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Combinatorial Designs

A. HARTMAN



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COMBINATORIAL DESIGNS

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COMBINATORIAL DESIGNS— A TRIBUTE TO HAIM HANANI

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SPECIAL VOLUME

COMBINATORIAL DESIGNS – A TRIBUTE TO HAIM HANANI

Guest Editor: A. HARTMAN

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Prof. Haim Hanani

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COMBINATORIAL DESIGNS – A TRIBUTE TO HAIM HANANI

PREFACE

This volume is dedicated to a mathematician who laid the ground work for the modern study of combinatorial design theory. Haim Hanani pioneered the techniques for constructing designs and the theory of pairwise balanced designs, leading directly to Wilson's Existence Theorems. He also has lead the way in the study of resolvable designs, covering and packing problems, latin squares, 3-designs, and other combinatorial configurations. All this is made more remarkable by the fact that Haim's first paper in design theory (the existence theorem for Steiner quadruple systems) appeared only in 1960. His encyclopaedic papers are widely referenced, and his genius for construction is known and respected throughout the design theory community.

Haim Hanani was born in Poland in 1912; he studied mathematics in Vienna and Warsaw from 1929–34, graduating with an M.A. from the University of Warsaw. In 1935 he emigrated to Israel and was awarded the Hebrew University's first Ph.D. in Mathematics in 1938. His dissertation was on the four colour problem. While a student he joined the National Military Organization (IZL), an underground force fighting for the establishment of a Jewish state in the land of Israel. He was imprisoned by the British authorities in 1944 and exiled to Eritrea, and then to Kenya, returning to Israel only in 1949 after Israel's independence. In 1955 he was appointed to the faculty of the Technion in Haifa. During the period from 1969–73 he served as the first rector of Ben Gurion University in Beersheba, and in 1979 he was awarded an honorary doctorate for his work in founding the university. In 1980 he was appointed Professor Emeritus at the Technion. Throughout his career he has held numerous administrative posts in the Technion and in professional and government agencies. He is on the editorial board of *Discrete Mathematics*, *Journal of Combinatorial Theory* and the *European Journal of Combinatorics*.

I would like to take this opportunity to express my gratitude to Professor Hanani for his contributions to mathematics, and to wish him a long, fruitful and healthy life on his seventy-fifth birthday. This volume of research and survey papers is a fitting tribute to a founding father, from his mathematical sons and daughters.

Alan Hartman
Toronto, Ontario
July, 1988

ACKNOWLEDGEMENTS

I would like to thank all the people who assisted in the preparation of this volume, particularly the authors and referees of the papers. It is due to the extremely high standard of these people that this volume contains such a wealth of excellent papers. I would like to thank Peter Hammer and the staff at North-Holland for their support of the project.

I would also like to express my thanks to Eric Mendelsohn, and the faculty and staff at the University of Toronto where most of the editorial work was done. Special thanks are due to Karin Smith for her dedicated help in preparing the manuscript.

OBITUARY: SHMUEL SCHREIBER (1920–1988)

It is with great sadness that we note the passing of Shmuel Schreiber. Shmuel's last two papers appear in this volume, and were completed only days before his death. He was born in Romania, arriving in (then) Palestine in 1940. He received his Master's degree from the Hebrew University in 1947. His career was not in academia, so his time for research was limited; nevertheless his papers on Steiner triple systems and finite algebras remain as important works. His presence at combinatorial meetings in Israel was inspiring, his questions and problems always challenging, and his infectious enthusiasm for mathematics was remarkable. He will be greatly missed by the Israeli mathematical community and the combinatorial theorists of the world who had the privilege to know him.

