems to me to be language. And surely nothing of less extent than reason itself could have been the source of an opinion so universally received. The truth of this appears as from other reasons, so also from the plain confession of the ablest patrons of abstract ideas, who acknowledge that they are made in order to naming; from which it is a clear consequence, that if there had been no such thing as speech or universal signs, there never had been any thought of abstraction. See B.3. C.6. Sect. 39 *and elsewhere of the Essay on Human Understanding*. Let us therefore examine the manner wherein words have contributed to the origin of that mistake. First then, 'tis thought that every name hath, or ought to have, one only precise and settled signification, which inclines men to think there are certain *abstract, determinate ideas*, which constitute the true and only immediate signification of each general name. And that it is by the mediation of these abstract ideas, that a general name comes to signify any particular thing. Whereas, in truth, there is no such thing as one precise and definite signification annexed to any general name, they all signifying indifferently a great number of particular ideas. All which doth evidently follow from what has been already said, and will clearly appear to anyone by a little reflexion. To this it will be objected, that every name that has a definition, is thereby restrained to one certain signification. For example, a *triangle* is defined to be a *plane surface comprehended by three right lines*; by which that name is limited to denote one certain idea and no other. To which I answer, that in the definition it is not said whether the surface be great or small, black or white, nor whether the sides are long or short, equal or unequal, nor with what angles they are inclined to each other; in all which there may be great variety, and consequently there is no one settled idea which limits the signification of the word *triangle*. 'Tis one thing for to keep a name constantly to the same definition, and another to make it stand everywhere for the same idea: the one is necessary, the other useless and impracticable.

19 But to give a farther account how words came to produce the doctrine of abstract ideas, it must be observed that it is a received opinion, that language has no other end but the communicating our ideas, and that every significant name stands for an idea. This being so, and it being withal certain, that names, which yet are not thought altogether insignificant, do not always mark out particular conceivable ideas, it is straightway concluded that they stand for abstract notions. That there are many names in use amongst speculative men, which do not always suggest to others determinate particular ideas, is what nobody will deny. And a little attention will discover, that it is not necessary (even in the strictest reasonings) significant names which stand for ideas should, every time they are used, excite in the understanding the ideas they are made to stand for: in reading and discoursing, names being for the most part used as letters are in *algebra*, in which though a particular quantity be marked by each letter, yet

to proceed right it is not requisite that in every step each letter suggest to your thoughts, that particular quantity it was appointed to stand for.

20 Besides, the communicating of ideas marked by words is not the chief and only end of language, as is commonly supposed. There are other ends, as the raising of some passion, the exciting to, or deterring from an action, the putting the mind in some particular disposition; to which the former is in many cases barely subservient, and sometimes entirely omitted, when these can be obtained without it, as I think doth not infrequently happen in the familiar use of language. I entreat the reader to reflect with himself, and see if it doth not often happen either in hearing or reading a discourse, that the passions of fear, love, hatred, admiration, disdain, and the like arise, immediately in his mind upon the perception of certain words, without any ideas coming between. At first, indeed, the words might have occasioned ideas that were fit to produce those emotions; but, if I mistake not, it will be found that when language is once grown familiar, the hearing of the sounds or sight of the characters is oft immediately attended with those passions, which at first were wont to be produced by the intervention of ideas, that are now quite omitted. May we not, for example, be affected with the promise of a *good thing*, though we have not an idea of what it is? Or is not the being threatened with danger sufficient to excite a dread, though we think not of any particular evil likely to befall us, nor yet frame to ourselves an idea of danger in abstract? If anyone shall join ever so little reflection of his own to what has been said, I believe it will evidently appear to him, that general names are often used in the propriety of language without the speaker's designing them for marks of ideas in his own, which he would have them raise in the mind of the hearer. Even proper names themselves do not seem always spoken, with a design to bring into our view the ideas of those individuals that are supposed to be marked by them. For example, when a Schoolman tells me *Aristotle hath said it*, all I conceive he means by it, is to dispose me to embrace his opinion with the deference and submission which custom has annexed to that name. And this effect may be so instantly produced in the minds of those who are accustomed to resign their judgment to the authority of that philosopher, as it is impossible any idea either of his person, writings, or reputation should go before. Innumerable examples of this kind may be given, but why should I insist on those things, which everyone's experience will, I doubt not, plentifully suggest unto him?

21 We have, I think, shewn the impossibility of *abstract ideas*. We have considered what has been said for them by their ablest patrons; and endeavoured to shew they are of no use for those ends, to which they are thought necessary. And lastly, we have traced them to the source from whence they flow, which appears to be language. It cannot be denied that words are of excellent use, in that by their means all that stock of knowledge which has been purchased by the joint labours of inquisitive men in all ages and nations, may be drawn into the view and made the possession of one single person. But at the same time it must be owned that most parts of knowledge have been strangely perplexed and darkened by the abuse of words, and general ways of speech

wherein they are delivered. Since therefore words are so apt to impose on the understanding, whatever ideas I consider, I shall endeavour to take bare and naked into my view, keeping out of my thoughts, so far as I am able, those names which long and constant use hath so strictly united with them; from which I may expect to derive the following advantages.

22 First, I shall be sure to get clear of all controversies purely verbal; the springing up of which weeds in almost all the sciences has been a main hindrance to the growth of true and sound knowledge. Secondly, this seems to be a sure way to extricate myself out of that fine and subtle net of *abstract ideas*, which has so miserably perplexed and entangled the minds of men, and that with this peculiar circumstance, that by how much the finer and more curious was the wit of any man, by so much the deeper was he like to be ensnared, and faster held therein. Thirdly, so long as I confine my thoughts to my own ideas divested of words, I do not see how I can easily be mistaken. The objects I consider, I clearly and adequately know. I cannot be deceived in thinking I have an idea which I have not. It is not possible for me to imagine, that any of my own ideas are alike or unlike, that are not truly so. To discern the agreements or disagreements there are between my ideas, to see what ideas are included in any compound idea, and what not, there is nothing more requisite, than an attentive perception of what passes in my own understanding.

23 But the attainment of all these advantages doth presuppose an entire deliverance from the deception of words, which I dare hardly promise myself; so difficult a thing it is to dissolve an union so early begun, and confirmed by so long a habit as that betwixt words and ideas. Which difficulty seems to have been very much increased by the doctrine of *abstraction*. For so long as men thought abstract ideas were annexed to their words, it doth not seem strange that they should use words for ideas: it being found an impracticable thing to lay aside the word, and retain the abstract idea in the mind, which in itself was perfectly inconceivable. This seems to me the principal cause, why those men who have so emphatically recommended to others, the laying aside all use of words in their meditations, and contemplating their bare ideas, have yet failed to perform it themselves. Of late many have been very sensible of the absurd opinions and insignificant disputes, which grow out of the abuse of words. And in order to remedy these evils they advise well, that we attend to the ideas signified, and draw off our attention from the words which signify them. But how good soever this advice may be, they have given others, it is plain they could not have a due regard to it themselves, so long as they thought the only immediate use of words was to signify ideas, and that the immediate signification of every general name was a *determinate, abstract idea.*

24 But these being known to be mistakes, a man may with greater ease prevent his being imposed on by words. He that knows he has no other than particular ideas, will not puzzle himself in vain to find out and conceive the abstract idea, annexed to any name. And he that knows names do not always stand for ideas, will spare himself the labour of looking for ideas, where there are none to be had. It were therefore to be wished that everyone would use his utmost

endeavours, to obtain a clear view of the ideas he would consider, separating from them all that dress and encumbrance of words which so much contribute to blind the judgment and divide the attention. In vain do we extend our view into the heavens, and pry into the entrails of the earth, in vain do we consult the writings of learned men, and trace the dark footsteps of antiquity; we need only draw the curtain of words, to behold the fairest tree of knowledge, whose fruit is excellent, and within the reach of our hand.

