MEASURE AND INTEGRATION THEORY ON INFINITE-DIMENSIONAL SPACES

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Xia Dao-Xing

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Measure and Integration Theory on Infinite-Dimensional Spaces

ABSTRACT HARMONIC ANALYSIS

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FOREWORD

The present book is a compendium of results which are mostly of fairly recent vintage, and the theory discussed herein is very much in a state of flux. Moreover, the original book seems to have been compiled and published rather hurriedly. Thus, there were a great number of inaccuracies in the original, ranging from typographical errors to very substantial gaps in the mathematical reasoning. I have made some effort to correct these inaccuracies; in many cases, I have altered and expanded proofs without burdening the reader with a tedious explanation of how and where the revised version deviates from the original. In many cases where doubts or difficulties still remain, I have called attention to these by footnotes. However, I cannot claim either completeness or consistency in this editorial work. Especially as regards Chapters III and IV, I feel dubious as to how much expenditure of effort would be justified in revising or developing the theory, at least until such time as more applications may be demonstrated; in this connection, the appearance of a subsequent volume, as indicated in the author's preface, would be most enlightening.

ELMER J. BRODY

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PREFACE

The study of measures and integrals on infinite-dimensional spaces arose from the theory of stochastic processes, particularly the theory of Wiener processes. In recent years, the subject has been intimately connected with research on characteristic functionals, limit theorems, sample spaces, and generalized stochastic processes. Even more noteworthy is the fact that questions of integration on infinite-dimensional spaces have, during the last ten-odd years, appeared in many scientific fields, such as quantum mechanics, quantum field theory, statistical physics, thermodynamics of irreversible processes, turbulence theory, atomic reactor computations, and coding problems. However, the application of integration on infinite-dimensional spaces to these fields has encountered many profound difficulties, and a lack of adequate techniques. Thus, it seems that further study of this new subject is amply justified.

Heretofore, there have appeared no introductory books on this topic, either in this country or abroad. As far as the author knows, there has been only a volume of lecture notes, "Integration of Functionals," written by K. O. Friedrichs, H. N. Shapiro, *et al.*, 1957, and still unpublished. Moreover, except for Wiener integrals, the mathematical theory of measures and integrals on infinite-dimensional spaces largely began to develop only after 1956. As the mathematical background involved in the literature of this theory is rather extensive, the novice is likely to find the going somewhat difficult. Therefore, the author has been so bold as to write the present book with the hope of smoothing the way for Chinese comrades undertaking research in this direction.

This volume is primarily devoted to introducing abstract harmonic analysis. It essentially consists of three parts. The first part is concerned with the representation of positive functionals and operator rings (Chapter II), which constitutes the basis of abstract harmonic analysis. Although this topic cannot be regarded as lying entirely within the domain of infinite-dimensional measure and integration theory, the two are intimately related. The second part deals with abstract harmonic analysis on pseudo-invariant measure spaces (Chapters III and IV); except for just a few theorems, the results given here were, for the most part, obtained in China. This kind of harmonic analysis may provide tools for the further investigation of measure and integration on infinite-dimensional spaces. In the third part, we discuss a mathematical problem arising in quantum field theory, i.e., the representation of commutation relations in Bose–Einstein fields (Chapter VI); here, applications of the theory developed in the first two parts are given. In addition, one chapter (Chapter V) is devoted to another important example of measure theory on infinite-dimensional spaces, i.e., Gaussian measures.

In a subsequent volume, we shall deal with the so-called continual integral problems which appear frequently in the applications of integration theory on infinite-dimensional spaces, as well as functional variational equations and various other applications.

We assume that the reader is familiar with the treatise of Halmos [1], or its equivalent, and has the basic knowledge of functional analysis which may be found in ordinary textbooks on that subject. It is also expected that the reader has some acquaintance with the basic notions of topological spaces, topological groups, and linear topological spaces; in this connection, he may consult, for example, Guan Zhao-zhi [1]. Chapter I and Appendices I, II of the present book also provide some supplementary background material.

Owing to the author's limitations, and the rather short time taken to write this book, its shortcomings are undoubtedly numerous, and errors inevitable. The reader's criticisms will be welcomed.

Part of the manuscript of this book was read by Professor Zheng Ceng-tong of Zhongshan University, who offered valuable comments. The teachers and research students of the Functional Analysis Group, Function Theory Teaching and Research Section, Fudan University Mathematics Department, also offered valuable opinions, especially Comrade Yan Shao-zong. For these contributions, I hereby express my thanks.

XIA DAO-XING

CHAPTER

I

SOME SUPPLEMENTARY BACKGROUND IN MEASURE THEORY

The measure-theoretic concepts and results used in this book may, for the most part, be found in Halmos' *Measure Theory*, and will be directly applied in the sequel without additional explanation. However, certain supplementary measure-theoretic results, not included in Halmos' book, will be introduced in the present chapter; these results will also be essential in the subsequent chapters.