Alan Hartman
Toronto, Ontario
July, 1988

LIST OF PUBLICATIONS by Haim Hanani

1. Über wesentlich unplättbar Kurven im dreidimensionalen Raume, *Fundamenta Mathematicae* 23 (1934) 135–142.
2. A contribution to the four color problem, thesis for the Ph.D. degree, submitted to the Hebrew University, Jerusalem, 1938 (in Hebrew).
3. A contribution to the four color problem, *Am. J. Math.* 64 (1942) 36–54.
4. Sur les changements des signes d'une série à termes complexes. *Comptes rendus des séances de l'Académie des Sciences* 225 (1947) 516–518 (with A. Dvoretzky).
5. On the number of straight lines determined by n points, *Riveon Lematematika* 5 (1951) 10–11 (in Hebrew, English Summary).
6. On sums of series of complex numbers, *Pacific J. Math.* 3 (1953) 695–709.
7. On the number of lines and planes determined by d points, *Scientific Publications, Technion, I.I.T. Haifa*, 6 (1954/5) 58–63.
8. Enumeration of rational numbers, *Riveon Lematematika* 9 (1955) 23–24 (in Hebrew, English Summary).
9. On the number of monotonic subsequences, *Bull. Res. Council Israel* 7F (1957/8) 11–13.
10. A generalization of the Banach and Mazur game, *Trans Amer. Math. Soc.* 94 (1960), 86–102.
11. On quadruple systems, *Canad. J. Math.* 12 (1960) 145–157.
12. A note on Steiner triple systems, *Math. Scandinavica* 8 (1960) 154–156.
13. The existence and construction of balanced incomplete block-designs, *Annals Math. Statistics* 32 (1961) 361–386.
14. Some characterizations of a class of unavoidable compact sets in the game of Banach and Mazur, *Pacific J. Math.* 11 (1961) 945–959, (with M. Reichbach).
15. On C_1 -summability of series, *Michigan Math. J.* 9 (1962) 1–14 (with P. Erdős).
16. On some tactical configurations, *Canad. J. Math.* 15 (1963) 702–722.
17. The sphere in the image, *Israel J. Math.* 1 (1963) 188–195 (with E. Netanyahu and M. Reichaw-Reichbach).
18. On a limit theorem in combinatorial analysis, *Publicationes Mathematicae* 10 (1963) 10–13 (with P. Erdős).
19. On covering of balanced incomplete block designs, *Canad. J. Math.* 16 (1964) 615–625.
20. On the lottery problem, *Publications of the Mathematical Institute of the Hungarian Academy of Sciences*, 9, series A (1964) 155–158 (with D. Orenstein and V.T. Sós).
21. A balanced incomplete block design, *Annals Math. Statistics* 36 (1965) 711.
22. Some combinatorial configurations, *Trans. N.Y. Acad. Sci., Ser. II*, 30 (1968) 421–426.
23. Eigenvalues of infinite matrices, *Colloquium Mathematicum* 19 (1968) 89–101. (With E. Netanyahu and M. Reichaw).
24. On the number of orthogonal latin squares, *J. Combin. Theory* 8 (1970) 247–271.
25. On balanced incomplete block designs with large number of elements, *Canad. J. Math.* 22 (1970) 61–65.
26. Truncated finite planes, *Proc. Symposia in Pure Mathematics*, A.M.S. 19 (1971) 115–120.
27. On balanced incomplete block designs with blocks having five elements, *J. Combin. Theory* 12 (1972) 184–201.
28. On resolvable designs, *Discrete Math.* 3 (1972) 343–357 (with D.K. Ray-Chaudhuri and R.M. Wilson).
29. On resolvable balanced incomplete block designs, *J. Combin. Theory* 17 (1974) 275–289.
30. On transversal designs, *Combinatorics, Proceedings of the NATO Advanced Study Institute (Breukelen, The Netherlands, 1974)* 43–53.
31. Balanced incomplete block designs and related designs, *Discrete Math.* 11 (1975) 255–369.
32. A combinatorial identity, *Fibonacci Quarterly* 14 (1976) 49–51.
33. Resolvable designs, *Colloquio Internazionale sulle Teorie Combinatorie, Atti dei Convegni Lincei, Roma* 17 (1976) 249–252.
34. Group divisible designs with block-size four, *Discrete Math.* 30 (1977) 1–10 (with A.E. Brouwer and A. Schrijver).
35. A class of three-designs, *J. Combin. Theory* 26 (1979) 1–19.

