THE THEORY *of* GAMBLING *and* STATISTICAL LOGIC

REVISED EDITION

Richard A. Epstein

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Gambling and Statistical Logic

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Academic Press
San Diego New York Boston
London Sydney Tokyo Toronto

This book is printed on acid-free paper. ♾

ACADEMIC PRESS, INC.
A Division of Harcourt Brace & Company
525 B Street, Suite 1900
San Diego, California 92101-4495

United Kingdom Edition published by
ACADEMIC PRESS LIMITED
24-28 Oval Road, London NWI 7DX

Library of Congress Cataloging-in-Publication Data

Epstein, Richard A.
The theory of gambling and statistical logic.

Includes bibliographies and indexes.
1. Games of chance (Mathematics) 2. Statistical decision. I. Title.
QA271.E67 1977 519 76-27440
ISBN 0-12-240761-X

PRINTED IN THE UNITED STATES OF AMERICA
95 96 97 QW 9 8 7 6 5 4 3 2 1

It is remarkable that a science which began with the consideration of games of chance should have become the most important object of human knowledge. . . . The most important questions of life are, for the most part, really only problems of probability.

PIERRE SIMON, MARQUIS DE LAPLACE

Théorie Analytique des Probabilités, 1812

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PREFACE TO REVISED EDITION

Gambling being one of the few constants of the human condition, little change in its practice or basic theory could be expected over the decade since publication of the first edition of this book. Thus, most of the subject matter has retained, and will continue to retain, its currency.

Two areas alone can claim substantial progress. First, further analysis of security prices has supplied new insights into the mountainous jumble of quotations that spew forth from the stock exchanges. R. A. Levy at the University of Chicago Center for Research in Security Prices verified deviations from the random-walk theory. The Black–Scholes option model laid a concrete foundation for an edifice of investigations into warrant-hedging and commodity option-hedging. Edward O. Thorp, following "Beat the Dealer" with "Beat the Market," led the most practical, well-armed, and successful assault on the vaults of the securities and commodities markets.

Second, surprisingly, the game of Blackjack revealed a yet finer structure to mathematicians probing its intricacies. In particular, Peter Griffin of California State University, Sacramento, devised, analyzed, and evaluated card-counting systems in a comprehensive examination of memory-based strategies. Casino managements have reacted, thus far, by reducing the number of single-deck Blackjack games and emphasizing the play of four decks.

The discipline of computerized chess was predicted to produce a program competitive with master levels of skill. Embarrassingly, the best computer program compares with a B-rated human player. The prediction for the past decade is herein repeated—with less than full confidence—for the next decade.

Special appreciation is due to Dr. Thorp and Professor Griffin for their cooperation and contributions and to Julian Braun of the IBM Corporation for additional Blackjack data.

PREFACE TO FIRST EDITION

Gambling is a near universal pastime. Perhaps more people with less knowledge succumb to its lure than to any other sedentary avocation. As Balzac averred, "the gambling passion lurks at the bottom of every heart."

In our most puritan society, gambling—like other pleasures—is either taxed, restricted to certain hours, or forbidden altogether. Yet the impulse to gamble remains an eternal aspect of the apparent irrationality of man. It finds outlets in business, war, politics, in the formal overtures of the gambling casinos, and in the less ceremonious exchanges among individuals of differing opinions.

To some, the nature of gambling appears illusive, only dimly perceivable through a curtain of numbers. In others it inspires a quasi-religious emotion: true believers are governed by such mystical forces as "luck," "fate," and "chance." To yet others, gambling is the algorithmic Circe in whose embrace lies the road map to El Dorado: the "foolproof" system. Even mathematicians have fallen prey to the clever casuistry of gambling fallacies. Special wards in lunatic asylums could well be populated with mathematicians who have attempted to predict random events from finite data samples.

It is, then, the intent of this book to dissipate the mystery, myths, and misconceptions that abound in the realm of gambling and statistical phenomena.

The mathematical theory of gambling enjoys a distinguished pedigree. For several centuries, gamblers' pastimes provided both the impetus and the only concrete basis for the development of the concepts and methods of probability theory. Today, games of chance are used to isolate in pure form the logical structures underlying real-life systems while games of skill provide testing grounds for the study of multistage decision processes in practical contexts. We can readily confirm A. M. Turing's conviction that games constitute an ideal model system leading toward the development of machine intelligence.