At some points in this book, we shall require the discussion of measures which are not σ-finite.[1] However, non-σ-finite measures in general are not well behaved (e.g., the Radon–Nikodyn theorem is not generally valid for such measures). Therefore, we shall in §1.2 investigate *localizable* measures, which are not necessarily σ-finite, but which do retain certain desirable properties of σ-finite measures. The measures ordinar-

[1] *Translator's note:* The term σ-finite, as used by the author, means *totally* σ-finite in the sense of Halmos [1]. This distinction is an important one in certain parts of this book. For example, according to the author's terminology, a Haar measure is pseudo-σ-finite, but not necessarily σ-finite.

ily used on groups are localizable, so that localizable measures, in fact, constitute a fairly broad class. Some rather deeper properties of localizable measures will be introduced in §2.4.

In §1.3 we shall introduce the Kolmogorov theorem. This is a fundamental theorem concerning the construction of measures on infinite-dimensional spaces from given measures on finite-dimensional spaces. We shall present this theorem in a very general form, related to the notion of a projective limit of locally convex linear topological spaces; in this form, it can be used for the construction of measures on locally convex linear topological spaces, starting from given measures on Banach spaces.

In §1.4, we introduce Kakutani inner measure, which plays an important role in the study of equivalence of measures on product spaces, as well as in the study of quasi-invariant measures.

§1.1. Some Measure-Theoretic Concepts

1° Extension and Restriction of Measures

We shall introduce certain generalizations of the usual notion (see Halmos [1]) of "measurable set."

Definition 1.1.1.[2] Let $(G, \mathfrak{B})$ be a measurable space. Let $A \subset G$, and suppose that, for every $B \in \mathfrak{B}$, we have $A \cap B \in \mathfrak{B}$. We then say that A is *measurable* with respect to $(G, \mathfrak{B})$. We denote the totality of such measurable sets by $\tilde{\mathfrak{B}}$.

Clearly, $\mathfrak{B} \subset \tilde{\mathfrak{B}}$, and $\tilde{\mathfrak{B}}$ is a σ-algebra on G. If $\mathfrak{B}$ is an algebra, then $\mathfrak{B} = \tilde{\mathfrak{B}}$.

Let f be a real (complex) function on G. If, for every Borel set A of the real line (complex plane), we have $\{g \mid f(g) \in A\} \in \tilde{\mathfrak{B}}$, we say that f is a *measurable function* on $(G, \mathfrak{B})$.

Definition 1.1.2. Let $(G, \mathfrak{B}, \mu)$ be a measure space. Define a set function $\tilde{\mu}$ on $(G, \tilde{\mathfrak{B}})$, as follows. For $A \in \tilde{\mathfrak{B}}$,

$$\tilde{\mu}(A) = \sup_{B \in \mathfrak{B}} \mu(A \cap B).$$

We call $\tilde{\mu}$ the *extension* of μ.

It is easily seen that, if $A \in \mathfrak{B}$, then $\tilde{\mu}(A) = \mu(A)$. Consequently, we shall, in the sequel, denote $\tilde{\mu}$ simply by μ, without danger of confusion.

[2] *Translator's note:* It should be recalled that, in Halmos' terminology, a σ-ring $\mathfrak{B}$ need not be a σ-algebra, that is, $\mathfrak{B}$ need not contain G itself.

In what follows, any measure space $(G, \mathfrak{B}, \mu)$ will, whenever necessary, be extended to $(G, \tilde{\mathfrak{B}}, \mu)$.

Again, extend $(G, \tilde{\mathfrak{B}}, \mu)$ to a complete measure space $(G, \mathfrak{B}^*, \mu^*)$. If f is measurable with respect to $(G, \mathfrak{B}^*)$, we say that f is a ***measurable function*** on $(G, \mathfrak{B}, \mu)$.

If $B \in \tilde{\mathfrak{B}}$ and $\mu(B) = 0$, we call B a ***μ-null set***, or simply a ***null set***.

Definition 1.1.3. Let $(G, \mathfrak{B}, \mu)$ be a measure space, $A \subset G$, and let

$$\mathfrak{B}_A = \{E \cap A \mid E \in \mathfrak{B}\}.$$

We call $\mathfrak{B}_A$ the ***restriction*** of $\mathfrak{B}$ to A.

If there exists a $C \in \mathfrak{B}$ such that the inner measure

$$\mu_*(C - A) = 0, \tag{1.1.1}$$

we may define a set function μ_A on $\mathfrak{B}_A$ as follows. For $E \in \mathfrak{B}$,

$$\mu_A(A \cap E) = \mu(E \cap C). \tag{1.1.2}$$

We call μ_A the ***restriction***[3] of μ to A.

