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5

ASYMPTOTIC ANALYSIS FOR PERIODIC STRUCTURES

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Introduction.

1. In Mechanics, Physics, Chemistry and Engineering, in the study of composite materials, macroscopic properties of crystalline or polymer structures, nuclear reactor design, etc., one is led to the study of boundary value problems in media with periodic structure.

If the period of the structure is small compared to the size of the region in which the system is to be studied, then an asymptotic analysis is called for: to obtain an asymptotic expansion of the solution in terms of a small parameter ε which is the ratio of the period of the structure to a typical length in the region. In other words, to obtain by systematic expansion procedures the passage from a microscopic description to a macroscopic description of the behavior of the system.

2. In mathematical terms, the above problems can be formulated, typically, as follows. A family of partial differential operators A^{ε}, depending on the small parameter ε, is given. The partial differential operators may be time independent or time dependent, steady or of evolution type, linear or nonlinear, etc. These operators have coefficients which are periodic (or sometimes almost periodic) functions in all or in some variables with periods proportional to ε. Since ε is assumed to be small, we have a family of operators with rapidly oscillationg coefficients.

In a domain Ω, we have a boundary value problem

(1) $$A^{\varepsilon}u_{\varepsilon} = f \text{ in } \Omega ,$$

(2) u_{ε} subject to appropriate boundary conditions ,

which we assume is well set when $\varepsilon > 0$ is fixed. The problem is now to obtain, if possible, an expansion[†] of u_ε

(3) $$u_\varepsilon = u_0 + \varepsilon u_1 + \dots ,$$

which would be asymptotic in general, or at least obtain the first term of this expansion along with a convergence theorem as $\varepsilon \to 0$.

3. The type of results that one obtains in many cases is that with a suitable definition of convergence (necessarily of a weak type as we shall see i.e., convergence of suitable averages), u_ε converges as $\varepsilon \to 0$ to u_0 where u_0 is the solution of

(4) $$\mathcal{A} u_0 = f \text{ in } \Omega ,$$

(5) u_0 subject to appropriate boundary conditions .[††]

In (4) $\mathcal{A}$ is, in general,[†††] a partial differential operator with simple[††††] coefficients; it is called the homogenized operator of the family A^ε because, in a well defined sense, we approximate u_ε by u_0 which satisfies an equation with simple coefficients. The coefficients of $\mathcal{A}$ are called, by definition, the effective coefficients or effective parameters that describe the macroscopic properties of the underlying medium.

[†] The form of the expansion may be more complicated than (3).

[††] Which, of course, depend on the conditions imposed on u_ε.

[†††] We shall see examples where the A^ε are partial differential operators but $\mathcal{A}$ is an integrodifferential operator. Also $\mathcal{A}$ itself may depend on ε, so we write sometimes $\mathcal{A}^\varepsilon$.

[††††] In many cases they are constants.

The most important aspect of the passage from (1), (2) to (4), (5) is the explicit analytical construction of $\mathcal{A}$ (i.e., its coefficients) and not merely the assertion that such an operator exists. Throughout this book we give analytical formulas for the construction of the coefficients of $\mathcal{A}$. This construction requires, typically, the solution of a boundary value problem within a single period cell. We call this the cell problem.

Thus, the solution of (1), (2) when ε is small is replaced by the solution of a cell problem and then of (4), (5). It is hardly surprising that both the cell problem and (4), (5) must be treated numerically since, except in trivial cases, exact solutions in closed form are not available. However, whereas the direct numerical solution of (1), (2), when ε is small, is an ill-conditioned and complicated computational problem, the solution of the cell problem plus (4), (5) is, usually, a standard problem in numerical analysis.

4. In order to obtain $\mathcal{A}$ from A^{ε}, in this book we use four methods which we now describe. The distinctions between these methods are not, however, sharp or clear-cut and the description that follows should be considered as a rough one.

The first method is based on the construction of asymptotic expansions using multiple scales. The use of multiple scales is well known in many specialized contexts (but may not be clearly articulated) as well as in modern perturbation theory. For many problems in ordinary differential equations, multiple scale methods give the same results as the well known method of averaging.† In the present context there are at least two natural spatial length scales.