25 Unless we take care to clear the first principles of knowledge, from the embarras and delusion of words, we may make infinite reasonings upon them to no purpose; we may draw consequences from consequences, and be never the wiser. The farther we go, we shall only lose ourselves the more irrecoverably, and be the deeper entangled in difficulties and mistakes. Whoever therefore designs to read the following sheets, I entreat him to make my words the occasion of his own thinking, and endeavour to attain the same train of thoughts in reading, that I had in writing them. By this means it will be easy for him to discover the truth or falsity of what I say. He will be out of all danger of being deceived by my words, and I do not see how he can be led into an error by considering his own naked, undisguised ideas.

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118 Hitherto of natural philosophy: we come now to make some inquiry concerning that other great branch of speculative knowledge, to wit, *mathematics*. These, how celebrated soever they may be, for their clearness and certainty of demonstration, which is hardly any where else to be found, cannot nevertheless be supposed altogether free from mistakes; if in their principles there lurks some secret error, which is common to the professors of those sciences with the rest of mankind. Mathematicians, though they deduce their theorems from a great height of evidence, yet their first principles are limited by the consideration of quantity: and they do not ascend into any inquiry concerning those transcendental maxims, which influence all the particular sciences, each part whereof, mathematics not excepted, doth consequently participate of the errors involved in them. That the principles laid down by mathematicians are true, and their way of deduction from those principles clear and incontestable, we do not deny. But we hold, there may be certain erroneous maxims of greater extent than the object of mathematics, and for that reason not expressly mentioned, though tacitly supposed throughout the whole progress of that science; and that the ill effects of those secret unexamined errors are diffused through all the branches thereof. To be plain, we suspect the mathematicians are, as well as other men, concerned in the errors arising from the doctrine of abstract general ideas, and the existence of objects without the mind.

119 *Arithmetic* hath been thought to have for its object abstract ideas of *number*. Of which to understand the properties and mutual habitudes is supposed no mean part of speculative knowledge. The opinion of the pure and

intellectual nature of numbers in abstract, hath made them in esteem with those philosophers, who seem to have affected an uncommon fineness and elevation of thought. It hath set a price on the most trifling numerical speculations which in practice are of no use, but serve only for amusement: and hath therefore so far infected the minds of some, that they have dreamt of mighty *mysteries* involved in numbers, and attempted the explication of natural things by them. But if we inquire into our own thoughts, and consider what hath been premised, we may perhaps entertain a low opinion of those high flights and abstractions, and look on all inquiries about numbers, only as so many *difficiles nugæ*, so far as they are not subservient to practice, and promote the benefit of life.

120 Unity in abstract we have before considered in *Sect.* 13, from which and what hath been said in the Introduction, it plainly follows there is not any such idea. But number being defined a *collection of units*, we may conclude that, if there be no such thing as unity or unit in abstract, there are no ideas of number in abstract denoted by the numerical names and figures. The theories therefore in arithmetic, if they are abstracted from the names and figures, as likewise from all use and practice, as well as from the particular things numbered, can be supposed to have nothing at all for their object. Hence we may see, how entirely the science of numbers is subordinate to practice, and how jejune and trifling it becomes, when considered as a matter of mere speculation.

121 However since there may be some, who, deluded by the specious shew of discovering abstracted verities, waste their time in arithmetical theorems and problems, which have not any use: it will not be amiss, if we more fully consider, and expose the vanity of that pretence; and this will plainly appear, by taking a view of arithmetic in its infancy, and observing what it was that originally put men on the study of that science, and to what scope they directed it. It is natural to think that at first, men, for ease of memory and help of computation, made use of counters, or in writing of single strokes, points or the like, each whereof was made to signify an unit, that is, some one thing of whatever kind they had occasion to reckon. Afterwards they found out the more compendious ways, of making one character stand in place of several strokes, or points. And lastly, the notation of the Arabians or Indians came into use, wherein by the repetition of a few characters or figures, and varying the signification of each figure according to the place it obtains, all numbers may be most aptly expressed: which seems to have been done in imitation of language, so that an exact analogy is observed betwixt the notation by figures and names, the nine simple figures answering the nine first numeral names and places in the former, corresponding to denominations in the latter. And agreeably to those conditions of the simple and local value of figures, were contrived methods of finding from the given figures or marks of the parts, what figures and how placed, are proper to denote the whole or *vice versa*. And having found the sought figures, the same rule or analogy being observed throughout, it is easy to read them into words; and so the number becomes perfectly known. For then the number of any particular things is said to be known, when we know the name or figures (with their due arrangement) that according to the standing analogy belong to

them. For these signs being known, we can by the operations of arithmetic, know the signs of any part of the particular sums signified by them; and thus computing in signs (because of the connection established betwixt them and the distinct multitudes of things, whereof one is taken for an unit), we may be able rightly to sum up, divide, and proportion the things themselves that we intend to number.

122 In *arithmetic* therefore we regard not the *things* but the *signs*, which nevertheless are not regarded for their own sake, but because they direct us how to act with relation to things, and dispose rightly of them. Now agreeably to what we have before observed, of words in general (*Sect.* 19. *Introd.*) it happens here likewise, that abstract ideas are thought to be signified by numeral names or characters, while they do not suggest ideas of particular things to our minds. I shall not at present enter into a more particular dissertation on this subject; but only observe that it is evident from what hath been said, those things which pass for abstract truths and theorems concerning numbers, are, in reality, conversant about no object distinct from particular numerable things, except only names and characters; which originally came to be considered, on no other account but their being *signs*, or capable to represent aptly, whatever particular things men had need to compute. Whence it follows, that to study them for their own sake would be just as wise, and to as good purpose, as if a man, neglecting the true use or original intention and subserviency of language, should spend his time in impertinent criticisms upon words, or reasonings and controversies purely verbal.

123 From numbers we proceed to speak of *extension*, which considered as relative, is the object of geometry. The *infinite* divisibility of *finite* extension, though it is not expressly laid down, either as an axiom or theorem in the elements of that science, yet is throughout the same every where supposed, and thought to have so inseparable and essential a connection with the principles and demonstrations in geometry, that mathematicians never admit it into doubt, or make the least question of it. And as this notion is the source from whence do spring all those amusing geometrical paradoxes, which have such a direct repugnancy to the plain common sense of mankind, and are admitted with so much reluctance into a mind not yet debauched by learning: so is it the principal occasion of all that nice and extreme subtlety, which renders the study of *mathematics* so difficult and tedious. Hence if we can make it appear, that no finite extension contains innumerable parts, or is infinitely divisible, it follows that we shall at once clear the science of geometry from a great number of difficulties and contradictions, which have ever been esteemed a reproach to human reason, and withal make the attainment thereof a business of much less time and pains, than it hitherto hath been.