Lemma 1.1.1.[4] Let $(G, \mathfrak{B}, \mu)$ be a measure space, and let A be a subset of G satisfying condition (1.1.1) for a given $C \in \mathfrak{B}$. Then the restriction μ_A of μ to A is well defined, and $(A, \mathfrak{B}_A, \mu_A)$ is a measure space.

PROOF. We need only prove that μ_A is well defined; the rest is obvious.

Let $E, F \in \mathfrak{B}$, with $A \cap E = A \cap F$. To justify the definition of μ_A, we need only show that

$$\mu(E \cap C) = \mu(F \cap C). \tag{1.1.3}$$

We may assume that $E \subset F$, for otherwise, we could replace F by $E \cup F$. Then, from $A \cap E = A \cap F$ and $E \subset F$, it follows that

$$A \cap (F - E) = 0,$$

whence $C - A \supset (C \cap F) - (C \cap E)$. But $\mu_*(C - A) = 0$, therefore $\mu((F \cap C) - (E \cap C)) = 0$, so that (1.1.3) holds.]

[3] *Translator's note:* Notice that μ_A depends upon the choice of C.

[4] See Halmos [1].

2° The Function Space $\mathfrak{L}_k^2(\Omega)$

We shall have occasion to use certain abstract functions taking values in a Hilbert space. We first introduce the following notions.

Definition 1.1.4. Let H be a Hilbert space, $\Omega = (G, B, \mu)$ a measure space, and f an abstract function on Ω such that (i) for every $g \in G$, $f(g) \in H$, (ii) for every $u \in H$, the numerical valued function $(f(g), u)$, $g \in G$, is a measurable function on Ω, and (iii) the range of values $\{f(g) \mid g \in G\}$ is contained in a separable subspace of H. We say that such an f is *measurable*, and denote the totality of such functions by $M(H, \Omega)$.

It is easily seen that $M(H, \Omega)$ forms a linear space with respect to ordinary addition of functions and multiplication by constants.

Lemma 1.1.2. Let $\{e_\lambda, \lambda \in \Lambda\}$ be a complete orthonormal system in the Hilbert space H. Then, a necessary and sufficient condition for f to belong to $M(H, \Omega)$ is that there exist a sequence $\{\lambda_n\} \subset \Lambda$ and a sequence of measurable functions f_{λ_n} on Ω such that

$$f(g) = \sum_{n=1}^{\infty} f_{\lambda_n}(g) e_{\lambda_n}. \tag{1.1.4}$$

Proof. Assume that f satisfies the above condition. Then, the values of f are contained in the separable subspace spanned by $\{e_{\lambda_n}, n = 1, 2, \ldots\}$, and $(f(g), u) = \sum f_{\lambda_n}(g)(e_k, u)$ is measurable. Conversely, suppose that f is measurable, let M be a separable closed linear subspace containing the range of f, and let $\{\varphi_k\}$ be a complete orthonormal system in M. For each k, there is a sequence $\{\lambda_n^{(k)}\} \subset \Lambda$, such that

$$\varphi_k = \sum_n (\varphi_k, e_{\lambda_n^{(k)}}) e_{\lambda_n^{(k)}}.$$

Therefore, the range of f is contained in the separable closed linear subspace spanned by $\{e_{\lambda_n^{(k)}}, k, n = 1, 2, \ldots\}$. Since $(f, e_{\lambda_n^{(k)}})$ is a measurable function on Ω, and since

$$f(g) = \sum_{n,k=1}^{\infty} (f, e_{\lambda_n^{(k)}}) e_{\lambda_n^{(k)}},$$

the condition of the lemma is satisfied.]

Corollary 1.1.3. If $\varphi, f \in M(H, \Omega)$, then $(f(g), \varphi(g))$ is a measurable function on Ω. In particular, $\|f(g)\|^2$ is a measurable function on Ω.

PROOF. By Lemma 1.1.2, there is a sequence $\{e_{\lambda_n}\}$ such that (1.1.4) holds, therefore,

$$(f(g), \varphi(g)) = \sum (f(g), e_{\lambda_n})\overline{(\varphi(g), e_{\lambda_n})},$$

whence it follows at once that $(f(g), \varphi(g))$ is a measurable function.]

Definition 1.1.5. Let H be a Hilbert space, and let $\Omega = (G, \mathfrak{B}, \mu)$ be a measure space. Let $\mathfrak{L}^2(H, \Omega)$ be the totality of functions in $M(H, \Omega)$ which satisfy the condition

$$\int_G \|f(g)\|^2 \, d\mu(g) < \infty, \tag{1.1.5}$$

and define an inner product on $\mathfrak{L}^2(H, \Omega)$ as follows[5]:

$$(f, \varphi) = \int_G (f(g), \varphi(g)) \, d\mu(g). \tag{1.1.6}$$

We let $L_2(\Omega)$ (or $L^2(\Omega)$) denote the usual space of measurable quadratically integrable functions on Ω.

