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Graph Theory in Memory of G.A. Dirac

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Preface

From 2 to 7 June 1985, a meeting was held at Sandbjerg, Denmark, in memory of the graph theorist Gabriel Andrew Dirac who died the year before. Attendance was by invitation only, and 55 mathematicians from 14 countries participated in lectures and discussions on graph theory related to the work of Dirac.

This volume contains contributions in honour of the memory of Dirac from participants and others, and should not be seen only as proceedings from the meeting.

All the papers have been refereed, and we wish to thank all the referees who have devoted their time to the project. We also thank the people who have helped typing the manuscripts: Susanne Albæk, Karin B. Andersen, Martin Hare Hansen, Allan L. Jensen, Frank Jensen, Susanne Kæseler and Astrid Pedersen. Professor W. T. Trotter Jr. is thanked for his careful editing of the contributions to the problem session.

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Denmark, July 1988

Lars Døvling Andersen
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Gabriel Andrew Dirac

The editors

With the passing of Gabriel Andrew Dirac on July 20, 1984, at the age of 59, the international community of graph theorists lost one of its main figures, and Danish university life lost a most stimulating and generous colleague and teacher.

Denmark had an early tradition in graph theory through the work of Julius Petersen, but after his death in 1910 graph theory was soon forgotten in this country (however Petersen's highschool textbooks were used as late as in the early 1960'ies).

With Dirac's move to Aarhus University graph theory was again taken up at a serious level in Denmark. At Aarhus University, in 1966–67 and from 1970, Dirac attracted a large number of students. His lectures were catching. He presented graph theory as a general and important mathematical theory and did so with rigour and meticulous care. Dirac's thorough style, influenced by the book of König (of which he had a very high opinion), was a delight for his students.

Dirac was a fascinating person. He had an unconventional view of many matters, he had a penetrating and sharp mind and did not take very much for granted. This held not only for mathematics, but also in everyday political and social life. Therefore it was very enriching to know him and learn from him, even if one did not always agree fully with him.

Dirac was a truly international figure. He was born on March 13, 1925, in Budapest, moved to England in 1937 when his mother married the great physicist and Nobel prize laureate P. A. M. Dirac. After wartime service in the aircraft industry and mathematical studies at the universities of Cambridge and London (obtaining his doctorate in 1951 under Richard Rado), he held university positions in London, Toronto, Vienna, Hamburg, Ilmenau, Dublin, Swansea and finally Aarhus. He was a British subject, but also knew Hungarian, German, French and Danish cultures and languages very well.

Besides mathematics and a happy family life with his wife Rosemari Dirac and four children Meike, Barbara, Holger and Annette, Dirac's great passion was fine art. He took an interest in numerous other things and was a very knowledgeable and multi-faceted person.

Dirac is among the most quoted graph theorists. He developed methods of great originality and made many fundamental discoveries. A survey of his main contributions is contained in the obituary by C. Thomassen [*J. Graph Theory* **9** (1985), 303–318].

By arranging the meeting at Sandbjerg in South Jutland, Denmark, in the summer of 1985, we wished to honour Dirac as a pioneer of graph theory, as an inventive and deep researcher for 35 years, as the most excellent teacher we have known, and last but not least as a good friend with a never ending interest in his students and friends. Dirac continues to be a rich source of inspiration, in mathematical research and presentation, as well as in the way we think about many other aspects of life.

We wish to thank Dirac's family and all participants at Sandbjerg for making the meeting a worthy memory of G. A. Dirac and his achievements.

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Hamilton Cycles in Metacirculant Graphs with Prime Power Cardinal Blocks

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Dedicated to the memory of G. A. Dirac

It is shown that every connected metacirculant graph with prime power cardinal blocks, other than the Petersen graph, has a Hamilton cycle.

1 Introduction

Metacirculant graphs were introduced in [2] as a logical generalization of the Petersen graph for the primary reason of providing a class of vertex-transitive graphs in which there might be some new non-hamiltonian, connected vertex-transitive graphs. The reader is referred to [2] for their basic properties although their construction is now described.