124 Every particular finite extension, which may possibly be the object of our thought, is an *idea* existing only in the mind, and consequently each part thereof must be perceived. If therefore I cannot perceive innumerable parts in any finite extension that I consider, it is certain they are not contained in it: but it is evident, that I cannot distinguish innumerable parts in any particular line,

surface, or solid, which I either perceive by sense, or figure to my self in my mind: wherefore I conclude they are not contained in it. Nothing can be plainer to me, than that the extensions I have in view are no other than my own ideas, and it is no less plain, that I cannot resolve any one of my ideas into an infinite number of other ideas, that is, that they are not infinitely divisible. If by *finite extension* be meant something distinct from a finite idea, I declare I do not know what that is, and so cannot affirm or deny any thing of it. But if the terms *extension*, *parts*, and the like, are taken in any sense conceivable, that is, for ideas; then to say a finite quantity or extension consists of parts infinite in number, is so manifest a contradiction, that every one at first sight acknowledges it to be so. And it is impossible it should ever gain the assent of any reasonable creature, who is not brought to it by gentle and slow degrees, as a converted Gentile to the belief of *transubstantiation*. Ancient and rooted prejudices do often pass into principles: and those propositions which once obtain the force and credit of a *principle*, are not only themselves, but likewise whatever is deducible from them, thought privileged from all examination. And there is no absurdity so gross, which by this means the mind of man may not be prepared to swallow.

125 He whose understanding is prepossessed with the doctrine of abstract general ideas, may be persuaded, that (whatever be thought of the ideas of sense), extension in *abstract* is infinitely divisible. And one who thinks the objects of sense exist without the mind, will perhaps in virtue thereof be brought to admit, that a line but an inch long may contain innumerable parts really existing, though too small to be discerned. These errors are grafted as well in the minds of *geometricians*, as of other men, and have a like influence on their reasonings; and it were no difficult thing, to shew how the arguments from geometry made use of to support the infinite divisibility of extension, are bottomed on them. At present we shall only observe in general, whence it is that the mathematicians are all so fond and tenacious of this doctrine.

126 It hath been observed in another place, that the theorems and demonstrations in geometry are conversant about universal ideas. *Sect.* 15. *Introd.* Where it is explained in what sense this ought to be understood, to wit, that the particular lines and figures included in the diagram, are supposed to stand for innumerable others of different sizes: or in other words, the geometer considers them abstracting from their magnitude: which doth not imply that he forms an abstract idea, but only that he cares not what the particular magnitude is, whether great or small, but looks on that as a thing indifferent to the demonstration: hence it follows, that a line in the scheme, but an inch long, must be spoken of, as though it contained ten thousand parts, since it is regarded not in it self, but as it is universal; and it is universal only in its signification, whereby it represents innumerable lines greater than it self, in which may be distinguished ten thousand parts or more, though there may not be above an inch in it. After this manner the properties of the lines signified are (by a very usual figure) transferred to the sign, and thence through mistake thought to appertain to it considered in its own nature.

127 Because there is no number of parts so great, but it is possible there may be a line containing more, the inch-line is said to contain parts more than any assignable number; which is true, not of the inch taken absolutely, but only for the things signified by it. But men not retaining that distinction in their thoughts, slide into a belief that the small particular line described on paper contains in it self parts innumerable. There is no such thing as the ten-thousandth part of an *inch*; but there is of a *mile* or *diameter of the earth*, which may be signified by that inch. When therefore I delineate a triangle on paper, and take one side not above an inch, for example, in length to be the *radius*: this I consider as divided into ten thousand or an hundred thousand parts, or more. For though the ten-thousandth part of that line considered in it self, is nothing at all, and consequently may be neglected without any error or inconveniency; yet these described lines being only marks standing for greater quantities, whereof it may be the ten-thousandth part is very considerable, it follows, that to prevent notable errors in practice, the *radius* must be taken of ten thousand parts, or more.

128 From what hath been said the reason is plain why, to the end any theorem may become universal in its use, it is necessary we speak of the lines described on paper, as though they contained parts which really they do not. In doing of which, if we examine the matter throughly, we shall perhaps discover that we cannot conceive an inch it self as consisting of, or being divisible into a thousand parts, but only some other line which is far greater than an inch, and represented by it. And that when we say a line is *infinitely divisible*, we must mean a line which is *infinitely great*. What we have here observed seems to be the chief cause, why to suppose the infinite divisibility of finite extension hath been thought necessary in geometry.

129 The several absurdities and contradictions which flowed from this false principle might, one would think, have been esteemed so many demonstrations against it. But by I know not what *logic*, it is held that proofs *à posteriori* are not to be admitted against propositions relating to infinity. As though it were not impossible even for an infinite mind to reconcile contradictions. Or as if any thing absurd and repugnant could have a necessary connection with truth, or flow from it. But whoever considers the weakness of this pretence, will think it was contrived on purpose to humour the laziness of the mind, which had rather acquiesce in an indolent scepticism, than be at the pains to go through with a severe examination of those principles it hath ever embraced for true.

130 Of late the speculations about infinites have run so high, and grown to such strange notions, as have occasioned no small scruples and disputes among the geometers of the present age. Some there are of great note, who not content with holding that finite lines may be divided into an infinite number of parts, do yet farther maintain, that each of those infinitesimals is it self subdivisible into an infinity of other parts, or infinitesimals of a second order, and so on *ad infinitum*. These, I say, assert there are infinitesimals of infinitesimals of infinitesimals, without ever coming to an end. So that according to them an inch doth not barely contain an infinite number of parts, but an infinity of an infinity

of an infinity *ad infinitum* of parts. Others there be who hold all orders of infinitesimals below the first to be nothing at all, thinking it with good reason absurd, to imagine there is any positive quantity or part of extension, which though multiplied infinitely, can ever equal the smallest given extension. And yet on the other hand it seems no less absurd, to think the square, cube, or other power of a positive real root, should it self be nothing at all; which they who hold infinitesimals of the first order, denying all of the subsequent orders, are obliged to maintain.

131 Have we not therefore reason to conclude, that they are *both* in the wrong, and that there is in effect no such thing as parts infinitely small, or an infinite number of parts contained in any finite quantity? But you will say, that if this doctrine obtains, it will follow the very foundations of geometry are destroyed: and those great men who have raised that science to so astonishing an height, have been all the while building a castle in the air. To this it may be replied, that whatever is useful in geometry and promotes the benefit of human life, doth still remain firm and unshaken on our principles. That science considered as practical, will rather receive advantage than any prejudice from what hath been said. But to set this in a due light, may be the subject of a distinct inquiry. For the rest, though it should follow that some of the more intricate and subtle parts of *speculative mathematics* may be pared off without any prejudice to truth; yet I do not see what damage will be thence derived to mankind. On the contrary, it were highly to be wished, that men of great abilities and obstinate application would draw off their thoughts from those amusements, and employ them in the study of such things as lie nearer the concerns of life, or have a more direct influence on the manners.