Theorem 1.1.4. Let $\{e_\lambda , \lambda \in \Lambda\}$ be a complete orthonormal system in H, and let $H_\lambda = \{f(g)\, e_\lambda \mid f \in L_2(\Omega)\}$. Then

$$\mathfrak{L}^2(H, \Omega) = \sum_{\lambda \in \Lambda} \oplus H_\lambda . \tag{1.1.7}$$

PROOF. Let $f \in \mathfrak{L}^2(H, \Omega)$. By Lemma 1.1.2, there is a sequence $\{\lambda_n\} \subset \Lambda$ such that (1.1.4) holds. Since $|f_{\lambda_k}(g)| \leqslant \|f(g)\|$, it follows that $f_{\lambda_k} \in L_2(\Omega)$, that is, $f_{\lambda_k}(g)\, e_{\lambda_k} \in H_{\lambda_k}$. Therefore, $f \in \sum_{k=1}^{\infty} \oplus H_{\lambda_k}$. This shows that f belongs to the right-hand side of (1.1.7).]

Notice that, if $f_{\lambda_k}(\cdot)\, e_{\lambda_k} \in H_{\lambda_k}$, $k = 1, 2, \ldots$, and if

$$\sum \|f_{\lambda_k}(\cdot) e_{\lambda_k}\|^2 < \infty,$$

then, forming $f \in M(H, \Omega)$ in accordance with (1.1.4), we have

$$\int_\Omega \|f(g)\|^2 \, d\mu(g) = \sum \int_\Omega |f_{\lambda_k}(g)|^2 \, d\mu(g) = \sum \|f_{\lambda_k}(\cdot) e_{\lambda_k}\|^2 < \infty.$$

Hence, $f \in \mathfrak{L}^2(H, \Omega)$. From this, we easily deduce the following result.

[5] By Corollary 1.1.3, $(f(g), \varphi(g))$, $g \in G$, is a measurable function on Ω. Moreover, by condition (1.1.5), $\int_G \|f(g)\| \, \|\varphi(g)\| \, d\mu(g) < \infty$, whence

$$\int_G |(f(g), \varphi(g))| \, d\mu(g) \leqslant \int_G \|f(g)\| \, \|\varphi(g)\| \, d\mu(g) < \infty,$$

so that (f, φ) is well defined. It is then easily verified that (f, φ) is an inner product on $\mathfrak{L}^2(H, \Omega)$.

Corollary 1.1.5. $\mathfrak{L}^2(H, \Omega)$ is a Hilbert space.[6]

Clearly, the concrete form of H has little bearing upon the properties of $\mathfrak{L}^2(H, \Omega)$; what is important is the dimension of H, that is, the cardinality of a complete orthonormal basis for H. If H is k-dimensional, we shall write H_k for H and denote $\mathfrak{L}^2(H, \Omega)$ by $\mathfrak{L}_k{}^2(\Omega)$. In particular, when $k = 1$, H_1 may be identified with the real line (or the complex plane), and $\mathfrak{L}_1{}^2(\Omega)$ is simply $L^2(\Omega)$.

As usual, $L^p(\Omega)$ (or $L_p(\Omega)$), $p \geqslant 1$, will denote the Banach space of all pth power integrable measurable real-valued (or complex-valued) functions on Ω, with the usual linear operations and the norm

$$\|f\|_p = \left(\int_G |f(x)|^p \, d\mu(x)\right)^{1/p}.$$

Also, $L_\infty(\Omega)$ (or $L^\infty(\Omega)$) will denote the Banach space consisting of all the essentially bounded measurable functions on Ω, with the usual operations and the norm

$$\|f\|_\infty = \inf_{\mu(E)=0} \sup_{x \in G-E} |f(x)|.$$

3° Determining Sets

Definition 1.1.6. Let $(G, \mathfrak{B})$ be a measurable space, where $\mathfrak{B}$ is a σ-algebra. Let $\mathfrak{D}$ be a family of measurable functions on $(G, \mathfrak{B})$, and suppose that there exists no σ-algebra $\mathfrak{B}_1 \subset \mathfrak{B}$, $\mathfrak{B}_1 \neq \mathfrak{B}$, such that $\mathfrak{D}$ constitutes a family of measurable functions on $(G, \mathfrak{B}_1)$. We then say that $\mathfrak{D}$ is a *determining set* (of functions) on $(G, \mathfrak{B})$, and that $\mathfrak{B}$ is the σ-algebra determined by $\mathfrak{D}$ on G.

It is easily seen that, if G is a set and $\mathfrak{D}$ is any family of functions on G, then there exists a unique σ-algebra $\mathfrak{B}$ determined by $\mathfrak{D}$. In fact, we need only let $\mathfrak{B}$ be the smallest σ-algebra which contains all sets of the form $f^{-1}(C)$, where $f \in \mathfrak{D}$ and C is a Borel set in the complex plane.