Let m and n be two positive integers and α be a unit in the ring of integers modulo n. Let $B_i = \{u_{i,0}, u_{i,1}, \ldots, u_{i,n-1}\}$ for $i = 0, 1, \ldots, m-1$ be m mutually disjoint sets of vertices each of cardinal n and called the *blocks* of the graph being defined. Choose sets of integers $S_j \subseteq \{0, 1, \ldots, n-1\}$ for $j = 0, 1, \ldots, \lfloor \frac{m}{2} \rfloor$ with the property that $\alpha^m S_j = S_j$ for all $j = 0, 1, \ldots, \lfloor \frac{m}{2} \rfloor$, where by definition $\alpha^m S_j = \{\, \alpha^m b : b \in S_j \,\}$. The set S_0 has the additional property that $0 \notin S_0$ and $b \in S_0$ implies $n - b \in S_0$. If m is even, then $S_{m/2}$ has the additional property that $\alpha^{m/2} S_{m/2} = -S_{m/2}$ where the negative is taken modulo n.

We place edges in the graph as follows. The vertex $u_{0,0}$ is adjacent to $u_{j,k}$ for $k \in S_j$, $j = 0, 1, \ldots, \lfloor \frac{m}{2} \rfloor$. The remaining edges are obtained by letting the group generated by the two permutations $\varrho = (u_{0,0} u_{0,1} \cdots u_{0,n-1})$ $(u_{1,0} u_{1,1} \cdots u_{1,n-1}) \cdots (u_{m-1,0} u_{m-1,1} \cdots u_{m-1,n-1})$ and τ, where $\tau(u_{i,k}) =$

$u_{i+1,\alpha k}$, act on the edges incident with $u_{0,0}$. The resulting graph is a *metacirculant* and is designated by $G(m, n, \alpha, S_0, S_1, \ldots, S_{\lfloor m/2 \rfloor})$. The Petersen graph is $G(2, 5, 2, \{1, 4\}, \{0\})$.

It was shown in [3] that if m is odd and n is a prime, then $G(m, n, \alpha, S_0, S_1, \ldots, S_{\lfloor m/2 \rfloor})$ is hamiltonian as long as it is connected. The proof depends heavily on D. Marušič's result [8] which states that every connected Cayley graph on a group which is a semidirect product of a prime order group by an odd order abelian group is hamiltonian. It was shown in [1] that if m is even and n is a prime, then $G(m, n, \alpha, S_0, S_1, \ldots, S_{m/2})$ is hamiltonian as long as it is connected and not the Petersen graph. This time the proof depends heavily on E. Durnberger's generalization [6] of Marušič's result in which the former is able to replace odd order abelian by abelian. Recently, D. Witte and K. Keating [7] have generalized the Marušič and Durnberger results to obtain the following theorem which is crucial for the proof of the main theorem below.

Theorem 1. *If G is a connected Cayley graph on a group whose commutator subgroup is cyclic and of prime power order, then G has a Hamilton cycle.*

2 Main result

The purpose of this section is to provide a proof of the following result.

Theorem 2. *Let $G = G(m, n, \alpha, S_0, S_1, \ldots, S_{\lfloor m/2 \rfloor})$ be a metacirculant graph. If n is a prime power, G is connected, and G is not the Petersen graph, then G possesses a Hamilton cycle.*

Proof. Let G be a metacirculant graph as described in the hypotheses. The proof proceeds by a sequence of reductions.

$$\textit{One may assume } n = p^e, \ \ p \textit{ a prime, and } e \geq 2. \tag{1}$$

The result has already been proved for n a prime in [1] and [3]. Thus, assumption (1) is valid.

$$\textit{One may assume } \alpha^m \not\equiv 1 \pmod{p^e}. \tag{2}$$

It is shown in [2] that $\alpha^m \equiv 1 \pmod{p^e}$ implies that G is a Cayley graph on the group H generated by ϱ and τ which were described in the introduction. If $\alpha = 1$, then H is abelian and G has a Hamilton cycle as shown by several people independently (see, for example, [5]). If H is not abelian, then the group $\langle \varrho \rangle$ generated by ϱ contains the commutator subgroup of H and G has a Hamilton cycle by Theorem 1.

Let d be an integer, $1 \le d \le e$, and define a quotient graph G/p^d as follows. It has vertices $w_{i,j}$ for $0 \le i \le m$ and $0 \le j \le p^{e-d}-1$ where one can think of $w_{i,j}$ as corresponding to the set of vertices $\{\, u_{i,t} : t \equiv j \pmod{p^{e-d}}$ and $u_{i,t} \in G \,\}$. Then $w_{i,j}$ is adjacent to $w_{r,s}$ if and only if some vertex $u_{i,t}$ in G is adjacent to some vertex $u_{r,u}$ in G where $t \equiv j \pmod{p^{e-d}}$ and $u \equiv s \pmod{p^{e-d}}$.