36. Decomposition of Hypergraphs into Octahedra, *Trans. N.Y. Acad. Sci.* 319 (1979) 260–264.
37. On three-designs of small order, *Discrete Math.* 45 (1983) 75–97 (with A. Hartman and E.S. Kramer).
38. On the original Steiner systems, *Discrete Math.* 51 (1984) 309–310.
39. BIRDs with blocksize 7, this volume.

RESOLVABLE GROUP DIVISIBLE DESIGNS WITH BLOCK SIZE 3

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Dedicated to Professor Haim Hanani on the occasion of his 75th birthday.

Let v be a non negative integer, let λ be a positive integer, and let K and M be sets of positive integers. A *group divisible design*, denoted by $\text{GD}[K, \lambda, M, v]$, is a triple (X, Γ, β) where X is a set of points, $\Gamma = \{G_1, G_2, \dots\}$ is a partition of X , and β is a class of subsets of X with the following properties. (Members of Γ are called *groups* and members of β are called *blocks*.)

1. The cardinality of X is v .
2. The cardinality of each group is a member of M .
3. The cardinality of each block is a member of K .
4. Every 2-subset $\{x, y\}$ of X such that x and y belong to distinct groups is contained in precisely λ blocks.
5. Every 2-subset $\{x, y\}$ of X such that x and y belong to the same group is contained in no block.

A group divisible design is *resolvable* if there exists a partition $\Pi = \{P_1, P_2, \dots\}$ of β such that each part P_i is itself a partition of X . In this paper we investigate the existence of resolvable group divisible designs with $K = \{3\}$, M a singleton set, and all λ . The case where $M = \{1\}$ has been solved by Ray-Chaudhuri and Wilson for $\lambda = 1$, and by Hanani for all $\lambda > 1$. The case where M is a singleton set, and $\lambda = 1$ has recently been investigated by Rees and Stinson. We give some small improvements to Rees and Stinson's results, and give new results for the cases where $\lambda > 1$. We also investigate a class of designs, introduced by Hanani, which we call frame resolvable group divisible designs and prove necessary and sufficient conditions for their existence.

1. Introduction

Let v be a non negative integer, let λ be a positive integer, and let K and M be sets of positive integers. A *group divisible design*, denoted by $\text{GD}[K, \lambda, M, v]$, is a triple (X, Γ, β) where X is a set of points, $\Gamma = \{G_1, G_2, \dots\}$ is a partition of X , and β is a class of subsets of X with the following properties. (Members of Γ are called *groups* and members of β are called *blocks*.)

1. The cardinality of X is v .
2. The cardinality of each group is a member of M .
3. The cardinality of each block is a member of K .

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4. Every 2-subset $\{x, y\}$ of X such that x and y belong to distinct groups is contained in precisely λ blocks.
5. Every 2-subset $\{x, y\}$ of X such that x and y belong to the same group is contained in no block.

When $M = \{m\}$ or $K = \{k\}$ are singleton sets we shorten the notation for $\text{GD}[K, \lambda, M, v]$ to $\text{GD}[k, \lambda, m, v]$.

A group divisible design is *resolvable* if there exists a partition $\Pi = \{P_1, P_2, \dots, P_r\}$ of β such that each part P_i is itself a partition of X . The parts P_i are called *parallel classes*, and the partition Π is called a *resolution*. The number r of parallel classes in a resolvable $\text{GD}[k, \lambda, m, v]$ is given by $r = \lambda(v - m)/(k - 1) = \lambda m(u - 1)/(k - 1)$, where u is the number of groups.

Group divisible designs are generalizations of many combinatorial design structures, we give a short list below.

A *pairwise balanced design* $\text{B}(K, \lambda, v)$ is equivalent to a $\text{GD}[K, \lambda, 1, v]$.

A *balanced incomplete block design* $\text{B}(k, \lambda, v)$ is equivalent to a $\text{GD}[k, \lambda, 1, v]$.

A *transversal design* $\text{T}(k, \lambda, m)$ is equivalent to a $\text{GD}[k, \lambda, m, km]$.

The main purpose of this paper is to investigate the existence of resolvable group divisible designs with parameters $\text{GD}[3, \lambda, m, v]$. Note that the existence of group divisible designs with block size 3 has been settled by Hanani [7] who proved the following.