Definition 1.1.7. Let $\Omega = (G, \mathfrak{B}, \mu)$ be a measure space and $\mathfrak{D}$ a family of measurable functions on Ω. For any σ-finite set A of Ω, let $\mathfrak{B}_A$ denote the σ-algebra determined by $\mathfrak{D}$ on A. Suppose that, for every such A, every measurable set of Ω which is contained in A differs from some set of $\mathfrak{B}_A$ by a μ-null set. We then say that $\mathfrak{D}$ is a *determining set* (of functions) on Ω.

[6] *Translator's note:* Of course, Theorem 1.1.4 is meaningful only in the context of this corollary.

Clearly, if $\mathfrak{D}$ is a determining set on $(G, \mathfrak{B})$, then $\mathfrak{D}$ is also a determining set on $(G, \mathfrak{B}, \mu)$.

Lemma 1.1.6. Let $\mathfrak{D}$ be a family of bounded measurable real-valued functions on the measure space Ω, such that $\mathfrak{D}$, with respect to the usual operations, forms an algebra containing the unit element 1. Suppose also that $\mathfrak{D}$ is a determining set on Ω.

Choose any $\rho \in L^1(\Omega)$, $\rho \geqslant 0$, and let $L^2(\Omega, \rho)$ be the space consisting of all measurable real-valued functions f on Ω which satisfy the condition

$$\|f\| = \left(\int_G |f(g)|^2 \rho(g)\, d\mu(g)\right)^{1/2} < \infty.$$

Then $\mathfrak{D}$ is dense in $L^2(\Omega, \rho)$ with respect to the norm $\|f\|$.

PROOF. Let $\mathfrak{S}$ be the totality of sets of the form[7]

$$\bigcap_{j=1}^{n} \{x \mid f_j(x) \in (a_j, b_j]\}, \qquad f_j \in \mathfrak{D}, \tag{1.1.8}$$

and let $\mathfrak{F}$ be the collection of all finite unions of sets in $\mathfrak{S}$. Then $\mathfrak{F}$ is an algebra. In fact, it is obvious that $G \in \mathfrak{F}$, and that the union and intersection of any finite number of sets of $\mathfrak{F}$ also belongs to $\mathfrak{F}$. To show that $\mathfrak{F}$ is an algebra, it only remains to prove that the complement of any set of $\mathfrak{F}$ also belongs to $\mathfrak{F}$. It obviously suffices to prove that the complement of any set of $\mathfrak{S}$ belongs to $\mathfrak{F}$, but this fact follows at once from the formula

$$G - \bigcap_{j=1}^{n} \{x \mid f_j(x) \in (a_j, b_j]\}$$

$$= \bigcup_{j=1}^{n} \{x \mid f_j(x) \in (-\infty, a_j]\} \bigcup_{j=1}^{n} \{x \mid f_j(x) \in (b_j, \infty)\}.$$

Hence, $\mathfrak{F}$ is an algebra.

Let $\mathfrak{D}^0$ be the closure of $\mathfrak{D}$ in $L^2(\Omega, \rho)$. Then, since $\mathfrak{D}$ is linear, $\mathfrak{D}^0$ is a closed linear subspace. We now proceed to show that, for every $E \in \mathfrak{F}$, the characteristic function C_E of E belongs to $\mathfrak{D}^0$. Let $f_1, \ldots, f_n \in \mathfrak{D}$. Then, there is a positive number ξ such that, for all $g \in G$, $|f_j(g)| \leqslant \xi$, $j = 1, 2, \ldots, n$. On the interval $[-\xi, \xi]$, define the functions

$$\psi_j(x) = \begin{cases} 1, & x \in (a_j, b_j] \cap [-\xi, \xi], \\ 0, & x \in [-\xi, \xi] - (a_j, b_j]. \end{cases}$$

[7] Here, $a_j < b_j$, and, when $b_j = \infty$, then $(a_j, b_j]$ means the totality of real numbers greater than a_j.