> *The quotient graph G/p^d is isomorphic to the metacirculant graph $G(m, p^{e-d}, \alpha', S_0', S_1', \ldots, S_{\lfloor m/2 \rfloor}')$ where* (3)
>
> - $\alpha' \equiv \alpha \pmod{p^{e-d}}$ *and* $1 \le \alpha' \le p^{e-d}-1$, *and*
> - $S_i' = \{\, j : 0 \le j < p^{e-d}$ and $j \equiv k \pmod{p^{e-d}},\ \ k \in S_i \,\}$.

It is easy to verify that (3) holds. The quotient graph G/p^e was called G/ϱ in [1, 3].

If $S_0 = \emptyset$, then G has a Hamilton cycle. (4)

Suppose first that $\alpha^m \not\equiv 1 \pmod p$. The quotient graph G/p^e is a circulant graph with symbol $S = \{\, j : S_j \neq \emptyset$ or $S_{m-j} \neq \emptyset$ for $1 \le j \le m-1 \,\}$. It is connected because G is connected and, accordingly, Chen and Quimpo's main theorem in [5] implies that G/p^e is edge-hamiltonian for $m \ge 3$.

Let G_i denote the subgraph induced on B_i by G for $i = 0, 1, \ldots, m-1$. Since $S_0 = \emptyset$, each $G_i = \overline{K}_{p^e}$, the complement of the complete graph with p^e vertices. This implies that some S_i, $i \neq 0$, must contain an element $b \not\equiv 0 \pmod p$ because G is connected. Thus, the edge $w_0 w_i$ is in the quotient graph G/p^e (the second subscript is dropped in this case as each block of G/p^e is a singleton).

Let $w_0 w_i \ldots w_0$ be a Hamilton cycle of G/p^e containing the edge $w_0 w_i$ when $m \ge 3$. The case $m = 2$ will be dealt with shortly. Now lift this cycle of G/p^e to a path in G by starting at $u_{0,0}$ and taking the edge to $u_{i,b}$. Then from $u_{i,b}$ take an edge to a vertex $u_{j,c}$ if $w_i w_j$ is the next edge of the cycle in G/p^e. Continue in this way until returning to a vertex $u_{0,a} \in B_0$ and call the resulting path P.

If $a \not\equiv 0 \pmod p$, then the juxtaposition of the paths P, $\varrho^a(P)$, $\varrho^{2a}(P)$, $\ldots$, $\varrho^{(n-1)a}(P)$ results in a Hamilton cycle. If $a \equiv 0 \pmod p$, the preceding juxtapositions will not work. Instead, the path P must be modified to produce another path P' that returns to a vertex $u_{0,a'}$ with $a' \not\equiv 0 \pmod p$. Since $b \not\equiv 0 \pmod p$ and $\alpha^m \not\equiv 1 \pmod p$, then $\alpha^m b \not\equiv b \pmod p$ and

$\alpha^m b \in S_i$ by the property that $\alpha^m S_i = S_i$. Thus, start P' with the edge $u_{0,0}u_{i,\alpha^m b}$ rather than $u_{0,0}u_{i,b}$ and obtain all other edges of P' by taking an edge parallel to each corresponding edge of P where parallel means the image under the appropriate power of ϱ. The path P' will then terminate at $u_{0,a'}$ where $a' = a - b + \alpha^m b$. Now $a' \not\equiv 0 \pmod{p}$ when $a \equiv 0 \pmod{p}$ and P' gives rise to a Hamilton cycle in G.

In the case that $m = 2$, notice that $-\alpha b \in S_1$ because $\alpha S_1 = -S_1$ (recall that $\alpha^{m/2} S_{m/2} = -S_{m/2}$ in a metacirculant graph when m is even). Thus, the 2-path $u_{0,0}u_{1,b}u_{0,b+\alpha b}$ is in G. Now $b + \alpha b \equiv 0 \pmod{p}$ implies that $\alpha \equiv -1 \pmod{p}$ which, in turn, implies that $\alpha^2 \equiv 1 \pmod{p}$. But this contradicts the present case that $\alpha^m \not\equiv 1 \pmod{p}$ and leads to the conclusion that $b + \alpha b \not\equiv 0 \pmod{p}$. Therefore, juxtapositions of this 2-path under powers of α as above produces a Hamilton cycle in G. This completes the proof of (4) when $\alpha^m \not\equiv 1 \pmod{p}$.