†N. N. Bogoliubov and Yu. Mitropolsky, Asymptotic methods in non-linear mechanics, Gordon and Breach, New York, 1961.

One measuring variations within one period cell (the fast scale) and the one measuring variations within the region of interest (the slow scale).

The use of multiple space scales (along with multiple time scales) to treat systematically boundary value problems with rapidly varying periodic structure was introduced and exploited by us. Its effectiveness in this context was anticipated by J. B. Keller.[†] The method was also used by E. Larsen[††] in transport theory (we do not treat transport problems here).

The second method is based on energy estimates. Since the coefficients of A^{ε} are rapidly oscillating, derivatives of the coefficients are multiplied by powers of ε^{-1}. This makes it difficult to obtain estimates independent of ε. One must then pass to the limit in a weak sense and for this one uses integration by parts and suitable test functions. The prototype of this argument is due to L. Tartar.

Very often we use the two methods together, especially in Chapters 1 and 2. The multiple scales method is used to obtain the operator $\mathcal{A}$ and expansions under liberal regularity conditions on the data and the coefficients. It is also used to construct special test functions in order to pass to the limit by the energy methods yielding convergence results (without expansions) under minimal regularity hypotheses.

[†] Private communication; see also, S. Kogelman and J. B. Keller, SIAM J. Appl. Math., 24 (1973), pp. 352-361, for a general expansion procedure using multiple space scales.

[††] J. Math. Phys., 16 (1975), pp. 1421-1427.

The third method is based on probabilistic arguments and works whenever the problem admits a probabilistic formulation or has a probabilistic origin. Again, one can construct expansions using multiple scales when a lot of regularity is assumed. With the probabilistically natural notions of weak solution and weak convergence one can also obtain convergence results, without expansions, using test functions constructed by multiple scales under minimal regularity conditions.

The fourth method is based on the spectral decomposition of operators with periodic coefficients, the so-called expansion in Bloch waves. This method is not intended as an alternative to the above methods since its applicability is more restricted. It is, however, indispensable in the study of high frequency wave propagation in rapidly varying periodic media. By high frequency we mean here that another length scale becomes relevant in the problem, namely the typical wavelength, which is now assumed to be small and comparable to the period of the structure of the medium.

5. The material in this book is organized in three units that have been written so that they could be read independently by readers with more specialized interests.

Chapters 1 and 2 form the first unit which deals with elliptic, parabolic and hyperbolic (but not high frequency) problems with emphasis on the energy methods (methods 1 and 2 mostly). Sections 1, 2 and 3 of Chapter 1 are basic to the whole book, however, and should be at least looked at by readers more interested in Chapters 3 or 4.

Chapter 3 is the second unit which treats problems probabilistically i.e., by method 3. A separate introduction to the contents is provided at the beginning of this chapter.

Chapter 4 is the third unit which deals with high frequency problems, i.e., method 4 mostly. Here again a separate introduction to the contents is provided at the beginning of the chapter.

In Chapter 1 we consider, among other things, elliptic systems, operators with coefficients that have multiple periodic structure (with coefficients that depend on several variables and are periodic with periods $\varepsilon, \varepsilon^2, \ldots, \varepsilon^N$, in each variable, respectively), some nonlinear operators and variational inequalities. Chapter 2 follows essentially the lines of Chapter 1, first for parabolic equations and next for hyperbolic operators. In the parabolic case we study operators with period of order ε in the space variables and of order ε^k in the time variable. For second order parabolic operators one has essentially the 3 cases $k < 2$, $k = 2$, $k > 2$, at least as far as the first term in the expansion (3) is concerned. In both Chapters 1 and 2 we give examples of partial differential operators for which the corresponding homogenized operator $\mathcal{A}$ is an integrodifferential operator.

6. A number of questions which can be analyzed by methods similar to the ones in this book are not studied here. In particular, for the analysis of transport problems we refer to a paper by us[†] and the references cited therein. For problems that deal with periodic distribution of holes, we refer to D. Cioranescu[††] and to our own papers.[†††]

A systematic treatment of the numerical and computational aspects of the problems considered here would go beyond the scope of this book. Some references are cited in the bibliography of Chapter 1.