It is easy to show that there exists a sequence of polynomials $\{p_{mj}\,;\, j = 1,..., n;\ m = 1, 2,...\}$, such that

$$\max_{|x|\leqslant\xi} |p_{mj}(x)| \leqslant 2, \tag{1.1.9}$$

and, for every j, and every $x \in [-\xi, \xi]$,

$$\lim_{m\to\infty} p_{mj}(x) = \psi_j(x). \tag{1.1.10}$$

If E is the set defined by (1.1.8), it is easily seen from (1.1.10) that

$$\lim_{m\to\infty} \prod_{j=1}^{n} p_{mj}(f_j(g)) = \prod_{j=1}^{n} \psi_j(f_j(g)) = C_E(g). \tag{1.1.11}$$

Since $\mathfrak{D}$ is an algebra containing the unit element 1, we have $\varphi_m(g) = \prod_{j=1}^{n} p_{mj}(f_j(g)) \in \mathfrak{D}$. By (1.1.9), $|\varphi_m(g)| \leqslant 2^n$. Hence, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{m\to\infty} \| \varphi_m - C_E \| = 0.$$

Thus, if $E \in \mathfrak{S}$, then $C_E \in \mathfrak{D}^0$. If $E_1, E_2 \in \mathfrak{S}$, then $E_1 \cap E_2 \in \mathfrak{S}$. Hence, by the formula

$$C_{E_1\cup E_2} = C_{E_1} + C_{E_2} - C_{E_1\cap E_2}\,,$$

and the linearity of $\mathfrak{D}^0$, it follows that $C_{E_1\cup E_2} \in \mathfrak{D}_0$. Consequently, $C_E \in \mathfrak{D}^0$ for every $E \in \mathfrak{F}$.

Suppose that $\mathfrak{D}^0 \neq L^2(\Omega, \rho)$. Then there is a nonzero vector $\varphi \in L^2(\Omega, \rho)$ such that $\varphi \perp \mathfrak{D}^0$. Hence, for every $E \in \mathfrak{F}$,

$$\int_E \varphi\rho \, d\mu = \int C_E \varphi\rho \, d\mu = 0. \tag{1.1.12}$$

By virtue of the countable additivity of the integral, it follows that (1.1.12) holds for every set E in the smallest σ-algebra $\mathfrak{F}_1$ containing $\mathfrak{F}$. Let $A = \{g \mid \rho(g) > 0\}$; then A is a σ-finite set of Ω. Since $\mathfrak{D}$ is a determining set, there exists, for every measurable set $F \subset A$, a set $E \in \mathfrak{F}_1$ such that $E \cap A$ differs from F by a μ-null set. Hence, by (1.1.12), we obtain

$$\int_F \varphi\rho \, d\mu = \int_{E\cap A} \varphi\rho \, d\mu = \int_E \varphi\rho \, d\mu = 0.$$

Therefore, $\varphi(g) = 0$ for almost all $g \in A$, that is, φ is the zero vector of $L^2(\Omega, \rho)$. This contradiction proves that $\mathfrak{D}^0 = L^2(\Omega, \rho)$.]

4° Measures on Product Spaces

Notice that, although Halmos [1] considers only products of countably many measure spaces, his method of treatment can also be used to define the product of arbitrarily many measure[8] spaces. We shall not give a detailed account of this construction here. In what follows, the product of the family of measure spaces $\{\Omega_\alpha = (G_\alpha, \mathfrak{B}_\alpha, \mu_\alpha), \alpha \in \mathfrak{A}\}$ will be denoted by $\mathsf{X}_{\alpha\in\mathfrak{A}} \Omega_\alpha$, or by $(\mathsf{X}_{\alpha\in\mathfrak{A}} G_\alpha, \mathsf{X}_{\alpha\in\mathfrak{A}} \mathfrak{B}_\alpha, \mathsf{X}_{\alpha\in\mathfrak{A}} \mu_\alpha)$. We proceed to mention a few obvious facts concerning such products.

Let $(G, \mathfrak{B}, \mu_k)$, $k = 1, 2$, and $(H, \mathfrak{F}, \nu_k)$, $k = 1, 2$, be measure spaces. Then, for the measure $\mu_1 \times \nu_1$ on $(G \times H, \mathfrak{B} \times \mathfrak{F})$ to be absolutely continuous with respect to $\mu_2 \times \nu_2$, it is necessary[9] and sufficient that $\mu_1 \ll \mu_2$, $\nu_1 \ll \nu_2$. In this case, we have[10]

$$\frac{d\mu_1 \times \nu_1(g, h)}{d\mu_2 \times \nu_2(g, h)} = \frac{d\mu_1(g)}{d\mu_2(g)} \frac{d\nu_1(h)}{d\nu_2(h)}. \tag{1.1.13}$$

Let $\{\Omega_n = (G_n, \mathfrak{B}_n, \mu_n),\ n = 1, 2, \ldots\}$ be a sequence of probability measure spaces, and let $\Omega = (G, \mathfrak{B}, \mu) = \mathsf{X}_{n=1}^{\infty} \Omega_n$. Let $\Omega^{(n)} = \mathsf{X}_{\nu=1}^{n} \Omega_\nu = (G^{(n)}, \mathfrak{B}^{(n)}, \mu^{(n)})$. For every $f \in L^2(\Omega^n)$, define a function on Ω by means of the correspondence $g \to f(g^{(n)})$, where $g = \{g_1, g_2, \ldots, g_n, \ldots\} \in G$ and $g^{(n)} = \{g_1, \ldots, g_n\} \in G^{(n)}$. This function clearly belongs to $L^2(\Omega)$. In this way, $L^2(\Omega^{(n)})$ is imbedded as a closed linear subspace of $L^2(\Omega)$. Let P_n be the operator projecting $L^2(\Omega)$ onto $L^2(\Omega^{(n)})$. We then have the following lemma.