From (2) and the proof just completed, it may be assumed there is an integer $d < e$ such that $\alpha^m \equiv 1 \pmod{p^d}$ but $\alpha^m \not\equiv 1 \pmod{p^{d+1}}$. The proof now proceeds by induction on the exponent e since the result is true when $n = p$. Consider the quotient metacirculant G/p^{e-d} whose vertices will be labelled $w_{i,j}$, $0 \le i < m$ and $0 \le j < p^d$, and whose twist multiplier is denoted α'. Since G is connected, G/p^{e-d} is connected and, by the induction hypothesis, contains a Hamilton cycle H'. Now H' must contain an edge $w_{i,j}w_{r,s}$ where $s - j \not\equiv 0 \pmod{p}$ and by suitably operating on H' with powers of ϱ and τ, it may be assumed without loss of generality that $w_{0,0}w_{k,b'}$ is an edge of H' with $b' \not\equiv 0 \pmod{p}$. Now $b' \in S'_k$ corresponds to some $b \in S_k$ and $b \not\equiv 0 \pmod{p}$.

Now lift H' to a path P in G that starts with the edge $u_{0,0}u_{k,b}$ and terminates with a vertex $u_{0,a}$ where $a \equiv 0 \pmod{p^d}$ because H' returns to $w_{0,0}$. Let W_0 be the set of vertices of B_0 that we collapsed to $w_{0,0}$, that is, $W_0 = \{\, u_{0,c} : c \equiv 0 \pmod{p^d} \,\}$. Now if $a \not\equiv 0 \pmod{p^{d+1}}$, then juxtapositions of P with images of P under powers of ϱ^a will produce a Hamilton cycle. If, on the other hand, $a \equiv 0 \pmod{p^{d+1}}$, then start P with $u_{0,0}u_{k,\alpha^m b}$ instead. Use parallel edges to replace the other edges of P to obtain P'. Then P' terminates at $u_{0,a'}$ where $a' = a - b + \alpha^m b$. Since $\alpha^m \equiv 1 \pmod{p^d}$, $a' \in W_0$ as required. But $a' \not\equiv 0 \pmod{p^{d+1}}$ for if $a' \equiv 0 \pmod{p^{d+1}}$ held, then $\alpha^m b - b \equiv 0 \pmod{p^{d+1}}$ would be true which would imply $\alpha^m \equiv 1 \pmod{p^{d+1}}$. Hence, P' can be used to produce a Hamilton cycle of G. This completes the proof of (4).

If $|S_0| = 1$, *then G has a Hamilton cycle.* (5)

If $|S_0| = 1$, then the prime p must be 2 and $n = 2^e$, $e > 1$. In addition,

$S_0 = \{2^{e-1}\}$ must hold so that each G_i consists of 2^{e-1} disjoint edges. Therefore, the quotient metacirculant $G/2$ has $S_0' = \emptyset$ and has a Hamilton cycle H' because of (4). Now lift H' as above to a path P in G starting at $u_{0,0}$ and terminating at either $u_{0,0}$ or $u_{0,2^{e-1}}$.

If P terminates at $u_{0,2^{e-1}}$, then juxtaposing P with $\alpha^{2^{e-1}}P$ yields a Hamilton cycle. If, on the other hand, P terminates at $u_{0,0}$, let P_s be the path $\alpha^{2^{e-1}}P$. Suppose that P begins with $u_{0,0}u_{i,b}$. Then P_s begins with $u_{0,2^{e-1}}u_{i,b+2^{e-1}}$. Now remove the two preceding edges that begin P and P_s, respectively, and insert the edges $u_{0,0}u_{0,2^{e-1}}$ and $u_{i,b}u_{i,b+2^{e-1}}$ ($\alpha^i S_0 = S_0$ as α must be odd). This amalgamates P and P_s into a single cycle which must be a Hamilton cycle since P and P_s each have length 2^{e-1}. Hence, (5) is true.

If $|S_0| \geq 3$, *then G has a Hamilton cycle.* (6)

Again recall the main result of Chen and Quimpo in [5] which states that a connected Cayley graph of degree at least three on an abelian group is Hamilton connected if it is not bipartite and Hamilton laceable if it is bipartite. First consider the case that p is an odd prime so that each component of G_i, $i = 0, 1, \ldots, m-1$, is not bipartite.