[†] J. Publ. RIMS, Kyoto Univ., Japan, 1978.

[††] Thesis, Paris, October 1977.

[†††] To appear.

7. The problems considered in this book have a rather long history. The first attempt to construct "effective parameters" for complicated media seems to go back to Poisson. We refer to I. Babuška[†] where a brief account of the historical development is given (along with several references). In Babuska's paper one sees clearly that the notion of "effective parameters" or "effective coefficients" depends very much on how one chooses to model a physical problem.

This means that a given physical problem may be modelled by imbedding it into a family of problems (parametrized by ε) of the form (1), (2) in many different ways. For example, modelling a problem by multiple periodic structure (cf. Chapter 1, Section 8 time variations proportional to ε^k with $k < 2$, $k = 2$ or $k > 2$, static or high frequency excitation, etc., constitutes a specific choice. The homogenized problems, and hence the effective parameters, are different in each case.

The modelling question is not, in this form,[††] a mathematical one and it is important to keep in mind that the definition of effective parameters is a relative one. The formulas change by changing the scaling of a problem, i.e., by adopting another family of problems (1), (2) in which to imbed a given physical problem.

[†]I. Babuška, Technical Note BN-821, July 1975, Institute of Fluid Dynamics and Appl. Math., Univ. of Maryland, College Park, Maryland 20742.

[††]The inverse problem: given a particular homogenization algorithm, find a family of problems (scaling) in some class, whose asymptotic limit is the given homogenized problem, is frequently interesting, difficult and need not have a "solution" in general.

For interesting partly heuristic argument leading to effective parameters, we refer to E. Sanchez Palencia.†

There are connections between the asymptotic problems considered here and the very general viewpoint of E. de Giorgi which is called Γ and *G convergence*. We refer to the work of E. de Giorgi and S. Spagnolo, S. Spagnolo,†† C. Sbordone,††† and to the references cited in a recent paper†††† of de Giorgi.

Some questions studied in Chapters 1 and 2 have also been studied by I. Babuska and N. S. Bakbalov (cf. the Bibliography at the end of Chapters 1 and 2).

A particular case of the probabilistic problems analyzed in Chapter 3 has been studied previously by M. I. Freidlin (cf. the bibliography of Chapter 3). The method of multiple scales that is followed frequently here was not used by Freidlin. His motivation seems to have been the generalization of the averaging method to stochastic equations with spatially rapidly oscillating periodic coefficients.

Some of the material of Chapters 1, 2 and 3 has been announced in notes at the C.R.A.S., Paris, and has been presented in various lectures by the authors, since 1975.

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†Int. J. Engng. Sci., 12 (1974), pp. 331-351.

††Boll. U.M.I., 8 (1973), pp. 391-411, and in Numerical Methods of Partial Differential Equations, III, B. Hubbard, editor, Academic Press, New York, 1976, pp. 469-498.

†††Annali Scuola Norm. Sup. Pisa, IV (1975), p. 617-638.

††††Boll. U.M.I., 5 (1977).

Some results in Chapter 4 are well known in solid-state physics (for example, the notion of effective mass). We give a systematic treatment here using the W.K.B. or geometrical optics methods† combined with multiple scale methods. A preliminary version of this chapter circulated among colleagues as a preprint during 1977.

8. A large number of questions remain open. Some of them are indicated in the text and follow the present lines of development. Of particular importance is the analysis of the behavior of solutions near boundaries and, possibly, any associated boundary layers. Relatively little seems to be known†† about this problem.

We wish to thank I. Babuska, J.B. Keller, F. Murat, E. Sanchez Palencia, and L. Tartar for several interesting discussions and useful advice while this work was being carried out.

We also would like to thank Dr. Breton, from Société Nationale des Industries Aérospatiales (SNIAS), for showing to us several very interesting examples arising in Industry.

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†A brief, self-contained introduction to these methods, along with references to the voluminous literature, is given in Chapter 4, Section 2. W.K.B. stands for Wentzel, Kramers and Brillouin who used these methods in the 1920's for the solution of some quantum mechanical problems. The methods were widely used much earlier by Liouville, Rayleigh and others but the terminology WKB persists.