132 If it be said that several theorems undoubtedly true, are discovered by methods in which infinitesimals are made use of, which could never have been, if their existence included a contradiction in it. I answer, that upon a thorough examination it will not be found, that in any instance it is necessary to make use of or conceive infinitesimal parts of finite lines, or even quantities less than the *minimum sensibile*: nay, it will be evident this is never done, it being impossible. ⟦And whatever mathematicians may think of fluxions or the differential calculus and the like, a little reflection will shew them, that in working by those methods, they do not conceive or imagine lines or surfaces less than what are perceivable to sense. They may, indeed, call those little and almost insensible quantities infinitesimals or infinitesimals of infinitesimals, if they please: but at bottom this is all, they being in truth finite, nor does the solution of problems require the supposing any other. But this will be more clearly made out hereafter.⟧[c]

133 By what we have premised, it is plain that very numerous and important errors have taken their rise from those false principles, which were impugned in the foregoing parts of this treatise. And the opposites of those erroneous tenets at the same time appear to be most fruitful principles, from whence do

[c] [This passage occurs in the first edition only.]

flow innumerable consequences highly advantageous to true philosophy as well as to religion. Particularly, *matter* or *the absolute existence of corporeal objects*, hath been shewn to be that wherein the most avowed and pernicious enemies of all knowledge, whether human or divine, have ever placed their chief strength and confidence. And surely, if by distinguishing the real existence of unthinking things from their being perceived, and allowing them a subsistence of their own out of the minds of spirits, no one thing is explained in Nature; but on the contrary a great many inexplicable difficulties arise: if the supposition of matter is barely precarious, as not being grounded on so much as one single reason: if its consequences cannot endure the light of examination and free inquiry, but screen themselves under the dark and general pretence of *infinites being incomprehensible*: if withal the removal of this *matter* be not attended with the least evil consequence, if it be not even missed in the world, but every thing as well, nay much easier conceived without it: if lastly, both *sceptics* and *atheists* are for ever silenced upon supposing only spirits and ideas, and this scheme of things is perfectly agreeable both to *reason* and *religion*: methinks we may expect it should be admitted and firmly embraced, though it were proposed only as an *hypothesis*, and the existence of matter had been allowed possible, which yet I think we have evidently demonstrated that it is not.

134 True it is, that in consequence of the foregoing principles, several disputes and speculations, which are esteemed no mean parts of learning, are rejected as useless. But how great a prejudice soever against our notions, this may give to those who have already been deeply engaged, and made large advances in studies of that nature: yet by others, we hope it will not be thought any just ground of dislike to the principles and tenets herein laid down, that they abridge the labour of study, and make human sciences more clear, compendious, and attainable, than they were before.

E. DE MOTU
(*BERKELEY 1721*)

In 1713 Berkeley left Dublin for London, where he published his *Three dialogues between Hylas and Philonous*, a popularized exposition of the doctrines of the *Principles of human knowledge*. For the next eight years he travelled on the Continent, wrote essays in defence of religion, and published little on philosophical topics. His *De motu sive de motus principio & natura, et de causa communicationis motuum*, written on the Continent at the end of this period of Berkeley's life, is his most sustained criticism of the foundations of Newtonian physics. It contains a subtle discussion (§§35–42) of the nature of physical axioms and of the relationship between mathematics and physics; the essay prefigures the critique of Newtonian mathematics in *The analyst*, and shows how Berkeley's general philosophical arguments—in particular his arguments against abstract ideas and material substance—could be applied to the foundations of natural science. For a discussion of Berkeley's philosophy of physics, see *Popper 1953–4*. The translation of the *De motu* is by Arthur Aston Luce, and is reprinted from *Berkeley 1948–57*, Vol. 4. References to *Berkeley 1721* should be to the paragraph numbers, which appeared in the original Luce translation.

OF MOTION

OR

THE PRINCIPLE AND NATURE OF MOTION AND THE CAUSE OF THE COMMUNICATION OF MOTIONS

1 In the pursuit of truth we must beware of being misled by terms which we do not rightly understand. That is the chief point. Almost all philosophers utter the caution; few observe it. Yet it is not so difficult to observe, where sense, experience, and geometrical reasoning obtain, as is especially the case in physics. Laying aside, then, as far as possible, all prejudice, whether rooted in linguistic usage or in philosophical authority, let us fix our gaze on the very nature of things. For no one's authority ought to rank so high as to set a value on his words and terms unless they are found to be based on clear and certain fact.

2 The consideration of motion greatly troubled the minds of the ancient philosophers, giving rise to various exceedingly difficult opinions (not to say absurd) which have almost entirely gone out of fashion, and not being worth a detailed discussion need not delay us long. In works on motion by the more recent and sober thinkers of our age, not a few terms of somewhat abstract and obscure signification are used, such as *solicitation of gravity, urge, dead forces*, etc., terms which darken writings in other respects very learned, and beget opinions at variance with truth and the common sense of men. These terms must be examined with great care, not from a desire to prove other people wrong, but in the interest of truth.

3 *Solicitation* and *effort* or *conation* belong properly to animate beings alone. When they are attributed to other things, they must be taken in a metaphorical sense; but a philosopher should abstain from metaphor. Besides, anyone who has seriously considered the matter will agree that those terms have no clear and distinct meaning apart from all affection of the mind and motion of the body.

4 While we support heavy bodies we feel in ourselves effort, fatigue, and discomfort. We perceive also in heavy bodies falling an accelerated motion towards the centre of the earth; and that is all the senses tell us. By reason,

however, we infer that there is some cause or principle of these phenomena, and that is popularly called *gravity*. But since the cause of the fall of heavy bodies is unseen and unknown, gravity in that usage cannot properly be styled a sensible quality. It is, therefore, an occult quality. But what an occult quality is, or how any quality can act or do anything, we can scarcely conceive—indeed we cannot conceive. And so men would do better to let the occult quality go, and attend only to the sensible effects. Abstract terms (however useful they may be in argument) should be discarded in meditation, and the mind should be fixed on the particular and the concrete, that is, on the things themselves.

5 *Force* likewise is attributed to bodies; and that word is used as if it meant a known quality, and one distinct from motion, figure, and every other sensible thing and also from every affection of the living thing. But examine the matter more carefully and you will agree that such force is nothing but an occult quality. Animal effort and corporeal motion are commonly regarded as symptoms and measures of this occult quality.

6 Obviously then it is idle to lay down gravity or force as the principle of motion; for how could that principle be known more clearly by being styled an occult quality? What is itself occult explains nothing. And I need not say that an unknown acting cause could be more correctly styled substance than quality. Again, *force*, *gravity*, and terms of that sort are more often used in the concrete (and rightly so) so as to connote the body in motion, the effort of resisting, *etc*. But when they are used by philosophers to signify certain natures carved out and abstracted from all these things, natures which are not objects of sense, nor can be grasped by any force of intellect, nor pictured by the imagination, then indeed they breed errors and confusion.

7 About general and abstract terms many men make mistakes; they see their value in argument, but they do not appreciate their purpose. In part the terms have been invented by common habit to abbreviate speech, and in part they have been thought out by philosophers for instructional purposes, not that they are adapted to the natures of things which are in fact singulars and concrete, but they come in useful for handing on received opinions by making the notions or at least the propositions universal.

8 We generally suppose that corporeal force is something easy to conceive. Those, however, who have studied the matter more carefully are of a different opinion, as appears from the strange obscurity of their language when they try to explain it. Torricelli says that force and impetus are abstract and subtle things and quintessences which are included in corporeal substance as in the magic vase of Circe.[1] Leibniz likewise in explaining the nature of force has this: 'Active primitive force which is ἐντελέχεια ἡ πρώτη corresponds to the soul or substantial form.' See *Acta Erudit. Lips.* Thus even the greatest men when they give

[1] Matter is nothing else than a magic vase of Circe, which serves as a receptacle of force and of the moments of the impetus. Force and the impetus are such subtle abstractions and such volatile quintessences that they cannot be shut up in any vessel except in the innermost substance of natural solids. See *Academic Lectures*.

way to abstractions are bound to pursue terms which have no certain significance and are mere shadows of scholastic things. Other passages in plenty from the writings of the younger men could be produced which give abundant proof that metaphysical abstractions have not in all quarters given place to mechanical science and experiment, but still make useless trouble for philosophers.