Theorem 1.1. *A group divisible design $\text{GD}[3, \lambda, m, v]$ exists if and only if*

$$\begin{aligned} v &\equiv 0 \pmod{m}, v \neq 2m, \\ \lambda(v - m) &\equiv 0 \pmod{2}, \text{ and} \\ \lambda v(v - m) &\equiv 0 \pmod{6}. \end{aligned}$$

For such a design to be resolvable an obvious additional necessary condition on the parameters is that

$$v \equiv 0 \pmod{3}.$$

We shall show that in the majority of cases the above conditions are also sufficient for the existence of resolvable designs $\text{GD}[3, \lambda, m, v]$. However, we do leave some cases where the necessary conditions are satisfied but the existence of the designs is undecided.

We begin by surveying the known existence theorems for resolvable group divisible designs with block size 3. The most celebrated existence problem for resolvable designs was first posed by Kirkman [9] in 1847, and is known as Kirkman's schoolgirl problem. This was solved by Ray-Chaudhuri and Wilson [11] in 1974 when they proved the following.

Theorem 1.2. *A resolvable group divisible design $\text{GD}[3, 1, 1, v]$ exists if and only if $v \equiv 3 \pmod{6}$.*

Another well studied problem for resolvable group divisible designs is the existence of resolvable transversal designs. A resolvable transversal design $\text{T}(3, 1, m)$ or resolvable $\text{GD}[3, 1, m, 3m]$ is equivalent to a pair of mutually orthogonal Latin squares of side m , and the following existence theorem was proved by Bose, Parker and Shrikhande [2, 3] in 1960.

Theorem 1.3. *A resolvable group divisible design $\text{GD}[3, 1, m, 3m]$ exists if and only if $m \notin \{2, 6\}$.*

Further progress was made on the case $m = 1$ by Hanani [6] when he proved.

Theorem 1.4. *A resolvable group divisible design $\text{GD}[3, 2, 1, v]$ exists if and only if $v \equiv 0 \pmod{3}$, and $v \neq 6$.*

An easy consequence of Theorems 1.2 and 1.4 is:

Theorem 1.5. *A resolvable group divisible design $\text{GD}[3, \lambda, 1, v]$ exists if and only if*

$$\begin{aligned} &\lambda \equiv 1 \pmod{2}, \text{ and } v \equiv 3 \pmod{6}, \text{ or} \\ &\lambda \equiv 0 \pmod{2}, \text{ and } v \equiv 0 \pmod{3}, \text{ and } v \neq 6, \text{ or} \\ &\lambda \equiv 0 \pmod{4}, \text{ and } v = 6. \end{aligned}$$

Proof. Theorems 1.2 and 1.4 cover the cases $\lambda = 1$ and $\lambda = 2$. For $\lambda > 2$ and $v \neq 6$ the designs are constructed by taking copies of the blocks and resolution classes of the designs with $\lambda \leq 2$. For $v = 6$ and $\lambda = 4j$ take j copies of all 3-subsets of a 6-set as blocks, and the resolution classes consist of a block and its complement.

Now let us assume that there exists a resolvable $\text{GD}[3, 4j + 2, 1, 6]$. We can assume that $X = \{0, 1, 2, 3, 4, 5\}$. Every resolution class contains two blocks, and these two blocks contain either 0 or 4 pairs $\{x, y\}$ such that $x \not\equiv y \pmod{2}$ (according to whether the resolution class is $\{\{0, 2, 4\}, \{1, 3, 5\}\}$ or not). There are a total of 9 such pairs, and thus $9(4j + 2)$ is a multiple of 4, a contradiction. \square

A resolvable group divisible design $\text{GD}[3, 1, 2, v]$ with $m = 2$ and $\lambda = 1$ has been referred to in the literature as a *nearly Kirkman triple system*, and the following existence theorem is mainly due to Baker and Wilson [1] with some final small cases solved in the papers of Brouwer [4] and Rees and Stinson [10]. (Note that a resolvable $\text{GD}[3, 1, 2, 6]$ is equivalent to a pair of orthogonal Latin squares of side 2, which do not exist by Theorem 1.3.)