It is also intended that a unified and complete theory be advanced. Thus it is necessary to establish formally the fundamental principles underlying the phenomena of gambling before citing examples illustrating their behavior. A majority of the requisite mathematical exposition for this goal has been elaborated and is available in the technical literature. Where deficiencies have prevented a comprehensive treatment, we have attempted to achieve the necessary developments. Mostly, however, this book constitutes a consolidation of knowledge found only in widely scattered sources.

The broad mathematical disciplines associated with the theory of gambling are probability theory and statistics, which are usually applied to those contests involving a statistical opponent ("nature"), and game theory, which is pertinent to games among "intelligent" contestants. To comprehend the operation of these disciplines normally requires only an understanding of the elementary mathematical tools—such as basic calculus. In a few isolated instances, rigorous proofs of certain fundamental statements dictate a descent into the pit of more abstract and esoteric mathematics (set theory, for example). When such instances arise, the reader not yet versed in advanced mathematical lore may gloss over the proofs with the assurance that their validity has been solidly substantiated.

If this book is successful, the reader previously susceptible to the extensive folklore of gambling will view the subject in a more rational light; the reader previously acquainted with the essentials of gambling theory will possess a more secure footing. The profits to be reaped from this knowledge are strongly dependent upon the individual. To any modestly intelligent person it is self-evident that the interests controlling operations of the gambling casinos are not, primarily, engaged in philanthropy. Furthermore, each of the principal games of chance or skill has been thoroughly analyzed by competent statisticians. Any inherent weakness, any obvious loophole, would have been uncovered long ago; and any of the multitude of miraculous "systems" which deserved its supernal reputation would have long since pauperized every gambling establishment in existence. The systems that promise something for nothing inevitably produce, instead, nothing for something.

It will also be self-evident to the reader that the laws of chance cannot be suspended despite all earnest supplications to the whim of Tyche or genuflections before the deities of the crap table. Such noumena cast small shadows on the real axis.

In the real world there is no "easy way" to assure a financial profit at the recognized games of chance or skill (if there were, the rules of play would soon be changed). An effort to understand the mathematics pertinent to each game, however, can produce a highly gratifying result. At least, it is gratifying to rationalize that we would rather lose intelligently than win ignorantly.

As with virtually all technical material, this book was not written in a vacuum. Much appreciation is due those who contributed both sympathetic ears and incisive criticism throughout that *annus mirabilis* required to complete the manuscript. In particular, we should like to award most honorable mention to the following individuals. Paul M. Cohen, of Litton Systems, is responsible for several shafts of light into the murky statistical depths. Dr. Douglas Anderson, of Purdue University, affixed his imprimatur of rigor to the corpus mathematica. Julian Braun, then of the Chicago IBM Datacenter, contributed gratuitously the programming effort underlying the Blackjack data presented in Chapter 7. And in a feat rivaling the decoding of the Knossos Linear B tablets, Mrs. Barbara Strouse, of Booz Allen Applied Research Inc., transcribed the original arcane chirographics into a legible text.

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CHAPTER ONE

KUBEIAGENESIS

Shortly after *pithecanthropus erectus* gained the ascendency, he turned his attention to the higher-order abstractions. He invented a concept that has since been variously viewed as a vice, a crime, a business, a pleasure, a type of magic, a disease, a folly, a weakness, a form of sexual substitution, an expression of the human instinct. He invented gambling.

Archeologists rooting in prehistoric sites have uncovered large numbers of roughly dice-shaped bones, called astragalia, that were apparently used in games some thousands of years ago. Whether our Stone Age predecessors cast these objects for prophecy or amusement or simply to win their neighbors' stone axes, they began a custom that has survived both evolution and revolution.

Although virtually every primitive culture engaged in some form of dice play, many centuries elapsed before thought was directed to the "fairness" of throwing dice or to the equal probability with which each face falls or should fall. The link between mathematics and gambling remained long unsuspected.

Most early civilizations were entrapped by the deep-rooted emotional appeal of absolute truth; they demanded Olympian certitude and could neither conceive nor accept the inductive reliability sought by modern physics. "Arguments from probabilities are impostors" was the doctrine expressed in the dialogue *Phaedo*. Carneades, in the Second Century B.C., was the first to shift from the traditional Greek rationalist position by developing an embryonic probability theory which distinguished three types of probability, or degrees of certainty. However, this considerable accomplishment (against the native grain) advanced the position of empiricist philosophy more than the understanding of events of chance.

Throughout the entire history of man preceding the Renaissance, all efforts toward explaining the phenomena of chance were characterized by comprehensive ignorance of the nature of probability. Yet gambling flourished in various forms almost continuously from the time Paleolithic man cast his polished knucklebones and painted pebbles. Lack of knowledge has rarely inhibited anyone from taking a chance.