Lemma 1.1.7. $\{P_n\}$ converges strongly to the identity operator I.

PROOF. Since we obviously have $P_1 \leqslant P_2 \leqslant \cdots \leqslant P_n \leqslant \cdots$, we need only prove that $Q = \bigcup_{n=1}^{\infty} L^2(\Omega^{(n)})$ is dense in $L^2(\Omega)$. Let $\mathfrak{D}$ denote the totality of bounded measurable real-valued functions in Q; clearly $\mathfrak{D}$ is a real algebra containing the unit element 1. Now, for any n, and any n-dimensional Borel set E, let

$$\tilde{E} = \{g \mid g = \{g_1, g_2, \ldots\} \in G, \quad \{g_1, \ldots, g_n\} \in E\}.$$

Then, the characteristic function $C_{\tilde{E}} \in L^2(\Omega^{(n)})$, hence $C_{\tilde{E}} \in \mathfrak{D}$. Moreover, the totality of such sets $\tilde{E}$ generates $\mathfrak{B}$. Consequently, $\mathfrak{D}$ is a determining set on Ω. Therefore, by Lemma 1.1.6, any real-valued function in $L^2(\Omega)$ can be approximated by elements of $\mathfrak{D}$. Since $\mathfrak{D} \subset Q$, it follows easily that, in either the real or complex case, Q is dense in $L^2(\Omega)$.]

[8] *Translator's note:* If the number of measure spaces is infinite, then, clearly, one must require that all but a finite number of them be probability spaces, that is, $\mu_\alpha(\Omega_\alpha) = 1$.

[9] *Translator's note:* If, say, μ_1 is identically zero, we can draw no conclusion regarding ν_1.

[10] *Translator's note:* Presumably, all the measures concerned are assumed to be σ-finite.

5° Direct Sums of Measures

Definition 1.1.8. Let $\Omega_\alpha = (G_\alpha, \mathfrak{B}_\alpha, \mu_\alpha)$, $\alpha \in \mathfrak{A}$ be a family of measure spaces, where $\{G_\alpha, \alpha \in \mathfrak{A}\}$ is a family of pairwise disjoint sets. Let $G = \bigcup_{\alpha\in\mathfrak{A}} G_\alpha$. Let $\mathfrak{B}$ be the totality of sets of the form

$$A = \bigcup_{\nu=1}^{\infty} A_{\alpha_\nu}, \qquad A_{\alpha_\nu} \in \mathfrak{B}_{\alpha_\nu}, \quad \{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\} \subset \mathfrak{A}. \tag{1.1.14}$$

Define the set function μ on $(G, \mathfrak{B})$ as follows: if A is of the form (1.1.14), then

$$\mu(A) = \sum_{\nu=1}^{\infty} \mu_{\alpha_\nu}(A_{\alpha_\nu}).$$

Then we say that $\Omega = (G, \mathfrak{B}, \mu)$ is the *direct sum* of the measure spaces $\{\Omega_\alpha, \alpha \in \mathfrak{A}\}$. If, moreover, $\mu_\alpha(G_\alpha) < \infty$ for all $\alpha \in \mathfrak{A}$, we say that $\{G_\alpha, \alpha \in \mathfrak{A}\}$ is a *partition* of Ω.

Clearly, the Ω described in Definition 1.1.8 is a measure space.

Lemma 1.1.8. Let $\Omega = (G, \mathfrak{B}, \mu)$ be the direct sum of the family of measure spaces $\{(G_\alpha, \mathfrak{B}_\alpha, \mu_\alpha), \alpha \in \mathfrak{A}\}$. Then $B \in \tilde{\mathfrak{B}}$ if and only if, for every $\alpha \in \mathfrak{A}$,

$$B \cap G_\alpha \in \tilde{\mathfrak{B}}_\alpha, \tag{1.1.15}$$

and, in that case,

$$\mu(B) = \sum_{\alpha\in\mathfrak{A}} \mu_\alpha(B \cap G_\alpha). \tag{1.1.16}$$

PROOF. If B satisfies condition (1.1.15), then, for any A of the form (1.1.14), we obviously have

$$A \cap B = \bigcup_{\nu=1}^{\infty} (A_{\alpha_\nu} \cap B),$$

whence, by (1.1.15), $A_{\alpha_\nu} \cap B \in \mathfrak{B}_{\alpha_\nu} \subset \mathfrak{B}$, and therefore $B \in \tilde{\mathfrak{B}}$. Conversely, if $B \in \tilde{\mathfrak{B}}$, $A_\alpha \in \mathfrak{B}_\alpha$, then

$$(B \cap G_\alpha) \cap A_\alpha = B \cap A_\alpha \in \mathfrak{B},$$

but $B \cap A_\alpha \subset G_\alpha$, hence $B \cap A_\alpha \in \mathfrak{B}_\alpha$, thus, (1.1.15) holds.