9 From that source derive various absurdities, such as that dictum: 'The force of percussion, however small, is infinitely great'—which indeed supposes that gravity is a certain real quality different from all others, and that gravitation is, as it were, an act of this quality, really distinct from motion. But a very small percussion produces a greater effect than the greatest gravitation without motion. The former gives out some motion indeed, the latter none. Whence it follows that the force of percussion exceeds the force of gravitation by an infinite ratio, *i.e.* is infinitely great. See the experiments of Galileo, and the writings of Torricelli, Borelli, and others on the definite force of percussion.

10 We must, however, admit that no force is immediately felt by itself, nor known or measured otherwise than by its effect; but of a dead force or of simple gravitation in a body at rest, no change taking place, there is no effect; of percussion there is some effect. Since, then, forces are proportional to effects, we may conclude that there is no dead force, but we must not on that account infer that the force of percussion is infinite; for we cannot regard as infinite any positive quantity on the ground that it exceeds by an infinite ratio a zero-quantity or nothing.

11 The force of gravitation is not to be separated from momentum; but there is no momentum without velocity, since it is mass multiplied by velocity; again, velocity cannot be understood without motion, and the same holds therefore of the force of gravitation. Then no force makes itself known except through action, and through action it is measured; but we are not able to separate the action of a body from its motion; therefore as long as a heavy body changes the shape of a piece of lead put under it, or of a cord, so long is it moved; but when it is at rest, it does nothing, or (which is the same thing) it is prevented from acting. In brief, those terms *dead force* and *gravitation* by the aid of metaphysical abstraction are supposed to mean something different from moving, moved, motion, and rest, but, in point of fact, the supposed difference in meaning amounts to nothing at all.

12 If anyone were to say that a weight hung or placed on the cord acts on it, since it prevents it from restoring itself by elastic force, I reply that by parity of reasoning any lower body acts on the higher body which rests on it, since it prevents it from coming down. But for one body to prevent another from existing in that space which *it* occupies cannot be styled the action of that body.

13 We feel at times the pressure of a gravitating body. But that unpleasant sensation arises from the motion of the heavy body communicated to the fibres and nerves of our body and changing their situation, and therefore it ought to be referred to percussion. In these matters we are afflicted by a number of serious prejudices, which should be subdued, or rather entirely exorcised by keen and continued reflection.

14 In order to prove that any quantity is infinite, we have to show that some, finite, homogeneous part is contained in it an infinite number of times. But dead force is to the force of percussion, not as part to the whole, but as the point to the line, according to the very writers who maintain the infinite force of percussion. Much might be added on this matter, but I am afraid of being prolix.

15 By the foregoing principles famous controversies which have greatly exercised the minds of learned men can be solved; for instance, that controversy about the proportion of forces. One side conceding that momenta, motions, and impetus, given the mass, are simply as the velocities, affirms that the forces are as the squares of the velocities. Everyone sees that this opinion supposes that the force of the body is distinguished from momentum, motion, and impetus, and without that supposition it collapses.

16 To make it still clearer that a certain strange confusion has been introduced into the theory of motion by metaphysical abstractions, let us watch the conflict of opinion about force and impetus among famous men. Leibniz confuses impetus with motion. According to Newton impetus is in fact the same as the force of inertia. Borelli asserts that impetus is only the degree of velocity. Some would make impetus and effort different, others identical. Most regard the motive force as proportional to the motion; but a few prefer to suppose some other force besides the motive, to be measured differently, for instance by the squares of the velocities into the masses. But it would be an endless task to follow out this line of thought.

17 *Force, gravity, attraction*, and terms of this sort are useful for reasonings and reckonings about motion and bodies in motion, but not for understanding the simple nature of motion itself or for indicating so many distinct qualities. As for attraction, it was certainly introduced by Newton, not as a true, physical quality, but only as a mathematical hypothesis. Indeed Leibniz when distinguishing elementary effort or solicitation from impetus, admits that those entities are not really found in nature, but have to be formed by abstraction.

18 A similar account must be given of the composition and resolution of any direct forces into any oblique ones by means of the diagonal and sides of the parallelogram. They serve the purpose of mechanical science and reckoning; but to be of service to reckoning and mathematical demonstrations is one thing, to set forth the nature of things is another.

19 Of the moderns many are of the opinion that motion is neither destroyed nor generated anew, but that the quantity of motion remains for ever constant. Aristotle indeed propounded that problem long ago, Does motion come into being and pass away, or is it eternal? *Phys.* Bk. 8. That sensible motion perishes is clear to the senses, but apparently they will have it that the same impetus and effort remains, or the same sum of forces. Borelli affirms that force in percussion is not lessened, but expanded, that even contrary impetus are received and retained in the same body. Likewise Leibniz contends that effort exists everywhere and always in matter, and that it is understood by reason where it is not evident to the senses. But these points, we must admit, are too abstract and obscure, and of much the same sort as substantial forms and entelechies.

20 All those who, to explain the cause and origin of motion, make use of the hylarchic principle, or of a nature's want or appetite, or indeed of a natural instinct, are to be considered as having said something, rather than thought it. And from these they[2] are not far removed who have supposed 'that the parts of the earth are self-moving, or even that spirits are implanted in them like a form' in order to assign the cause of the acceleration of heavy bodies falling. So too with him[3] who said 'that in the body besides solid extension, there must be something posited to serve as starting-point for the consideration of forces'. All these indeed either say nothing particular and determinate, or if there is anything in what they say, it will be as difficult to explain as that very thing it was brought forward to explain.

21 To throw light on nature it is idle to adduce things which are neither evident to the senses, nor intelligible to reason. Let us see then what sense and experience tell us, and reason that rests upon them. There are two supreme classes of things, body and soul. By the help of sense we know the extended thing, solid, mobile, figured, and endowed with other qualities which meet the senses, but the sentient, percipient, thinking thing we know by a certain internal consciousness. Further we see that those things are plainly different from one another, and quite heterogeneous. I speak of things known; for of the unknown it is profitless to speak.

22 All that which we know to which we have given the name *body* contains nothing in itself which could be the principle of motion or its efficient cause; for impenetrability, extension, and figure neither include nor connote any power of producing motion; nay, on the contrary, if we review singly those qualities of body, and whatever other qualities there may be, we shall see that they are all in fact passive and that there is nothing active in them which can in any way be understood as the source and principle of motion. As for gravity we have already shown above that by that term is meant nothing we know, nothing other than the sensible effect, the cause of which we seek. And indeed when we call a body heavy we understand nothing else except that it is borne downwards, and we are not thinking at all about the cause of this sensible effect.

23 And so about body we can boldly state as established fact that it is not the principle of motion. But if anyone maintains that the term *body* covers in its meaning occult quality, virtue, form, and essence, besides solid extension and its modes, we must just leave him to his useless disputation with no ideas behind it, and to his abuse of names which express nothing distinctly. But the sounder philosophical method, it would seem, abstains as far as possible from abstract and general notions (if *notions* is the right term for things which cannot be understood).