In the sixteenth century came the first reasoned considerations relating games of chance to a rudimentary theory of probability. Gerolamo Cardano (1501–1576), physician, philosopher, scientist, astrologer, religionist, gambler, murderer, was responsible for the initial attempt to organize the concept of chance events into a cohesive discipline. In *Liber de Ludo Aleae, the Book on Games of Chance* (published posthumously in 1663) he expressed a rough concept of mathematical expectation, derived power laws for the repetition of events, and conceived the definition of probability as a frequency ratio. Cardano described the "circuit" to be the totality of permissible outcomes of an event. The "circuit" was 6 for one die, 36 for two dice, 216 for three dice. He then defined the probability of occurrence of a particular outcome as the sum of possible ways of achieving that outcome divided by the "circuit." Cardano investigated the probabilities of casting astragals and undertook to explain the relative occurrence of card combinations, notably for the game of Primero (loosely similar to Poker). A century before de Méré he posed the problem of the number of throws of two dice necessary to obtain at least one roll of two aces with an even chance (he answered 18 rather than the correct value of 24.6). Of this brilliant but erratic Milanese, the English historian Henry Morley wrote, "He was a genius, a fool, and a charlatan who embraced and amplified all the superstition of his age, and all its learning." Cardano was imbued with a sense of mysticism; his undoing came through a pathological belief in astrology. In a publicized event he cast the horoscope of the frail fifteen-year-old Edward VI of England, including specific predictions for the fifty-fifth year, third month, and seventeenth day of the monarch's life. Edward was inconsiderate enough to expire the following year at the age of sixteen. Undismayed, Cardano then had the temerity to cast the horoscope of Jesus Christ, an act not viewed with levity by sixteenth century theologians. Finally, when the self-predicted day of his own death arrived, with his health showing no signs of faltering, he redeemed his reputation by an act of suicide.

Following Cardano, several desultory assaults were launched on the mysteries of gambling. Kepler issued a few words on the subject, and shortly after the turn of the seventeenth century Galileo wrote a small treatise titled, *Considerazione sopra il Giuoco dei Dadi.*† A group of gambling noblemen of the Florence court had consulted Galileo to determine why the total 10 appears more often than 9 in throws of 3 dice. The famous physicist showed that 27 cases out of 216 possible total the number 10, while the number 9 occurs 25 times out of 216.

Then, in 1654, came the most significant event in the theory of gambling as the discipline of mathematical probability emerged from its chrysalis. A noted gambler and roué, Antoine Gombauld, the Chevalier de Méré, Sieur de Baussay, posed to his friend the Parisian mathematician Blaise Pascal, the following problem: "Why do the odds differ in throwing a 6 in four rolls of one die as opposed to throwing two 6's in 24 rolls of two dice?" In subsequent correspondence with Pierre de Fermat (then a jurist in Toulouse) to answer this question, Pascal constructed the foundations on which the theory of probability rests today. In the discussion of various gambling problems, Pascal's conclusions and calculations were occasionally incorrect, while Fermat achieved greater accuracy by considering both dependent and independent probabilities. Deriving a solution to the "Problem of Points" (two players are lacking x and y points, respectively, to win a game; if the game is interrupted, how should the stakes be divided between them?), Pascal developed an approach similar to the calculus of finite differences. Pascal was an inexhaustible genius from childhood; much of his mathematical work was begun at the age of sixteen. At nineteen he invented and constructed the first calculating machine in history.‡ He is also occasionally credited with the invention of the roulette wheel. Whoever of the two great mathematicians contributed more, Fermat and Pascal were first, based on considerations of games of chance, to place the theory of probability in a mathematical framework.

Curiously, the remaining half of the seventeenth century witnessed little interest in or extension of the work of Pascal and Fermat. In 1657 Christian Huygens published a treatise titled, *De Ratiociniis in Ludo Aleae* (Competition in Games of Chance), wherein he deals with the probability of certain dice combinations and originates the concept of "mathematical expectation." Leibnitz also produced work on probabilities, neither notable nor rigorous: he stated that the sums of 11 and 12, cast with

† Originally, "Sopra le Scoperte de i Dadi."