Suppose that the components of G_0, and thus each G_i as well, have cardinal p^d, $d > 1$. Then the quotient metacirculant G/p^d has $S_0' = \emptyset$ and by (4) has a Hamilton cycle H'. Each vertex of H' corresponds to a component of some G_i and each edge of H' corresponds to a set of parallel edges joining the vertices of the two components that correspond to the endvertices of the edge. By the Chen-Quimpo Theorem, the subgraph induced on each component is Hamilton connected because the components are circulant graphs. It is then obvious that H' lifts to a Hamilton cycle of G.

Now consider the case that $p = 2$. If the components of G_0 are Hamilton connected, then the preceding argument works. Suppose that the components of G_0 have cardinal 2^d, $d > 2$, and induce bipartite graphs, that is, the graphs induced by the components are Hamilton laceable. Again the quotient metacirculant $G/2^d$ has a Hamilton cycle H' by (4). Each vertex of H' corresponds to a bipartite graph induced by a component of G. If w is a vertex of H', let A_w and B_w, respectively, denote the bipartition sets of the component corresponding to w.

The cycle H' will be lifted in a different manner than either of the two earlier methods. Let $w_{0,0}$ be the vertex of H' whose corresponding component contains $u_{0,0}$ and let $u_{0,0} \in A_{w_{0,0}}$. Choose a vertex $u_{0,b} \in B_{w_{0,0}}$. There exists a Hamilton path in the component joining $u_{0,0}$ and $u_{0,b}$ by the

Chen-Quimpo Theorem. Now let w' be the next vertex of H' following $w_{0,0}$. This means that $u_{0,0}$ is adjacent to a vertex x in either $A_{w'}$ or $B_{w'}$, but whichever is the case, $u_{0,b}$ is then adjacent to a vertex y of the opposite bipartition set because of the action of ϱ^{e-d+1}. There is a Hamilton path joining x and y in the component D corresponding to w'. Join x to $u_{0,0}$, join y to $u_{0,b}$ and remove from the Hamilton path in D an edge with endvertices x' and y' distinct from x and y. This is possible because $|D| \geq 4$. Now take edges from x' and y' to vertices in opposite bipartition sets in the component corresponding to the next vertex of H'. Join them with a Hamilton path and remove an appropriate edge for extending the partial cycle in G into the next component. Continue working along H' in this way until reaching the last vertex of H' before $w_{0,0}$. Do not remove an edge from the Hamilton path in this component. The result is a Hamilton cycle in G. The proof of (6) is complete.

This leaves the case that $|S_0| = 2$. Since $\alpha^m S_0 = S_0$, $\alpha^m \equiv \pm 1 \pmod{p^e}$ and (2) precludes $\alpha^m \equiv 1 \pmod{p^e}$. It is not surprising that this remaining case requires the most intricate proof because the Petersen graph satisfies $|S_0| = 2$ and $\alpha^2 \equiv -1 \pmod 5$ where $m = 2$.

When $|S_0| = 2$, G has a Hamilton cycle. (7)

Let the components of G_0 have p^d vertices, $1 \leq d \leq e$, so that they are cycles of length p^d. The quotient metacirculant G/p^{e-d} has $S_0' = \emptyset$ and must be connected. It follows from (4) that G/p^{e-d} has a Hamilton cycle H'. Notice that each vertex of H' corresponds to a cycle of length p^d in G and an edge of H' corresponds to one or more parallel 1-factors in G joining the two p^d-cycles that correspond to the endvertices of the edge. Hence, the following observation is true.

The subgraph of G corresponding to an edge and its two endvertices in H' contains a graph isomorphic to a generalized Petersen graph. (8)

The notation $G(n,k)$ will be used for the generalized Petersen graph with vertices $u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}$ and edge set $\{\, u_iv_i, u_iu_{i+1}, v_iv_{i+1} : i = 0, 1, \ldots, n-1$ with subscripts reduced modulo $n \,\}$. Since both cycles in the generalized Petersen graph referred to in (8) have length p^d, $\gcd\{n,k\} = 1$. Recall that K. Bannai [4] has proved that $G(n,k)$, when $\gcd\{n,k\} = 1$, has a Hamilton cycle if and only if $G(n,k)$ is not isomorphic to $G(n,2)$ with $n \equiv 5 \pmod 6$. But in the latter case there is a compensating result. Namely, in [1] it was shown that there are Hamilton paths in $G(n,2)$, when $n \equiv 5 \pmod 6$, from u_0 to every v_i, $i \neq 0$. In fact, it is