24 The contents of the idea of body we know; but what we know in body is agreed not to be the principle of motion. But those who as well maintain something unknown in body of which they have no idea and which they call

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[2] Borelli.
[3] Leibniz.

the principle of motion, are in fact simply stating that the principle of motion is unknown, and one would be ashamed to linger long on subtleties of this sort.

25 Besides corporeal things there is the other class, *viz.* thinking things, and that there is in them the power of moving bodies we have learned by personal experience, since our mind at will can stir and stay the movements of our limbs, whatever be the ultimate explanation of the fact. This is certain, that bodies are moved at the will of the mind, and accordingly the mind can be called, correctly enough, a principle of motion, a particular and subordinate principle indeed, and one which itself depends on the first and universal principle.

26 Heavy bodies are borne downwards, although they are not affected by any apparent impulse; but we must not think on that account that the principle of motion is contained in them. Aristotle gives this account of the matter, 'Heavy and light things are not moved by themselves; for that would be a characteristic of life, and they would be able to stop themselves.' All heavy things by one and the same certain and constant law seek the centre of the earth, and we do not observe in them a principle or any faculty of halting that motion, of diminishing it or increasing it except in fixed proportion, or finally of altering it in any way. They behave quite passively. Again, in strict and accurate speech, the same must be said of percussive bodies. Those bodies as long as they are being moved, as also in the very moment of percussion, behave passively, exactly as when they are at rest. Inert body so acts as body moved acts, if the truth be told. Newton recognizes that fact when he says that the force of inertia is the same as impetus. But body, inert and at rest, does nothing; therefore body moved does nothing.

27 Body in fact persists equally in either state, whether of motion or of rest. Its existence is not called its action; nor should its persistence be called its action. Persistence is only continuance in the same way of existing, which cannot properly be called action. Resistance which we experience in stopping a body in motion we falsely imagine to be its action, deluded by empty appearance. For that resistance which we feel is in fact passion in ourselves, and does not prove that body acts, but that we are affected; it is quite certain that we should be affected in the same way, whether that body were to be moved by itself, or impelled by another principle.

28 Action and reaction are said to be in bodies, and that way of speaking suits the purposes of mechanical demonstrations; but we must not on that account suppose that there is some real virtue in them which is the cause or principle of motion. For those terms are to be understood in the same way as the term *attraction*; and just as attraction is only a mathematical hypothesis, and not a physical quality, the same must be understood also about action and reaction, and for the same reason. For in mechanical philosophy the truth and the use of theorems about the mutual attraction of bodies remain firm, as founded solely in the motion of bodies, whether that motion be supposed to be caused by the action of bodies mutually attracting each other, or by the action of some agent different from the bodies, impelling and controlling them. Similarly the traditional formulations of rules and laws of motions, along with

the theorems thence deduced remain unshaken, provided that sensible effects and the reasonings grounded in them are granted, whether we suppose the action itself or the force that causes these effects to be in the body or in the incorporeal agent.

29 Take away from the idea of body extension, solidity, and figure, and nothing will remain. But those qualities are indifferent to motion, nor do they contain anything which could be called the principle of motion. This is clear from our very ideas. If therefore by the term *body* be meant that which we conceive, obviously the principle of motion cannot be sought therein, that is, no part or attribute thereof is the true, efficient cause of the production of motion. But to employ a term, and conceive nothing by it is quite unworthy of a philosopher.

30 A thinking, active thing is given which we experience as the principle of motion in ourselves. This we call *soul*, *mind*, and *spirit*. An extended thing also is given, inert, impenetrable, movable, totally different from the former and constituting a new genus. Anaxagoras, wisest of men, was the first to grasp the great difference between thinking things and extended things, and he asserted that the mind has nothing in common with bodies, as is established from the first book of Aristotle's *De Anima*. Of the moderns Descartes has put the same point most forcibly. What was left clear by him others have rendered involved and difficult by their obscure terms.

31 From what has been said it is clear that those who affirm that active force, action, and the principle of motion are really in bodies are adopting an opinion not based on experience, are supporting it with obscure and general terms, and do not well understand their own meaning. On the contrary those who will have mind to be the principle of motion are advancing an opinion fortified by personal experience, and one approved by the suffrages of the most learned men in every age.

32 Anaxagoras was the first to introduce *nous* to impress motion on inert matter. Aristotle, too, approves that opinion and confirms it in many ways, openly stating that the first mover is immovable, indivisible, and has no magnitude. And he rightly notes that to say that every mover must be movable is the same as to say that every builder must be capable of being built. *Phys*. Bk. 8. Plato, moreover, in the Timaeus records that this corporeal machine, or visible world, is moved and animated by mind which eludes all sense. To-day indeed Cartesian philosophers recognize God as the principle of natural motions. And Newton everywhere frankly intimates that not only did motion originate from God, but that still the mundane system is moved by the same actus. This is agreeable to Holy Scripture; this is approved by the opinion of the schoolmen; for though the Peripatetics tell us that nature is the principle of motion and rest, yet they interpret *natura naturans* to be God. They understand of course that all the bodies of this mundane system are moved by Almighty Mind according to certain and constant reason.

33 But those who attribute a vital principle to bodies are imagining an obscure notion and one ill suited to the facts. For what is meant by being endowed with the vital principle, except to live? And to live, what is it but to move oneself, to

stop, and to change one's state? But the most learned philosophers of this age lay it down for an indubitable principle that every body persists in its own state, whether of rest or of uniform movement in a straight line, except in so far as it is compelled from without to alter that state. The contrary is the case with mind; we feel it as a faculty of altering both our own state and that of other things, and that is properly called vital, and puts a wide distinction between soul and bodies.

34 Modern thinkers consider motion and rest in bodies as two states of existence in either of which every body, without pressure from external force, would naturally remain passive; whence one might gather that the cause of the existence of bodies is also the cause of their motion and rest. For no other cause of the successive existence of the body in different parts of space should be sought, it would seem, than that cause whence is derived the successive existence of the same body in different parts of time. But to treat of the good and great God, creator and preserver of all things, and to show how all things depend on supreme and true being, although it is the most excellent part of human knowledge, is, however, rather the province of first philosophy or metaphysics and theology than of natural philosophy, which to-day is almost entirely confined to experiments and mechanics. And so natural philosophy either presupposes the knowledge of God or borrows it from some superior science. Although it is most true that the investigation of nature everywhere supplies the higher sciences with notable arguments to illustrate and prove the wisdom, the goodness, and the power of God.

35 The imperfect understanding of this situation has caused some to make the mistake of rejecting the mathematical principles of physics on the ground that they do not assign the efficient causes of things. It is not, however, in fact the business of physics or mechanics to establish efficient causes, but only the rules of impulsions or attractions, and, in a word, the laws of motions, and from the established laws to assign the solution, not the efficient cause, of particular phenomena.

36 It will be of great importance to consider what properly a principle is, and how that term is to be understood by philosophers. The true, efficient and conserving cause of all things by supreme right is called their fount and principle. But the principles of experimental philosophy are properly to be called foundations and springs, not of their existence but of our knowledge of corporeal things, both knowledge by sense and knowledge by experience, foundations on which that knowledge rests and springs from which it flows. Similarly in mechanical philosophy those are to be called principles, in which the whole discipline is grounded and contained, those primary laws of motions which have been proved by experiments, elaborated by reason and rendered universal. These laws of motion are conveniently called principles, since from them are derived both general mechanical theorems and particular explanations of the phenomena.