††A. Bensoussan, J.L. Lions, G. Papanicolaou, Boundary Layer Analysis of the Dirichlet problem for elliptic equations with rapidly varying coefficients, Proceedings of the Kyoto Conference on Stochastic differential equations, Kyoto July 1976, to be published.

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TABLE OF CONTENTS

Chapter 1 : Elliptic Operators

Orientation.

Sections 1, 2, 3 are basic for the reading of all other parts of the book. Section 1 presents the simplest "model" problem ; the multiple scale method is applied to this problem in Section 2 and an "energy" proof of convergence is given in Section 3.

The reader mainly interested in Probalistic Methods (Chapter 3) should read Section 5 (after Sections 1, 2, 3), some parts of Chapter 2 (indicated in the Orientation of Chapter 2) and then proceed with Chapter 3. The same can be said for those mostly interested in high frequency wave propagation, before reading Chapter 4.

Section 4 gives L^p estimates, $p > 2$ (not too large) ; these estimates are not indispensable ; we use them in some parts of Section 8, but - as we show in that Section - one can avoid these estimates by making some stronger regularity assumptions on the coefficients.

Section 5 gives "correctors" which are improving the approximation. These correctors (or variant of them) are indispensable in numerical computations (not reported in this book).

Sections 6, 7, 9, 10, 11 give extensions and variants to elliptic equations of higher order or to some elliptic systems of interest in the applications.

Section 8 treats the case when there are more than 2 different scales ; this Section is technically difficult but the final result is quite simple. Some even more general situations (but for second order operators only) are considered in Chapter 3.

Sections 12, 13, 14 give variants ; Section 14 shows (among other things) that one has to be quite careful when working with problems which involve at the same time "homogenization" and "singular perturbations". Section 15 gives examples where the homogenized operator or Partial Differential Operators (local operators) is a non local (pseudo-differential) operator. Sections 16 and 17 study the homogenization of some non linear problems, in particular of some Variational Inequalities.

1. Setting of the "model" problem.

1.1 Setting of the problem (I).

Let $\mathcal{O}$ be a bounded open set of R^n; $\mathcal{O}$ is assumed to be bounded to simplify the exposition but this hypothesis is by no means indispensable; we shall return to this point on several occasions.

In $\mathcal{O}$ we are going to consider various boundary value problems associated to operators A^ε which are uniformly elliptic when $\varepsilon \to 0$ and the coefficients of A^ε are rapidly oscillating (with "period" ε). More precisely, we define

$$Y = \prod_{j=0}^{n}]0,y_j^o[\subset R^n ;$$

a function $f: R^n \to R^m$ is said to be Y-periodic if it admits period y_j^o in the direction y_j, $j = 1,\ldots,n$; we consider functions a_{ij}, $i,j = 1,\ldots,n$, such that

$$(1.1) \quad \begin{aligned} & a_{ij}(y) \in R , \quad a_{ij} \text{ is Y-periodic} , \quad a_{ij} \in L^\infty(R^n) , \\ & a_{ij}(y)\xi_i\xi_j \geq \alpha\xi_i\xi_i , \quad \alpha > 0 , \quad \text{a.e. in } y .^\dagger \end{aligned}$$

We do not assume, for the time being, that $a_{ij} = a_{ji}$.

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†We adopt here and in what follows the summation convention.

We also consider a_0 such that

(1.2)
$$a_0 \in L^\infty(R^n) \;, \quad a_0 \text{ is Y-periodic} \;,$$
$$a_0(y) \geq \alpha_0 > 0 \quad \text{a.e.}$$

To the functions a_{ij} and a_0 we associate the family of operators

(1.3)
$$A^\varepsilon = -\frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right) + a_0\left(\frac{x}{\varepsilon}\right) \;,$$

where ε is a "small" positive parameter.

Remark 1.1.

As it was said in the Introduction, operators (1.3) model the simplest possible situations of composite materials, when the period of the structure can be chosen to be ε.

Remark 1.2.

It is immediately seen from (1.1) and (1.2) that the family A^ε consists of second order elliptic operators, which are uniformly elliptic in ε.

Remark 1.3.

