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Leopoldo Nachbin passed away in April 1993. As Editor of the Mathematics Studies, he will be succeeded by Saul Lubkin.
The present book was recommended for publication by Leopoldo Nachbin.

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# SOLUTION OF CONTINUOUS NONLINEAR PDEs THROUGH ORDER COMPLETION

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This book is printed on acid-free paper.

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Dedicated to
Anne-Sophie
and
Myra-Sharon

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## FOREWORD

This work inaugurates a *new* and *general* solution method for *arbitrary continuous nonlinear* PDEs. The solution method is based on Dedekind *order completion* of usual spaces of smooth functions defined on domains in Euclidean spaces. However, the nonlinear PDEs dealt with need *not* satisfy any kind of monotonicity properties. Moreover, the solution method is completely *type independent*. In other words, it does not assume anything about the nonlinear PDEs, except for the *continuity* of their left hand term, which includes the unknown function. Furthermore, the right hand term of such nonlinear PDEs can in fact be any given *discontinuous* and *measurable* function.

One of the *advantages* and *novelties* of this solution method is that the resulting generalized solutions of arbitrary continuous nonlinear PDEs can be assimilated with usual *measurable functions* defined on Euclidean domains.

By requiring minimal smoothness conditions, the method based on order completion goes quite far beyond all other linear or nonlinear methods developed earlier in the literature in order to provide generalized solutions for PDEs. In addition, owing to the rather ultimate intuitive clarity of the concept of order, this method offers a new and basic insight into the mechanisms involved both in *existence* results and the *structure* of generalized solutions.

Finally, owing to the same clarity of the concept of order, this method leads to a new *numerical solution* method for large classes of nonlinear PDEs. This new numerical method - based on a direct approximation - is presented elsewhere.

The work is divided in three parts, followed by an appendix.

Part I, in Sections 2 - 5, presents the basic idea, as well as the general existence results obtained through the order completion method. Section 2 is consecrated to certain basic and rather elementary - although less than customary - approximation properties related to the local and global solution of arbitrary continuous nonlinear PDEs. In Sections 3 and 4 the relevant spaces of generalized functions are introduced. The basic existence results are presented in Theorems 5.1 - 5.3 and Corollary 5.1 in Section 5.

Section 6 offers a first clarification of certain basic aspects of the order completion method in solving PDEs. For that purpose, a few examples of PDEs and of their generalized solutions are presented, together with comments on the nature and meaning of such solutions.

Part I ends with a rather lengthy Section 7 which is dedicated to the *structure* of the generalized solutions obtained in Section 5. In subsection 7.1, it is shown that such generalized solutions for arbitrary continuous nonlinear PDEs can be assimilated with usual *measurable functions* defined on Euclidean domains.

In Subsection 7.2 a further result on the *structure* of the generalized solutions is presented by showing that the spaces of generalized functions $\hat{\mathcal{M}}^m_T(\Omega)$ have a *flabby sheaf* structure. This is a particularly important point. Indeed, as is known, Kaneko, the space $\mathcal{D}'(\Omega)$ of the L Schwartz distributions is *not* a flabby sheaf, this being one of its significant *disadvantages*. In other words, if $\Delta$ is an open subset of $\Omega$ and T is a distribution on $\Delta$, that is, $T \in \mathcal{D}'(\Delta)$, then in general, one *cannot* find a distribution $S \in \mathcal{D}'(\Omega)$ such that T is given by S restricted to $\Delta$. Contrary to that situation, the spaces of generalized functions $\hat{\mathcal{M}}^m_T(\Omega)$ are flabby sheaves, this fact being another *advantage* of the order completion method in solving nonlinear PDEs. Finally, Subsection 7.3 offers several technically more involved examples and counterexamples connected with the relationship between measurable functions and generalized solutions.

Part II gives a number of nontrivial applications of the existence results in Section 5.

First, in Section 8, the solution of the Cauchy problem for a large class of first order nonlinear systems is presented. These systems can have any finite number of independent variables. This application shows the *versatility* as well as the *easy* and *straightforward* nature of the order completion method in solving nonlinear PDEs, including systems.

In Section 9 one takes as if a step back, in order to have a deeper insight into the basic abstract mechanism of the order completion method when applied to obtaining *existence of solutions* for rather general equations which need not necessarily be PDEs. A further such insight, albeit a more specific one, is presented in Section 10.