37 A thing can be said to be explained mechanically then indeed when it is reduced to those most simple and universal principles, and shown by accurate

reasoning to be in agreement and connection with them. For once the laws of nature have been found out, then it is the philosopher's task to show that each phenomenon is in constant conformity with those laws, that is, necessarily follows from those principles. In that consist the explanation and solution of phenomena and the assigning their cause, *i.e.* the reason why they take place.

38 The human mind delights in extending and expanding its knowledge; and for this purpose general notions and propositions have to be formed in which particular propositions and cognitions are in some way comprised, which then, and not till then, are believed to be understood. Geometers know this well. In mechanics also notions are premised, *i.e.* definitions and first and general statements about motion from which afterwards by mathematical method conclusions more remote and less general are deduced. And just as by the application of geometrical theorems, the sizes of particular bodies are measured, so also by the application of the universal theorems of mechanics, the movements of any parts of the mundane system, and the phenomena thereon depending, become known and are determined. And that is the sole mark at which the physicist must aim.

39 And just as geometers for the sake of their art make use of many devices which they themselves cannot describe nor find in the nature of things, even so the mechanician makes use of certain abstract and general terms, imagining in bodies force, action, attraction, solicitation, *etc.* which are of first utility for theories and formulations, as also for computations about motion, even if in the truth of things, and in bodies actually existing, they would be looked for in vain, just like the geometers' fictions made by mathematical abstraction.

40 We actually perceive by the aid of the senses nothing except the effects or sensible qualities and corporeal things entirely passive, whether in motion or at rest; and reason and experience advise us that there is nothing active except mind or soul. Whatever else is imagined must be considered to be of a kind with other hypotheses and mathematical abstractions. This ought to be laid to heart; otherwise we are in danger of sliding back into the obscure subtlety of the schoolmen, which for so many ages, like some dread plague, has corrupted philosophy.

41 Mechanical principles and universal laws of motions or of nature, happy discoveries of the last century, treated and applied by aid of geometry, have thrown a remarkable light upon philosophy. But metaphysical principles and real efficient causes of the motion and existence of bodies or of corporeal attributes in no way belong to mechanics or experiment, nor throw light on them, except in so far as by being known beforehand they may serve to define the limits of physics, and in that way to remove imported difficulties and problems.

42 Those who derive the principle of motion from spirits mean by *spirit* either a corporeal thing or an incorporeal; if a corporeal thing, however tenuous, yet the difficulty recurs; if an incorporeal thing, however true it may be, yet it does not properly belong to physics. But if anyone were to extend natural philosophy beyond the limits of experiments and mechanics, so as to

cover a knowledge of incorporeal and inextended things, that broader interpretation of the term permits a discussion of soul, mind, or vital principle. But it will be more convenient to follow the usage which is fairly well accepted, and so to distinguish between the sciences as to confine each to its own bounds; thus the natural philosopher should concern himself entirely with experiments, laws of motions, mechanical principles, and reasonings thence deduced; but if he shall advance views on other matters, let him refer them for acceptance to some superior science. For from the known laws of nature very elegant theories and mechanical devices of practical utility follow; but from the knowledge of the Author of nature Himself by far the most excellent considerations arise, but they are metaphysical, theological, and moral.

43 So far about principles; now we must speak of the nature of motion. Motion, though it is clearly perceived by the senses, has been rendered obscure rather by the learned comments of philosophers than by its own nature. Motion never meets our senses apart from corporeal mass, space, and time. There are indeed those who desire to contemplate motion as a certain simple and abstract idea, and separated from all other things. But that very fine-drawn and subtle idea eludes the keen edge of intellect, as anyone can find for himself by meditation. Hence arise great difficulties about the nature of motion, and definitions far more obscure than the thing they are meant to illustrate. Such are those definitions of Aristotle and the school-men, who say that motion is the act 'of the movable in so far as it is movable, or the act of a being in potentiality in so far as it is in potentiality'. Such is the saying of a famous man[a] of modern times, who asserts that 'there is nothing real in motion except that momentary thing which must be constituted when a force is striving towards a change'. Again, it is agreed that the authors of these and similar definitions had it in mind to explain the abstract nature of motion, apart from every consideration of time and space; but how that abstract quintessence, so to speak, of motion, can be understood I do not see.

44 Not content with this they go further and divide and separate from one another the parts of motion itself, of which parts they try to make distinct ideas, as if of entities in fact distinct. For there are those who distinguish movement from motion, looking on the movement as an instantaneous element in the motion. Moreover, they would have velocity, conation, force, and impetus to be so many things differing in essence, each of which is presented to the intellect through its own abstract idea separated from all the rest. But we need not spend any more time on these discussions if the principles laid down above hold good.

45 Many also define motion by *passage*, forgetting indeed that passage itself cannot be understood without motion, and through motion ought to be defined. So very true is it that definitions throw light on some things, and darkness again on others. And certainly hardly anyone could by defining them make clearer or better known the things we perceive by sense. Enticed by the vain hope of

[a] [The identity of the 'famous man' is uncertain.]

doing so, philosophers have rendered easy things very difficult, and have ensnared their own minds in difficulties which for the most part they themselves produced. From this desire of defining and abstracting many very subtle questions both about motion and other things take their rise. Those useless questions have tortured the minds of men to no purpose; so that Aristotle often actually confesses that motion is 'a certain act difficult to know', and some of the ancients became such pastmasters in trifling as to deny the existence of motion altogether.

46 But one is ashamed to linger on minutiæ of this sort; let it suffice to have indicated the sources of the solutions; but this, too, I must add. The traditional mathematical doctrines of the infinite division of time and space have, from the very nature of the case, introduced paradoxes and thorny theories (as are all those that involve the infinite) into speculations about motion. All such difficulties motion shares with space and time, or rather has taken them over from that source.

47 Too much abstraction, on the one hand, or the division of things truly inseparable, and on the other hand composition or rather confusion of very different things have perplexed the nature of motion. For it has become usual to confuse motion with the efficient cause of motion. Whence it comes about that motion appears, as it were, in two forms, presenting one aspect to the senses, and keeping the other aspect covered in dark night. Thence obscurity, confusion, and various paradoxes of motion take their rise, while what belongs in truth to the cause alone is falsely attributed to the effect.

48 This is the source of the opinion that the same quantity of motion is always conserved; anyone will easily satisfy himself of its falsity unless it be understood of the force and power of the cause, whether that cause be called nature or *nous*, or whatever be the ultimate agent. Aristotle indeed (*Phys.* Bk. 8) when he asks whether motion be generated and destroyed, or is truly present in all things from eternity like life immortal, seems to have understood the vital principle rather than the external effect or change of place.

49 Hence it is that many suspect that motion is not mere passivity in bodies. But if we understand by it that which in the movement of a body is an object to the senses, no one can doubt that it is entirely passive. For what is there in the successive existence of body in different places which could relate to action, or be other than bare, lifeless effect?

50 The Peripatetics who say that motion is the one act of both the mover and the moved do not sufficiently divide cause from effect. Similarly those who imagine effort or conation in motion, or think that the same body at the same time is borne in opposite directions, seem to be the sport of the same confusion of ideas, and the same ambiguity of terms.