‡ The first calculating machine based on modern principles must be credited, however, to Charles Babbage (1830).

two dice, have equal probabilities (*Dissertatio de Arte Combinatoria*, 1666). John Wallis contributed a brief work on combinations and permutations, as did the Jesuit John Caramuel. John Arbuthnot, Francis Roberts, and Craig were responsible for occasional pronouncements on combinatorial analysis, but without special distinction. A shallow debut of the discipline of statistics was launched by John Graunt in his book on population growth, *Natural and Political Observations Made Upon the Bills of Mortality*. John de Witt analyzed the problem of annuities, and Edmund Halley published the first complete mortality tables.[†] By mathematical standards, however, none of these works can qualify as first-class achievements.

More important for the advancement of probabilistic comprehension was the destructive scepticism that arose during the Renaissance and Reformation. The doctrine of certainty in science, philosophy, and theology was severely attacked. In England, William Chillingworth promoted the view that man is unable to find absolutely certain religious knowledge. Rather, he asserted, a limited certitude based on common sense should be accepted by all reasonable men. Chillingworth's theme was later applied to scientific theory and practice by Glanville, Boyle, and Newton, and given a philosophical exposition by Locke.

Turning into the eighteenth century, the "Age of Reason" irrupted and the appeal of probability theory again attracted qualified mathematicians. The *Ars Conjectandi* (Art of Guessing) of Jacob Bernoulli developed the theory of permutations and combinations. One-fourth of the work (published posthumously in 1713) consists of solutions of problems relating to games of chance. Bernoulli wrote other treatises on dice combinations and was the first to consider the problem of duration of play. He analyzed various card games (for example, Trijaques) popular in his time and contributed to probability theory the famous theorem that by sufficiently increasing the number of observations, any preassigned degree of accuracy is attainable. Bernoulli's theorem was the first to express frequency statements within the formal framework of the probability calculus. Bernoulli envisioned the subject of probability from the most general point of view to that date. He predicted applications for the theory of probability outside the narrow range of problems relating to games of chance; the classical definition of probability is essentially derived from his work.

In 1708 Remond de Montmort published his work on chance titled, *Essai d'Analyse sur les Jeux d'Hazards*. This treatise was largely concerned

[†] Life insurance per se is an ancient practice. The first crude table of life expectancies was drawn up by Domitius Ulpianus circa A.D. 200.

with combinatorial analysis and dice and card probabilities. In connection with the game of Treize, or Rencontres, de Montmort was first to solve the matching problem (the probability that the value of a card coincides with the number expressing the order in which it is drawn). He also calculated the mathematical expectation of several intricate dice games: for example, Quinquenove, Hazard, Esperance, Trois Dez, Passe-dix, and Rafle. Of particular interest is his analysis of the card games le Her and le Jeu des Tas. Following de Montmort, Abraham de Moivre issued his work, *Doctrine of Chances*, which extended the knowledge of dice combinations, permutation theory, and card game analysis (specifically, the distribution of Honors in the game of Whist), and, most important, which proposed the first limit theorem. In this and subsequent treatises, he developed further the concepts of matching problems, duration of play, probability of ruin, mathematical expectation, and the theory of recurring series, and increased the value of Bernoulli's theorem by the aid of Stirling's theorem. On the basis of his approximation to the sum of terms of a binomial expansion (1733), he is commonly credited with the invention of the normal distribution. De Moivre lived much of his life among the London coffeehouse gamblers.[†] It was likely this environment that led him to write the *Doctrine of Chances*, which is virtually a gambler's manual.

Miscellaneous contributors to gambling and probability theory in the first half of the eighteenth century include Nicolas Bernoulli, Barbeyrac, Mairan, Nicole, and Thomas Simpson. The latter was responsible for introducing the idea of continuity into probability (1756). In general, these mathematicians analyzed contemporary card games and solved intricate dice problems. Very little additional understanding of probability was furnished to the *orbis scientiarum*.

Daniel Bernoulli developed the concepts of risk and mathematical expectation by the use of differential calculus. He also deserves credit, with de Moivre, for indicating the significance of the limit theorem to probability theory. Leonard Euler, far more renowned for contributions to other branches of mathematics, worked for some time on questions of probabilities and developed the theory of partitions, a subject first broached in a letter from Leibnitz to Johann Bernoulli (1669). He published a memoir titled, *Calcul de la Probabilité dans le Jeu de Rencontre*, in 1751 and calculated various lottery sequences and ticket-drawing combinations. Jean le Rond d'Alembert was best known for his opinions contrary to the scientific theories of his time. His errors and paradoxes abound in eighteenth

[†] His mathematics classes were held at Slaughter's Coffee House in St. Martin's Lane; one successful student was Thomas Bayes.

century mathematical literature. According to d'Alembert, the result of tossing three coins differs from three tosses of one coin. He also believed that tails is more probable after a long run of heads, and promulgated the doctrine that a very small probability is practically equivalent to zero. This idea leads to the progression betting system that bears his name. His analyses of card and dice games were equally penetrating.