Theorem 1.6. *A resolvable group divisible design $GD[3, 1, 2, v]$ exists if and only if $v \equiv 0 \pmod{6}$, and $v \geq 18$.*

Rees and Stinson also proved the following theorem, which is the state of the art for resolvable group divisible designs with $k = 3$, $\lambda = 1$, and arbitrary m .

Theorem 1.7. *A resolvable group divisible design $GD[3, 1, m, v]$ exists if and only if*

$$\begin{aligned} v &\equiv 0 \pmod{m}, v \neq 2m, \\ v - m &\equiv 0 \pmod{2}, \\ v &\equiv 0 \pmod{3} \text{ and,} \\ (m, v) &\notin \{(2, 6), (2, 12), (6, 18)\} \end{aligned}$$

with the possible exceptions of

$$\begin{aligned} (m, v) &\in \{(6, 66), (18, 198)\} \\ m &\equiv 6 \text{ or } 30 \pmod{36}, \text{ and } v = 14m \\ m &\equiv 2 \text{ or } 10 \pmod{12}, \text{ and } v = 6m. \end{aligned}$$

In this paper we improve on Rees and Stinson's result by removing the first two classes of exceptions, and some of the third class. We also prove a result similar to Theorem 1.7 with $\lambda > 1$. We denote the set of primes less than or equal to p by D_p . Our main result is the following.

Theorem 1.8. *A resolvable group divisible design $GD[3, \lambda, m, um]$ exists if and only if*

$$\begin{aligned} u &\neq 2, \\ \lambda m(u - 1) &\equiv 0 \pmod{2}, \\ um &\equiv 0 \pmod{3} \text{ and,} \\ (\lambda, m, u) &\notin \{(2j + 1, 2, 3), (1, 2, 6), (1, 6, 3), (4j + 2, 1, 6) : j = 0, 1, 2, \dots\} \end{aligned}$$

with the possible exceptions of the cases where $u = 6$ and $\lambda \not\equiv 0 \pmod{4}$. Moreover, there exist resolvable $GD[3, \lambda, m, 6m]$ for all odd λ and even m such that $m/2$ is divisible by a member of D_7 ; and there exist resolvable $GD[3, \lambda, m, 6m]$ for all $\lambda \equiv 2 \pmod{4}$ and all m divisible by a member of D_{19} , except possibly $m \in \{22, 26, 34, 38\}$.

A further configuration investigated in this paper has appeared in Hanani's paper [6] in a disguised form, and explicitly in Stinson's paper [12]. We have chosen to use the terminology *frame resolvable group divisible design* as a compromise between the terms currently in use. A group divisible design

(X, Γ, β) is said to be *frame resolvable* if there exists a partition $\Pi = \{P_1, P_2, \dots, P_f\}$ of β such that each P_i is itself a partition of $X \setminus G_j$ for some $G_j \in \Gamma$. The parts P_i are called *frame parallel classes*, and the partition Π is called a *frame resolution*.

Two obvious necessary conditions for the existence of a frame resolvable $\text{GD}[k, \lambda, m, v]$ are that $v \neq km$, and $v - m \equiv 0 \pmod{k}$. The number of frame parallel classes, f , is given by

$$f = \frac{\lambda v(v - m)}{k(k - 1)} \div \frac{v - m}{k} = \frac{\lambda v}{k - 1},$$

and hence an additional necessary condition is that $\lambda v \equiv 0 \pmod{k - 1}$. Note that the number of frame parallel classes which partition $X \setminus G_i$ for some fixed group G_i is given by $f - r = \lambda m / (k - 1)$ and we shall sometimes use this fact to index the frame resolution as $\Pi = \{P_{ij} : i = 1, 2, \dots, u; j = 1, 2, \dots, \lambda m / (k - 1)\}$ where u is the number of groups and P_{ij} is a partition of $X \setminus G_i$ for all j .

In the case $k = 3$ Stinson [12] has shown that the necessary conditions stated above are also sufficient when $\lambda = 1$, and his result is stated below.