51 Diligent attention in grasping the concepts of others and in formulating one's own is of great service in the science of motion as in all other things; and unless there had been a failing in this respect I do not think that matter for dispute could have come from the query, Whether a body is indifferent to motion and to rest, or not. For since experience shows that it is a primary law

of nature that a body persists exactly in 'a state of motion and rest as long as nothing happens from elsewhere to change that state', and on that account it is inferred that the force of inertia is under different aspects either resistance or impetus, in this sense assuredly a body can be called indifferent in its own nature to motion or rest. Of course it is as difficult to induce rest in a moving body as motion in a resting body; but since the body conserves equally either state, why should it not be said to be indifferent to both?

52 The Peripatetics used to distinguish various kinds of motion corresponding to the variety of changes which a thing could undergo. To-day those who discuss motion understand by the term only local motion. But local motion cannot be understood without understanding the meaning of *locus*. Now *locus* is defined by moderns as 'the part of space which a body occupies', whence it is divided into relative and absolute corresponding to space. For they distinguish between absolute or true space and relative or apparent space. That is they postulate space on all sides measureless, immovable, insensible, permeating and containing all bodies, which they call absolute space. But space comprehended or defined by bodies, and therefore an object of sense, is called relative, apparent, vulgar space.

53 And so let us suppose that all bodies were destroyed and brought to nothing. What is left they call absolute space, all relation arising from the situation and distances of bodies being removed together with the bodies. Again, that space is infinite, immovable, indivisible, insensible, without relation and without distinction. That is, all its attributes are privative or negative. It seems therefore to be mere nothing. The only slight difficulty arising is that it is extended, and extension is a positive quality. But what sort of extension, I ask, is that which cannot be divided nor measured, no part of which can be perceived by sense or pictured by the imagination? For nothing enters the imagination which from the nature of the thing cannot be perceived by sense, since indeed the imagination is nothing else than the faculty which represents sensible things either actually existing or at least possible. Pure intellect, too, knows nothing of absolute space. That faculty is concerned only with spiritual and inextended things, such as our minds, their states, passions, virtues, and such like. From absolute space then let us take away now the words of the name, and nothing will remain in sense, imagination, or intellect. Nothing else then is denoted by those words than pure privation or negation, *i.e.* mere nothing.

54 It must be admitted that in this matter we are in the grip of serious prejudices, and to win free we must exert the whole force of our minds. For many, so far from regarding absolute space as nothing, regard it as the only thing (God excepted) which cannot be annihilated; and they lay down that it necessarily exists of its own nature, that it is eternal and uncreate, and is actually a participant in the divine attributes. But in very truth since it is most certain that all things which we designate by names are known by qualities or relations, at least in part (for it would be stupid, to use words to which nothing known, no notion, idea or concept, were attached), let us diligently inquire whether it is possible to form any idea of that pure, real, and absolute space continuing to exist after

the annihilation of all bodies. Such an idea, moreover, when I watch it somewhat more intently, I find to be the purest idea of nothing, if indeed it can be called an idea. This I myself have found on giving the matter my closest attention; this, I think, others will find on doing likewise.

55 We are sometimes deceived by the fact that when we imagine the removal of all other bodies, yet we suppose our own body to remain. On this supposition we imagine the movement of our limbs fully free on every side; but motion without space cannot be conceived. None the less if we consider the matter again we shall find, 1st, relative space conceived defined by the parts of our body; 2nd, a fully free power of moving our limbs obstructed by no obstacle; and besides these two things nothing. It is false to believe that some third thing really exists, *viz.* immense space which confers on us the free power of moving our body; for this purpose the absence of other bodies is sufficient. And we must admit that this absence or privation of bodies is nothing positive.[4]

56 But unless a man has examined these points with a free and keen mind, words and terms avail little. To one who meditates, however, and reflects, it will be manifest, I think, that predications about pure and absolute space can all be predicated about nothing. By this argument the human mind is easily freed from great difficulties, and at the same time from the absurdity of attributing necessary existence to any being except to the good and great God alone.

57 It would be easy to confirm our opinion by arguments drawn, as they say *a posteriori*, by proposing questions about absolute space, *e.g.* Is it substance or accidents? Is it created or uncreated? and showing the absurdities which follow from either answer. But I must be brief. I must not omit, however, to state that Democritus of old supported this opinion with his vote. Aristotle is our authority for the statement, *Phys.* Bk. 1, where he has these words, 'Democritus lays down as principles the solid and the void, of which the one, he says, is as what is, the other as what is not.' That the distinction between absolute and relative space has been used by philosophers of great name, and that on it as on a foundation many fine theorems have been built, may make us scruple to accept the argument, but those are empty scruples, as will appear from what follows.

58 From the foregoing it is clear that we ought not to define the true place of the body as the part of absolute space which the body occupies, and true or absolute motion as the change of true or absolute place; for all place is relative just as all motion is relative. But to make this appear more clearly we must point out that no motion can be understood without some determination or direction, which in turn cannot be understood unless besides the body in motion our own body also, or some other body, be understood to exist at the same time. For *up*, *down*, *left*, and *right* and all places and regions are founded in some relation, and necessarily connote and suppose a body different from the body moved. So that if we suppose the other bodies were annihilated and,

[4] See the arguments against absolute space in my book on *The Principles of Human Knowledge* in the English tongue published ten years ago.

for example, a globe were to exist alone, no motion could be conceived in it; so necessary is it that another body should be given by whose situation the motion should be understood to be determined. The truth of this opinion will be very clearly seen if we shall have carried out thoroughly the supposed annihilation of all bodies, our own and that of others, except that solitary globe.

59 Then let two globes be conceived to exist and nothing corporeal besides them. Let forces then be conceived to be applied in some way; whatever we may understand by the application of forces, a circular motion of the two globes round a common centre cannot be conceived by the imagination. Then let us suppose that the sky of the fixed stars is created; suddenly from the conception of the approach of the globes to different parts of that sky the motion will be conceived. That is to say that since motion is relative in its own nature, it could not be conceived before the correlated bodies were given. Similarly no other relation can be conceived without correlates.

60 As regards circular motion many think that, as motion truly circular increases, the body necessarily tends ever more and more away from its axis. This belief arises from the fact that circular motion can be seen taking its origin, as it were, at every moment from two directions, one along the radius and the other along the tangent, and if in this latter direction only the impetus be increased, then the body in motion will retire from the centre, and its orbit will cease to be circular. But if the forces be increased equally in both directions the motion will remain circular though accelerated—which will not argue an increase in the forces of retirement from the axis, any more than in the forces of approach to it. Therefore we must say that the water forced round in the bucket rises to the sides of the vessel, because when new forces are applied in the direction of the tangent to any particle of water, in the same instant new equal centripetal forces are not applied. From which experiment it in no way follows that absolute circular motion is necessarily recognized by the forces of retirement from the axis of motion. Again, how those terms *corporeal forces* and *conation* are to be understood is more than sufficiently shown in the foregoing discussion.

61 A curve can be considered as consisting of an infinite number of straight lines, though in fact it does not consist of them. That hypothesis is useful in geometry; and just so circular motion can be regarded as arising from an infinite number of rectilinear directions—which supposition is useful in mechanics. Not, however, on that account must it be affirmed that it is impossible that the centre of gravity of each body should exist successively in single points of the circular periphery, no account being taken of any rectilineal direction in the tangent or the radius.

62 We must not omit to point out that the motion of a stone in a sling or of water in a whirled bucket cannot be called truly circular motion as that term is conceived by those who define the true places of bodies by the parts of absolute space, since it is strangely compounded of the motions, not alone of bucket or sling, but also of the daily motion of the earth round her own axis, of her monthly motion round the common centre of gravity of earth and moon,