Section 11 returns to nonlinear systems of PDEs and extends recent results to the case of *rough initial data* given by *measures*. These systems include, among others, the well known Carleman system.

In Section 12 a connection and comparison is established between the order completion method and the classical functional analytic method based on the completion of uniform spaces. Again, several important advantages of the order completion method become apparent.

Sections 13 and 14 complete Part II by presenting several extensions of the order completion method. It is particularly important to note that Section 13 introduces a major further *departure* in the use of the order completion method for the solution of arbitrary continuous nonlinear PDEs. Indeed, unlike in the previous Sections, in Section 13 the use of *non 'pull-back'* partial orders is presented, see (13.1), in order to obtain the existence of global generalized solutions in Theorem 13.1. It should be mentioned that the use of such non 'pull-back' partial orders is essential in Section 18, see (18.19). However, an appropriate development of the method of solving arbitrary continuous nonlinear PDEs through order completion in *non 'pull-back'* partial orders will require a more extensive treatment, which will be presented in a subsequent volume.

Part III is dedicated to what amounts to three different and independent approaches to the *group invariance* of *global generalized solutions* of large classes of nonlinear PDEs. The respective approaches achieve for the first time in the literature the following two results which constitute a major aim of Lie's original 1874 project:

- *global generalized solutions* can be obtained for large classes of linear and nonlinear PDEs, and

- the *group invariance* of these global generalized solutions can be studied within *finite dimensional* manifolds.

In this way, one also obtains three different and independent possible solutions to Hilbert's Fifth Problem when this problem is considered in its natural extended sense. A detailed presentation of related issues can be found in Subsection 16.1.

Section 16 presents the group invariance in the case of global generalized solutions of arbitrary *analytic* nonlinear PDEs, obtained through the global Cauchy-Kovalevskaia theorem, Rosinger [5-7]. One major *advantage* of this approach to group invariance is the following. All *projectable* Lie groups of symmetry for classical solutions will automatically extend to symmetry groups of the global generalized solutions. Related applications are given to delta wave solutions of semilinear hyperbolic equations. However, the group invariance goes beyond the case of projectable groups, as the application to Riemann solvers of the nonlinear shock wave equation indicates.

In Section 17 the group invariance is applied in the case of global generalized solutions obtained for large classes of linear or nonlinear PDEs based on the method in Colombeau [1,2,3]. Here however, one encounters limitations owing to the *growth conditions* required by the Colombeau method. Moreover, the mentioned extension of classical symmetry groups to symmetry groups of global generalized solutions cannot be performed in the easy way encountered in Sections 16 and 18.

Finally, Section 18 presents the group invariance of global generalized solutions obtained through the order completion method. This approach benefits from all the advantages of the approach in Section 16, and in addition, it appears to be still more simple and direct.

The work ends with an Appendix in which the basic Dedekind order completion results of 1937, obtained by MacNeille, is presented without proof. The Appendix also contains a number of further general results on order completion developed by the authors, according to the necessities of the theory in Parts I and II of the work.

This work was started in August 1990, during a three month visit of M Oberguggenberger at the Department of Mathematics and Applied Mathematics of the University of Pretoria, when a first version of a good deal of Parts I and II, as well as of the Appendix was completed in collaboration with E E Rosinger. The idea of solving general, not necessarily monotonous nonlinear PDEs through the Dedekind order completion of spaces of smooth functions was suggested by E E Rosinger, who also outlined certain main avenues, as well as details of the order completion method, in particular, the non 'pull-back', approach in Section 13. Most of the specific results in Sections 2-6 of Part I, as well as Sections 8-12 and 14 of Part II were contributed by M Oberguggenberger, who introduced the 'pull-back' partial orders (4.7) which are essential in the general existence results in Section 5, as well as in their further applications in Sections 8-11 and 14. Several revisions and extensions followed. During February - April 1991, H A Biagioni visited the mentioned department and had a thorough discussion of the first version of Part I and the Appendix with E E Rosinger. A number of corrections and improvements resulted. Later in 1991, E E Rosinger visited for five months the Institut für Mathematik und Geometrie, Universität Innsbruck, where in collaboration with M Oberguggenberger, Parts I, II and the Appendix were brought near to their completion. During this period Y-G Wang contributed several useful suggestions to Part I. Further improvements were obtained during a 1992 visit at Innsbruck by E E Rosinger. Certain suggestions related to the flabbiness of several sheaves used in this work, and aimed at improving Subsection 7.2, were contributed by J W de Roewer. Part III was done during 1993 by E E Rosinger, with help from two of his doctoral students, Y E Walus and M Rudolph.