In many cases, it will be possible to take

(1.4)
$$a_0 = 0$$

(one could even consider negative functions a_0, provided they are not too large).

The first problem we want to study in this book is, roughly speaking, as follows:

We consider the equation

$$A^{\varepsilon}u_{\varepsilon} = f \quad \text{in} \quad \mathcal{O},$$

(1.5)

u_{ε} subject to boundary conditions on $\Gamma = \partial\mathcal{O}$ †,

and we want to study the behavior of u_{ε} as $\varepsilon \to 0$.

A typical "result" we shall obtain (in a more precise form!) is: one can construct (with constructive formulas) a second order elliptic operator $\mathcal{A}$ such that $u_{\varepsilon} \to u$ (in an appropriate topology) where u is the solution of

$$\mathcal{A}u = f \quad \text{in} \quad \mathcal{O},$$

(1.6)

u is subject to boundary conditions (which of course will depend on those boundary conditions imposed on u_{ε}) .

The operator $\mathcal{A}$ is the so-called homogenized operator of the family A^{ε}.

We now proceed with a more precise formulation of the boundary conditions in (1.5).

1.2 Setting of the problem (II): boundary conditions.

We use a variational formulation. The presentation is partially imposed by the structure of (1.3) since, even assuming the a_{ij}'s regular (which will not be the case in many results obtained below), the functions $a_{ij}(\frac{x}{\varepsilon})$ would have derivatives of order $\frac{1}{\varepsilon}$; therefore the a priori estimates on u_{ε} which are independent of ε will not be based on the regularity (if any) of the coefficients. Therefore a variational formulation (of a weak type) is indispensable here.

Sobolev spaces. (cf. S. L. Sobolev [1], J. Nečas [1], J. L. Lions and E. Magenes [1], R. Adams [1]).

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†The boundary conditions are made precise in Section 1.2 below.

We shall denote by $H^1(\mathcal{O})$ the space

(1.7) $$H^1(\mathcal{O}) = \left\{v \mid v, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n} \in L^2(\mathcal{O})\right\};$$

provided with the norm given by

$$||v||^2_{H^1(\mathcal{O})} = |v|^2 + \sum^n \left|\frac{\partial v}{\partial x_i}\right|^2,$$

(1.8) $$|v|^2 = \int v^2 dx ,^\dagger$$

$H^1(\mathcal{O})$ is a Hilbert space .

We define

(1.9) $$H^1_0(\mathcal{O}) = \text{closure of } \mathcal{C}^\infty_0(\mathcal{O}) \text{ in } H^1(\mathcal{O}),$$

(1.10) $$\mathcal{C}^\infty_0(\mathcal{O}) = C^\infty \text{ functions with compact support in } \quad .$$

One has (cf. Bibliography)

(1.11) $$H^1_0(\mathcal{O}) = \{v \mid v \in H^1(\mathcal{O}), v = 0 \text{ on } \Gamma\},$$

and it is known (Poincaré's inequality) that on $H^1_0(\mathcal{O})$, the norms $||v||_{H^1(\mathcal{O})}$ and $\left(\sum_{i=1}^n \left|\frac{\partial v}{\partial x_i}\right|^2\right)^{1/2}$ are <u>equivalent</u>.

We shall introduce

(1.12)
$$V = \text{closed subspace of } H^1(\mathcal{O}), \text{ such that}$$
$$H^1_0(\mathcal{O}) \subseteq V \subseteq H^1(\mathcal{O}) .$$

<u>Bilinear form associated to A^ε.</u>

For $u,v \in H^1(\mathcal{O})$, we define

---

†For the time being, all functions are supposed to be <u>real valued</u>.

$$a^{\varepsilon}(u,v) = \int_{\mathcal{O}} a_{ij}^{\varepsilon}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} dx + \int_{\mathcal{O}} a_0^{\varepsilon} uv dx , \tag{1.13}$$

where we have set

$$a_{ij}^{\varepsilon}(x) = a_{ij}(x/\varepsilon) , \quad a_0^{\varepsilon}(x) = a_0(x/\varepsilon) . \tag{1.14}$$

We observe that, by virtue of (1.1), (1.2)

$$a^{\varepsilon}(v,v) \geq \min(\alpha,\alpha_0) ||v||^2_{H^1(\mathcal{O})} \quad \forall v \in H^1(\mathcal{O}) . \tag{1.15}$$

Remark 1.4.