Theorem 1.9. *A frame resolvable group divisible design $\text{GD}[3, 1, m, v]$ exists if and only if*

$$\begin{aligned} v &\equiv 0 \pmod{m}, v \neq 2m, 3m \\ v - m &\equiv 0 \pmod{3}, \text{ and} \\ m &\equiv 0 \pmod{2}. \end{aligned}$$

Hanani [6] has also shown that the necessary conditions are sufficient when $\lambda = 2$ and $m = 1$. His result is:

Theorem 1.10. *A frame resolvable group divisible design $\text{GD}[3, 2, 1, v]$ exists if and only if $v \equiv 1 \pmod{3}$.*

In the same paper Hanani also constructs frame resolvable $\text{GD}[3, 2, m, v]$ designs with $m \in \{3, 12, 24\}$ and infinitely many values of v . In this paper we extend the above results to prove:

Theorem 1.11. *A frame resolvable group divisible design $\text{GD}[3, \lambda, m, v]$ exists if and only if*

$$\begin{aligned} v &\equiv 0 \pmod{m}, v \neq 2m, 3m, \\ \lambda(v - m) &\equiv 0 \pmod{2}, \\ v - m &\equiv 0 \pmod{3}, \text{ and} \\ \lambda v &\equiv 0 \pmod{2}. \end{aligned}$$

In Section 2 we describe the major constructions necessary to prove Theorems 1.8 and 1.11. In Section 3 we prove these results, and the appendix contains the constructions of resolvable and frame resolvable designs with small parameters needed in the proofs.

2. Recursive constructions

In this section we show how to construct both resolvable and frame resolvable group divisible designs using the existence of designs with smaller values of the various parameters. Throughout the sequel we shall denote the set $\{0, 1, \dots, n-1\}$ by I_n . The first lemma shows how to increase λ without altering any of the other parameters.

Lemma 2.1 (Addition Lemma). *If there exist a (frame) resolvable $\text{GD}[K, \lambda, m, v]$ and a (frame) resolvable $\text{GD}[K, \mu, m, v]$ then there exists a (frame) resolvable $\text{GD}[K, \lambda + \mu, m, v]$.*

Proof. Take the union of the two postulated designs. \square

In most cases this lemma reduces our problem to consideration of only two cases namely $\lambda = 1$ or 2 . The next theorem is multiplicative on the number of points and the index λ . In general we will be using the theorem with $k_1 = k$ thus keeping the block size constant, but we shall also have occasion to set $k_1 \neq k$.

Theorem 2.2 (Multiplication Theorem). *If there exist a (frame) resolvable $\text{GD}[k_1, \lambda, m, v]$ and a resolvable $\text{GD}[k, \mu, g, k_1g]$ then there exists a (frame) resolvable $\text{GD}[k, \lambda\mu, mg, vg]$.*

Proof. Let (X, Γ, β) be a (frame) resolvable $\text{GD}[k_1, \lambda, m, v]$ with (frame) resolution $\Pi = \{\pi_1, \pi_2, \dots\}$. We construct a (frame) resolvable $\text{GD}[k, \lambda\mu, mg, vg]$ as follows. Let $X' = X \times I_g$. Let $\Gamma' = \{G \times I_g : G \in \Gamma\}$. For each block $B \in \beta$ we construct a resolvable $\text{GD}[k, \mu, g, k_1g]$ with point set $B \times I_g$, groups $\{x\} \times I_g$ for each $x \in B$, block set $\beta(B)$, and resolution $\Pi(B) = \{P(B, j) : j = 1, 2, \dots\}$. Now let $\beta' = \bigcup_{B \in \beta} \beta(B)$, and construct (frame) parallel classes $P'(i, j) = \bigcup_{B \in \pi_i} P(B, j)$. \square

To apply this theorem we generally use Theorem 1.3 which guarantees the existence of resolvable $\text{GD}[3, 1, g, 3g]$ for all $g \neq 2, 6$. Thus our problem usually reduces to consideration of the cases where $m = 1, 2, 3$, and 6 . The next theorem shows that the set $U = \{u : \text{there exists a frame resolvable } \text{GD}[k, \lambda, m, mu]\}$ is PBD-closed.