The authors should like to dedicate a most special, kind and grateful thought to the memory of Prof Leopoldo Nachbin, long time editor of the Mathematics Studies of North-Holland. Over the last decade and a half, Prof Nachbin proved to be by far the most ardent supporter and promoter of the newly emerged nonlinear theories of generalized functions. As one indication of his support and promotion of these theories, let us only mention the following. From the nine research monographs published in this field prior to the present one, five were published by Prof Nachbin during 1980 - 1990, in his mentioned North-Holland series of Mathematics Studies.

The authors are particularly grateful to Prof D Laugwitz for his appreciation of the method and results in this work.

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Last, and certainly not least, the authors wish to express their grateful appreciation to Drs Arjen Sevenster, Associate Publisher at Elsevier, whose generous help over the years has contributed to the publication of several of the research monographs on nonlinear theories of generalized functions.

The Authors

Innsbruck, Pretoria
August 1993

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'... Analysis is but Inequalities ...'

J. Dieudonné

'... Equality is also a fraud ...'

attributed to Lenin
by F. Treves

# PART I. GENERAL EXISTENCE OF SOLUTIONS THEORY

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## 1. INTRODUCTION

The basic existence and uniqueness results in Theorems 5.1 and 5.2, Corollary 5.1 and Theorem 5.3 in Section 5, obtained through Dedekind order completion, have been announced at the end of Rosinger [6], see p 370, and they come as a first step in a completely new approach which we could call the 'order first' method in the solution of general nonlinear PDEs, which need not be monotonous.

This 'order first' method will roughly mean that for proving the *existence of solutions* of large classes of nonlinear PDEs we shall *not* need functional analysis or algebra, and instead, we shall only use the Dedekind *order completion* of spaces of smooth functions defined on Euclidean spaces. Needless to say, however, that for *other* problems, such as for instance the regularity of solutions of nonlinear PDEs, we shall be ready, according to need and usefulness, to bring in any kind of mathematical methods available.

In Rosinger [1-8], a nonlinear theory of generalized functions, based on an 'algebra first' approach has been introduced and developed for the solution of rather general classes of continuous nonlinear PDEs. A particular, however quite natural and central case of that approach has been extensively pursued in Colombeau [1,2], while in Oberguggenberger [1] the general approach in Rosinger [1-8] was given additional power, by further extending the ranges of its algebraic framework. A recent account of many of the main ideas, methods and applications to linear and nonlinear PDEs of this 'algebra first' approach can be found in Oberguggenberger [2].

In short, the essence of this 'algebra first' method is to embed the L Schwartz distributions into suitable spaces of generalized functions given by quotient algebras, or sometime quotient vector spaces of the respective form

(1.1) $\mathcal{D}'(\Omega) \subset A = \mathcal{A}/\mathcal{I}$

(1.2) $\mathcal{D}'(\Omega) \subset E = \mathcal{S}/\mathcal{V}$

where one may have, for instance, that $\mathcal{A}$ is a subalgebra in $\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$, $\mathcal{I}$ is an ideal in $\mathcal{A}$, while $\mathcal{S}$ and $\mathcal{V}$ are vector subspaces in $(\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$, where $\Omega \subset \mathbb{R}^n$ is any open set.

Given a nonlinear PDE

$$(1.3) \qquad T(x,D)U(x) = f(x), \quad x \in \Omega$$

one can, under rather general conditions, extend the nonlinear partial differential operator $T(x,D)$ so that it will act between the spaces of generalized functions in (1.1) or (1.2), in any of the following ways