The Rev. Thomas Bayes (his revolutionary paper, "An Essay Towards Solving a Problem in the Doctrine of Chances," was published in 1763, two years after his death) contributed the theorem stating exactly how the probability of a certain "cause" changes as different events actually occur. Although the proof of his formula rests on an unsatisfactory postulate, he was the first to use mathematical probability inductively—that is, arguing from a sample of the population or from the particular to the general. Bayes's theorem has (with some hazard) been made the foundation of the theory of testing statistical hypotheses. Whereas Laplace later defined probability by means of the enumeration of discrete units, Bayes defined a continuous probability distribution by a formula for the probability between any pair of assigned limits. He did not, however, consider the metric of his continuum.

Lagrange added his contribution to probability theory and solved many of the problems previously posed by de Moivre. Beguelin, Buffon,† Johann Lambert, the later John Bernoulli, Mallet, and William Emerson wrote sundry articles on games of chance and the calculus of probability during the second half of the eighteenth century. The Marquis de Condorcet supplied the doctrine of credibility—that is, the statement that the mathematical measure of probability is an accurate measure of our degree of belief. Trembley, Malfatti, and E. Waring developed further the theory of combinations.

Quantitatively, the theory of probability is more indebted to Pierre Simon, Marquis de Laplace, than to any other mathematician. His great work, *Théorie Analytique des Probabilités*, was published in 1812 and was accompanied by a popular exposition, *Essai Philosophique sur les Probabilités*. These tomes represent an outstanding contribution to the subject, containing a multitude of new ideas, results, and analytic methods. Parts of a theory of equations in finite differences with one and two independent variables are proposed, the concomitant analysis being applied to problems of duration of play and random sampling. Laplace elaborated on the idea

† In his *Political Arithmetic*, Buffon considered the chance of the sun rising tomorrow. He calculated the probability as $1 - (\frac{1}{2})^x$, where x is the number of days the sun is known to have risen over past history. Another contribution from Buffon was the establishment of 1/10,000 as the lowest practical probability.

of inverse probabilities first considered by Bayes. In subsequent memoirs, he investigated applications of generating functions, solutions and problems in birth statistics and the French lottery (a set of 90 numbers, five of which are drawn at a time), and generalized the matching problem of Montmort. Although he employed overly intricate analysis and not always lucid reasoning, he provided the greatest advance in probabilistic methodology in history. To Laplace is credited the first scientifically reasoned deterministic interpretation of the universe.

It was mid-eighteenth century when the empiricist David Hume studied the nature of probability and concluded that chance events are possessed of a subjective nature. Although he initiated development of the logical structure of the causality concept, Hume was unable to grasp fully the significance of inductive inference. He was not sufficiently endowed mathematically to exploit the theory of probability in a philosophic framework. By the early nineteenth century, the understanding of chance events had risen above the previous, naïve level; it was now accepted that probability statements could be offered with a high degree of certainty if they were transformed into statistical statements. The analysis of games of chance had led to the clarification of the meaning of probability and had provided its epistemological foundations.

Following Laplace, Denis Poisson was first to define probability as "the limit of a frequency" (*Rechercher sur le Probabilité des Jugements*, 1837), and the Austrian philosopher and mathematician Bernhard Bolzano expressed the view that probability represents a statistical concept of relative frequency. Like Laplace, Augustus de Morgan held that probability refers to a state of mind. Knowledge, he wrote, is never certain and the measure of the strength of our belief in any given proposition is referred to as its probability (Laplace had stated, "Chance is but the expression of man's ignorance."). The brilliant mathematician Karl Friedrich Gauss applied probability theory to astronomical research. After Gauss, most mathematicians chose to disregard a subject that had become virtually a stepchild in Victorian Europe. The English logician John Venn, the French economist Augustin Cournot, and the American logician Charles Sanders Peirce all expanded upon Bolzano's relative frequency concept. Henri Poincaré wrote an essay on chance, as did Peirce. In *The Doctrine of Chances*, Peirce in effect states that anything that can happen will happen: for example, insurance companies must eventually become bankrupt. This view, if taken literally, is a misleading signpost on a convoluted footpath.

Generally, the neglected orphan probability theory was fostered more by nonmathematicians. These men expanded the subject to include the modern statistical methods that may return far greater dividends in the