Theorem 2.3 (PBD-closure Theorem). *If there exist a pairwise balanced design $B[K, 1, v]$ and for each $u \in K$ there exists a frame resolvable $GD[k, \lambda, m, mu]$ then there exists a frame resolvable $GD[k, \lambda, m, mv]$.*

Proof. Let (X, β) be a $B[K, 1, v]$. We construct a frame resolvable $GD[k, \lambda, m, mv]$ as follows. Let $X' = X \times I_m$. Let $\Gamma' = \{\{x\} \times I_m : x \in X\}$. For each block $B \in \beta$ of cardinality u we construct a frame resolvable $GD[k, \lambda, m, mu]$ with point set $B \times I_m$, groups $\{x\} \times I_m$ for each $x \in B$, and block set $\beta(B)$. Its frame resolution $\Pi(B) = \{P(B, x, j) : x \in B, j = 1, 2, \dots, \lambda m / (k - 1)\}$, is indexed so that $P(B, x, j)$ is a partition of $(B \setminus \{x\}) \times I_m$ for all j . Now let $\beta' = \bigcup_{B \in \beta} \beta(B)$, and construct frame parallel classes $P'(x, j) = \bigcup_{x \in B \in \beta} P(B, x, j)$, for all $x \in X$ and all $j = 1, 2, \dots, \lambda m / (k - 1)$. \square

With k, λ , and m fixed, this theorem reduces our existence problem for frame resolvable $GD[k, \lambda, m, mu]$ to finitely many values of u , using the known finite generating sets for U . An example of the kind of result we shall use is the following theorem of Drake and Larson [5].

Theorem 2.4. *For all $v \leq 4$ there exists a $B(K, 1, v)$ where $K = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 23\}$.*

The next theorem is similar to the PBD-closure theorem and it illustrates the interplay between frame resolvable and resolvable group divisible designs.

Theorem 2.5 (FR + 1-closure Theorem). *If there exist a group divisible design $GD[K, 1, M, v]$ and for each $g \in M$ there exists a resolvable $GD[k, \lambda, m, m(g + 1)]$ and for each $u \in K$ there exists a frame resolvable $GD[k, \lambda, m, mu]$ then there exists a resolvable $GD[k, \lambda, m, m(v + 1)]$.*

Proof. Let (X, Γ, β) be a $GD[K, 1, M, v]$. We construct a resolvable $GD[k, \lambda, m, m(v + 1)]$ as follows. Let $X' = (X \cup \{\infty\}) \times I_m$. Let $\Gamma' = \{\{x\} \times I_m : x \in X \cup \{\infty\}\}$. For each group $G \in \Gamma$ of cardinality g we construct a resolvable $GD[k, \lambda, m, m(g + 1)]$ with point set $(G \cup \{\infty\}) \times I_m$, groups $\{x\} \times I_m$ for each $x \in G \cup \{\infty\}$, and block set $\beta(G)$. Its resolution $\Pi(G) = \{\pi(G, x, j) : x \in G, j = 1, 2, \dots, \lambda m / (k - 1)\}$, is indexed arbitrarily by the ordered pairs (x, j) . This is possible since the number of parallel classes is $\lambda mg / (k - 1)$. For each block $B \in \beta$ of cardinality u we construct a frame resolvable $GD[k, \lambda, m, mu]$ with point set $B \times I_m$, groups $\{x\} \times I_m$ for each $x \in B$, and block set $\beta(B)$. Its frame resolution $\Pi(B) = \{P(B, x, j) : x \in B, j = 1, 2, \dots, \lambda m / (k - 1)\}$, is indexed so that $P(B, x, j)$ is a partition of $(B \setminus \{x\}) \times I_m$ for all j . Now let $\beta' = \bigcup_{G \in \Gamma} \beta(G) \cup \bigcup_{B \in \beta} \beta(B)$, and construct the following parallel classes. Let x be a member of X and let G be the unique group in Γ which contains x , now for each $j = 1, 2, \dots, \lambda m / (k - 1)$ define

$$P'(x, j) = \pi(G, x, j) \cup \bigcup_{x \in B \in \beta} P(B, x, j). \quad \square$$