Let $\Gamma_0 \subset \Gamma$ be a subset of Γ of positive surface measure (actually one could take Γ_0 of positive capacity), and let us assume that

$$V = \{v | v \in H^1(\mathcal{O}), v = 0 \text{ on } \Gamma_0\} . \tag{1.16}$$

The precise meaning of (1.16) is clear if Γ_0 is "smooth"; if not, V consists of the closure in $H^1(\mathcal{O})$ of functions which are zero in a (variable) neighborhood of Γ_0. Then if $a_0^{\varepsilon} = 0$, i.e. if

$$a^{\varepsilon}(u,v) = \int_{\mathcal{O}} a_{ij}^{\varepsilon}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} dx , \tag{1.17}$$

one has

$$a^{\varepsilon}(v,v) \geq c||v||^2_{H^1(\mathcal{O})} , \quad c > 0 , \quad \forall v \in V , \tag{1.18}$$

where c does not depend on ε.

Boundary value problem.

The function u_{ε} is defined as the solution of[†]

[†]We have existence and uniqueness of a solution by the Lax-Milgram's Lemma, since one has (1.15) (or (1.18)).

$$(1.19)\qquad \begin{aligned} & u_\varepsilon \in V , \\ & a^\varepsilon(u_\varepsilon, v) = (f,v) , \quad \forall\, v \in V , \end{aligned}$$

where

$$(1.20)\qquad (f,v) = \int_{\mathcal{O}} fv\,dx , \qquad f \in L^2(\mathcal{O}) \ .^{\dagger}$$

Examples.

Example 1.1.

$V = H_0^1(\mathcal{O})$. Then (1.19) is the classical Dirichlet's problem.

Example 1.2.

$V = H^1(\mathcal{O})$. Then (1.20) is the Neumann's problem; assuming the boundary Γ and the coefficients a_{ij}'s smooth enough, the problem is

$$(1.21)\qquad \begin{aligned} & A^\varepsilon u_\varepsilon = f \quad \text{in } \mathcal{O} , \\ & \frac{\partial u_\varepsilon}{\partial \nu_{A^\varepsilon}} = 0 \quad \text{on } \Gamma , \end{aligned}$$

where $\dfrac{\partial v}{\partial \nu_{A^\varepsilon}} = a_{ij}^\varepsilon \dfrac{\partial v}{\partial x_i} \cos(\nu, x_j)$, ν = outer normal to Γ.

Example 1.3.

V is given by (1.16). Then (1.19) means:

$$(1.22)\qquad \begin{aligned} & A^\varepsilon u_\varepsilon = f \quad \text{in } \mathcal{O} , \\ & u_\varepsilon = 0 \quad \text{on } \Gamma_0 , \\ & \frac{\partial u_\varepsilon}{\partial \nu_{A^\varepsilon}} = 0 \quad \text{on } \Gamma_1 = \Gamma - \Gamma_0 . \end{aligned}$$

---

† We shall consider more general right hand sides later on.

Example 1.4.

Let us suppose that

(1.23) $$V = \{v | v \in H^1(\mathcal{O}), v = \text{constant on } \Gamma\}$$

(where the value of the constant depends of course on v).

Then (1.19) means

(1.24)
$$A^\varepsilon u_\varepsilon = f \quad \text{in } \mathcal{O},$$
$$u_\varepsilon = c_\varepsilon \quad \text{on } \Gamma, \qquad c_\varepsilon = \text{constant (not given)},$$
$$\int_\Gamma \frac{\partial u_\varepsilon}{\partial \nu_{A^\varepsilon}} d\Gamma = 0 .$$

The first problem we want to study is now in its precise form to derive the behavior of u_ε, solution of (1.19), as $\varepsilon \to 0$.

Before entering the study of the general case, let us consider a very simple particular case.