It remains to prove (1.1.16). Let $A \in \mathfrak{B}$ be of the form (1.1.14). Then

$$\mu(B \cap A) = \sum \mu_{\alpha_i}(B \cap A_{\alpha_i}) \leqslant \sum \mu_\alpha(B \cap G_\alpha),$$

whence we get

$$\mu(B) = \sup_{A\in\mathfrak{B}} \mu(B \cap A) \leqslant \sum \mu_\alpha(B \cap G_\alpha). \tag{1.1.17}$$

If the right-hand side of (1.1.17) is a finite number, then there are at most countably many indices α such that $\mu_\alpha(G_\alpha \cap B) > 0$; denoting these indices by $\alpha_1, \ldots, \alpha_n, \ldots$, we have

$$\sum_\alpha \mu_\alpha(G_\alpha \cap B) = \sum \mu_{\alpha_i}(G_{\alpha_i} \cap B). \tag{1.1.18}$$

If the right-hand side of (1.1.17) is ∞, one can also find indices $\alpha_1, \ldots, \alpha_n, \ldots \in \mathfrak{A}$ such that (1.1.18) holds. However, $\mu_{\alpha_i}(G_{\alpha_i} \cap B) = \mu(G_{\alpha_i} \cap B)$, hence

$$\mu(B) \geqslant \sum_{\alpha_i} \mu(G_{\alpha_i} \cap B) \geqslant \sum \mu_{\alpha_i}(G_{\alpha_i} \cap B). \tag{1.1.19}$$

Combining (1.1.17), (1.1.18), and (1.1.19), we obtain (1.1.16).]

Corollary 1.1.9. Under the conditions of Lemma 1.1.8, let $B \in \tilde{\mathfrak{B}}$; then B is a μ-null set if and only if, for all $\alpha \in \mathfrak{A}$, $\mu_\alpha(B \cap G_\alpha) = 0$.

Example 1.1.1. Let $\Omega = (G, \mathfrak{B}, \mu)$ be a measure space. If there is a sequence G_n, $n = 1, 2, \ldots$, of disjoint sets of $\mathfrak{B}$, such that $G = \bigcup_{n=1}^{\infty} G_n$, then $\{G_n, n = 1, 2, \ldots\}$ is a partition of Ω.

In general, let $\Omega = (G, \mathfrak{B}, \mu)$ be a measure space, and let $\{G_\alpha, \alpha \in \mathfrak{A}\} \subset \mathfrak{B}$ be a family of disjoint sets such that $G = \bigcup_{\alpha \in \mathfrak{A}} G_\alpha$. If $\mathfrak{A}$ is not countable, then $\{G_\alpha, \alpha \in \mathfrak{A}\}$ is not necessarily a partition of Ω. For example, if G is the interval $[0, 1]$, and μ is the ordinary Lebesgue measure on G, then $\{\{\alpha\}, \alpha \in [0, 1]\}$ is not a partition of $[0, 1]$.

Let $\Omega_\alpha = (G_\alpha, \mathfrak{B}_\alpha, \mu_\alpha)$, $\alpha \in \mathfrak{A}$ be a family of measure spaces, and let $\Omega = (G, \mathfrak{B}, \mu)$ be the direct sum of this family. For each $\alpha \in \mathfrak{A}$, let f_α be a given measurable function on $\Omega_\alpha = (G_\alpha, \mathfrak{B}_\alpha, \mu_\alpha)$. Define a function f on G as follows: if $g \in G_\alpha$, then

$$f(g) = f_\alpha(g).$$

For any Borel set A and any $\alpha \in \mathfrak{A}$, the set

$$\{g \mid f(g) \in A\} \cap G_\alpha = \{g \mid f_\alpha(g) \in A\}$$

is measurable, hence f is measurable. Moreover, it is easy to prove the following lemma.

Lemma 1.1.10. Let $\Omega = (G, \mathfrak{B}, \mu)$ be the direct sum of the family of measure spaces $\{\Omega_\alpha = (G_\alpha, \mathfrak{B}_\alpha, \mu_\alpha), \alpha \in \mathfrak{A}\}$. Extend each $f_\alpha \in L^2(\Omega_\alpha)$ to a function on G by defining its values on $G - G_\alpha$ to be zero. Then,

$$L^2(\Omega) = \sum_{\alpha \in \mathfrak{A}} \oplus L^2(\Omega_\alpha).$$