$$(1.4) \qquad T(x,D) : A \rightarrow A'$$

$$(1.5) \qquad T(x,D) : E \rightarrow A$$

$$(1.6) \qquad T(x,D) : E \rightarrow E'$$

Here $A'$ and $E'$ are quotient algebras, respectively vector spaces similar to those in (1.1) and (1.2). Once extensions such as in (1.4) - (1.6) have been constructed, the solution of the nonlinear PDE in (1.3) can be obtained in that framework. The important fact to note is that, in view of the embeddings (1.1), (1.2), the solutions of nonlinear PDEs in (1.3) obtained within the frameworks in (1.4) - (1.6) will contain as particular cases the known classical as well as distributional solutions. Furthermore, one obtains solutions for important classes of earlier unsolved or unsolvable linear and nonlinear PDEs. For instance, the method in Rosinger [1-8], which develops the general nonlinear theory of extensions of the type (1.4) and (1.5), yields results such as a global version of the Cauchy-Kovalevskaia theorem, as well as an algebraic characterization for the solvability of arbitrary polynomial nonlinear PDEs with continuous coefficients. The method in Colombeau [1,2] is based on extensions of the type

$$(1.7) \qquad T(x,D) : \mathcal{G}(\Omega) \rightarrow \mathcal{G}(\Omega)$$

where $\mathcal{G}(\Omega)$ is a specific instance of the quotient algebras A in (1.1) which however enjoys many optimal properties. This method yields the solution of large classes of earlier unsolved or unsolvable linear and nonlinear PDEs, see Rosinger [5, pp 145-192]. In Oberguggenberger [1] instances of the very general extensions (1.6) have been considered in the case of nonlinear PDEs. More precisely, generalized solutions for systems of semilinear wave equations with rough initial data have been obtained in conditions which contain as particular cases some of the important earlier results, such as for instance in Rauch & Reed. Further extensions of these results have recently been presented in Gramchev [1,2] and Oberguggenberger & Wang.

It is important to note that the nonlinear theory of generalized functions in Rosinger [1-8], Colombeau [1,2] and Oberguggenberger [1,2] is nontrivial, since it had to overcome the constraints imposed by the so called Schwartz impossibility result, Schwartz. Furthermore, the interest in the generalized solutions obtained for various classes of nonlinear PDEs is not limited to the sphere of exact solutions. Indeed, the method in Colombeau [1,2] has led to important results in the numerical solution of nonlinear nonconservative systems of PDEs, see Biagioni and the literature cited there.

As mentioned in the Foreword, Chapter 1 and Final Remarks in Rosinger [6], the need for an 'algebra first' approach in the study of generalized solutions of nonlinear PDEs arises from the fact that the so called Schwartz impossibility result has its roots in a rather simple and basic algebraic conflict between discontinuity, multiplication and differentiation. A further detailed study of this basic conflict can be found in Oberguggenberger [2, pp XI, 1-36]. As a consequence of that situation, as well as of the sequential approach to weak and generalized solutions originated by Mikusinski, one is led to the construction of a nonlinear theory of generalized functions along the lines in (1.1) - (1.7). A typical feature of that construction is that functional analytic methods are hardly at all used at the beginning. Instead, one deals with algebraic properties of quotient vector spaces and quotient algebras of sequences of smooth functions on domains in Euclidean spaces.

The mentioned relegation of functional analysis to a secondary role should not come as a surprise. Indeed, the classical Cauchy-Kovalevskaia theorem for instance, has been proved without functional analytic methods, yet it is by far the most general and powerful nonlinear existence, uniqueness and regularity result. In fact, nearly one century of functional analysis has not been able to improve on that theorem on its own terms of nonlinear, type independent generality. And so it comes to pass that the Cauchy-Kowalevskaia theorem remains an unsurpassed maximal point. Still, the only 'hard' mathematics used in the proof of that theorem is the rather obvious manipulation of classical, Abel type majorants for complex power series, and in particular, the ability to sum up a geometric series. By the way, recently, a certain awareness starts to emerge, Evans, about the limitation of functional analytic methods in the study of nonlinear PDEs.

Coming back to the aim of this work, namely, the start of a completely new method in solving general nonlinear PDEs through Dedekind order completion, the following should be noted. Modern mathematics is a multilayered theory, in which successive layers include and enrich by further particularization the more basic and general ones. For instance, we can note the following succession of less and less basic layers:

- set theory
- binary relations
- order
- algebra
- topology
- functional analysis
- etc.

The 'algebra first' approach initiated and developed in Rosinger [1-8], brought with it a certain 'desescalation' from the rather exclusively functional analytic methods in the modern treatment of linear and nonlinear PDEs, to the algebraic methods in rings of continuous or smooth functions.

In this work, as announced in Rosinger [6, pp 369, 370], a further 'desescalation' is initiated, this time to an 'order first' approach.