1.3 An example: a one-dimensional problem.

We consider the case n = 1 in the above problem, for the Dirichlet's boundary condition (to fix ideas). Therefore if $\mathcal{O} =]x_0, x_1[$, we have

(1.25) $$-\frac{d}{dx}\left(a^\varepsilon(x)\frac{du_\varepsilon}{dx}\right) = f \quad \text{in} \quad \mathcal{O},$$

(1.26) $$u_\varepsilon(x_0) = u_\varepsilon(x_1) = 0,$$

where a(y) is Y periodic (i.e. admits a period y_0) and $a(y) \geq \alpha > 0$ a.e.

The variational formulation is

(1.27)
$$\int_\mathcal{O} a^\varepsilon \frac{du_\varepsilon}{dx}\frac{dv}{dx} dx = \int_\mathcal{O} fv dx, \quad \forall v \in V = H_0^1(\mathcal{O}),$$
$$u_\varepsilon \in H_0^1(\mathcal{O}) .$$

By taking $v = u_\varepsilon$ in (1.27) one immediately sees that

(1.28) $$||u_\varepsilon||_{H^1(\mathcal{O})} \le C \; .^{\dagger}$$

Therefore one can extract a subsequence, still denoted by u_ε, such that

(1.29) $$u_\varepsilon \to u \quad \text{in } H_0^1(\mathcal{O}) \text{ weakly} \; .$$

We also notice (this is a simple exercise) that

(1.30) $$a^\varepsilon \to \mathcal{m}(a) = \frac{1}{y_0}\int_0^{y_0} a(y)dy \quad \text{in } L^\infty(\mathcal{O}) \text{ weak star} \; .^{\dagger\dagger}$$

From (1.29), (1.30), (1.25) it is tempting to believe that in the limit one has

(1.31) $$-\frac{d}{dx}\left(\mathcal{m}(a)\,\frac{du}{dx}\right) = f \; ,$$

u satisfying the boundary conditions analogous to (1.26), i.e.

(1.32) $$u(x_0) = u(x_1) = 0 \; .$$

But this is untrue (in general). The correct answer is as follows: we introduce

(1.33) $$\xi^\varepsilon = a^\varepsilon \frac{du_\varepsilon}{dx} \; ;$$

since a^ε remains in a bounded set of $L^\infty(\mathcal{O})$ and since one has (1.28), ξ^ε is bounded in $L^2(\mathcal{O})$ and by (1.25) we have

(1.34) $$-\frac{d\xi^\varepsilon}{dx} = f \; ,$$

[†] Where here and in what follows, the c's denote constants which do not depend on ε.

[††] In general, if $g_\varepsilon, g \in L^\infty(\mathcal{O})$, $g_\varepsilon \to g$ in $L^\infty(\mathcal{O})$ weak star means $\int_{\mathcal{O}} g_\varepsilon \phi dx \to \int_{\mathcal{O}} g\phi dx$, $\phi \in L^1(\mathcal{O})$.

so that ξ^ε is bounded in $H^1(\mathcal{O})$. Since the identity mapping from $H^1(\mathcal{O}) \to L^2(\mathcal{O})$ is compact,[†] it follows that one can assume that

$$(1.35) \qquad \xi^\varepsilon \to \xi \quad \text{in } L^2(\mathcal{O}) \text{ strongly },$$

so that

$$(1.36) \qquad \frac{1}{a^\varepsilon}\,\xi^\varepsilon \to \mathcal{M}(\tfrac{1}{a})\,\xi \quad \text{in } L^2(\mathcal{O}) \ \underline{\text{weakly}}$$

(Since $\frac{1}{a^\varepsilon} \to \mathcal{M}(\frac{1}{a})$ in $L^\infty(\mathcal{O})$ weak star). But $\frac{1}{a^\varepsilon}\,\xi^\varepsilon = \frac{du_\varepsilon}{dx}$ so that (1.36) and (1.29) imply

$$\frac{du}{dx} = \mathcal{M}(\tfrac{1}{a})\,\xi .$$

On the other hand (1.36) gives $-\frac{d\xi}{dx} = f$, so that

$$(1.37) \qquad -\frac{d}{dx}\left(\frac{1}{\mathcal{M}(\frac{1}{a})}\,\frac{du}{dx}\right) = f .$$

<u>The homogenized operator associated to</u> A^ε is given by

$$(1.38) \qquad -\frac{1}{\mathcal{M}(\frac{1}{a})}\,\frac{d^2}{dx^2} = \mathcal{A} .$$

Since $\mathcal{A}$ is uniquely defined, $u_\varepsilon \to u$ in $H_0^1(\mathcal{O})$ weakly, without extracting a subsequence.

<u>Remark 1.5</u>.

We notice that

$$(1.39) \qquad \mathcal{M}(a) \geq \frac{1}{\mathcal{M}(\frac{1}{a})}, \quad \text{with strict inequality in general .}$$

[†]This is true if $\mathcal{O}$ is <u>bounded</u>. But in the unbounded case the <u>local</u> compactness is sufficient (i.e. the compactness of the injection mapping $H^1(\mathcal{O}) \to L^2(\mathcal{O}')$, $\mathcal{O}'$ bounded, $\overline{\mathcal{O}'} \subset \mathcal{O}$).

Remark 1.6.

The periodicity of a(y) did not play a fundamental role in the above result. We can more generally assume that†

$$(1.40)\qquad \begin{aligned} &a^{\varepsilon} \text{ remains in a bounded set of } L^{\infty}(\mathcal{O}) , \\ &a^{\varepsilon}(x) \geq \alpha > 0 \quad \text{a.e.} \end{aligned}$$

Then $\frac{1}{a^{\varepsilon}}$ also remains in a bounded set of $L^{\infty}(\mathcal{O})$. Therefore we can extract a subsequence $\frac{1}{a^{\varepsilon'}}$, such that

$$(1.41)\qquad \frac{1}{a^{\varepsilon'}} \to \mu \quad \text{in } L^{\infty}(\mathcal{O}) \text{ weak star .}$$

Then $u_{\varepsilon'} \to u$ in $H_0^1(\mathcal{O})$ weakly, where u is the solution of

$$(1.42)\qquad -\frac{d}{dx}\left(\frac{1}{\mu(x)}\frac{du}{dx}\right) = f .$$

But this time, contrary to the preceding case, the limit u is not unique.

2. Asymptotic expansions.

2.1 Orientation.

We introduce in this section a method based on asymptotic expansions using multiple scales (i.e. "slow" and "fast" variables). As we shall see all over this book, the method we are going to develop is the most convenient and the most useful to obtain the right answers. The justification of the formulas obtained by this

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†We do not assume that $a^{\varepsilon}(x) = a(x/\varepsilon)$.

method can sometimes be made directly, but in general other tools will be needed; these tools will be introduced later on in this book.

2.2 Asymptotic expansions using multiple scales.

We introduce functions $\phi(x,y)$, $x \in \mathcal{O}$, $y \in R^n$, which are Y-periodic in y, and we associate to $\phi(x,y)$ the function $\phi(x,x/\varepsilon)$.

We shall look for $u_\varepsilon = u_\varepsilon(x)$ in the form of the asymptotic expansion

$$u_\varepsilon(x) = u_0(x,x/\varepsilon) + \varepsilon u_1(x,x/\varepsilon) + \varepsilon^2 u_2(x,x/\varepsilon) + \cdots , \tag{2.1}$$

where the functions $u_j(x,y)$ are Y-periodic in y, $\forall\, x \in \mathcal{O}$.

Remark 2.1.

It is technically complicated to keep track of boundary conditions when seeking u_ε in the form (2.1) and this is actually the source of serious technical difficulties in justifying the method. The method will nevertheless give the "right answer" because it will turn out that, in this sort of problems, the boundary conditions are somewhat irrelevant.

The idea of the method is (simply) to insert (2.1) in equation (1.5) and to identify powers of ε.

In order to present these computations in a simple form, it is useful to consider first x and y as independent variables and to replace next y by x/ε.

Applied to a function $\phi(x,x/\varepsilon)$, the operator $\frac{\partial}{\partial x_j}$ becomes $\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}$. With this in mind, one can write

$$A^\varepsilon = \varepsilon^{-2}A_1 + \varepsilon^{-1}A_2 + \varepsilon^0 A_3 , \tag{2.2}$$

where