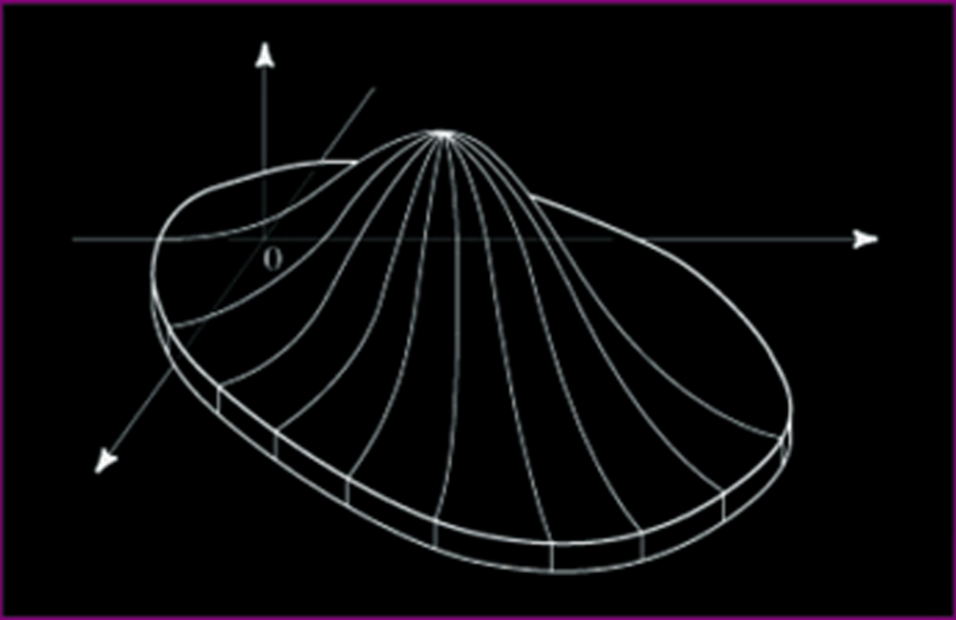
GEORGE G. ROUSSAS

A Course in Mathematical Statistics

SECOND EDITION



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Second Edition

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A Course in Mathematical Statistics

Second Edition

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To my wife and sons

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Contents

Preface to the Second Edition

This is the second edition of a book published for the first time in 1973 by Addison-Wesley Publishing Company, Inc., under the title *A First Course in Mathematical Statistics*. The first edition has been out of print for a number of years now, although its reprint in Taiwan is still available. That issue, however, is meant for circulation only in Taiwan.

The first issue of the book was very well received from an academic viewpoint. I have had the pleasure of hearing colleagues telling me that the book filled an existing gap between a plethora of textbooks of lower mathematical level and others of considerably higher level. A substantial number of colleagues, holding senior academic appointments in North America and elsewhere, have acknowledged to me that they made their entrance into the wonderful world of probability and statistics through my book. I have also heard of the book as being in a class of its own, and also as forming a collector's item, after it went out of print. Finally, throughout the years, I have received numerous inquiries as to the possibility of having the book reprinted. It is in response to these comments and inquiries that I have decided to prepare a second edition of the book.

This second edition preserves the unique character of the first issue of the book, whereas some adjustments are affected. The changes in this issue consist in correcting some rather minor factual errors and a considerable number of misprints, either kindly brought to my attention by users of the book or located by my students and myself. Also, the reissuing of the book has provided me with an excellent opportunity to incorporate certain rearrangements of the material.

One change occurring throughout the book is the grouping of exercises of each chapter in clusters added at the end of sections. Associating exercises with material discussed in sections clearly makes their assignment easier. In the process of doing this, a handful of exercises were omitted, as being too complicated for the level of the book, and a few new ones were inserted. In

Chapters 1 through 8, some of the materials were pulled out to form separate sections. These sections have also been marked by an asterisk (*) to indicate the fact that their omission does not jeopardize the flow of presentation and understanding of the remaining material.

Specifically, in Chapter 1, the concepts of a field and of a σ-field, and basic results on them, have been grouped together in Section 1.2*. They are still readily available for those who wish to employ them to add elegance and rigor in the discussion, but their inclusion is not indispensable. In Chapter 2, the number of sections has been doubled from three to six. This was done by discussing independence and product probability spaces in separate sections. Also, the solution of the problem of the probability of matching is isolated in a section by itself. The section on the problem of the probability of matching and the section on product probability spaces are also marked by an asterisk for the reason explained above. In Chapter 3, the discussion of random variables as measurable functions and related results is carried out in a separate section, Section 3.5*. In Chapter 4, two new sections have been created by discussing separately marginal and conditional distribution functions and probability density functions, and also by presenting, in Section 4.4*, the proofs of two statements, Statements 1 and 2, formulated in Section 4.1; this last section is also marked by an asterisk. In Chapter 5, the discussion of covariance and correlation coefficient is carried out in a separate section; some additional material is also presented for the purpose of further clarifying the interpretation of correlation coefficient. Also, the justification of relation (2) in Chapter 2 is done in a section by itself, Section 5.6*. In Chapter 6, the number of sections has been expanded from three to five by discussing in separate sections characteristic functions for the one-dimensional and the multidimensional case, and also by isolating in a section by itself definitions and results on moment-generating functions and factorial moment generating functions. In Chapter 7, the number of sections has been doubled from two to four by presenting the proof of Lemma 2, stated in Section 7.1, and related results in a separate section; also, by grouping together in a section marked by an asterisk definitions and results on independence. Finally, in Chapter 8, a new theorem, Theorem 10, especially useful in estimation, has been added in Section 8.5. Furthermore, the proof of Pólya's lemma and an alternative proof of the Weak Law of Large Numbers, based on truncation, are carried out in a separate section, Section 8.6*, thus increasing the number of sections from five to six.

In the remaining chapters, no changes were deemed necessary, except that in Chapter 13, the proof of Theorem 2 in Section 13.3 has been facilitated by the formulation and proof in the same section of two lemmas, Lemma 1 and Lemma 2. Also, in Chapter 14, the proof of Theorem 1 in Section 14.1 has been somewhat simplified by the formulation and proof of Lemma 1 in the same section.

Finally, a table of some commonly met distributions, along with their means, variances and other characteristics, has been added. The value of such a table for reference purposes is obvious, and needs no elaboration.

This book contains enough material for a year course in probability and statistics at the advanced undergraduate level, or for first-year graduate students not having been exposed before to a serious course on the subject matter. Some of the material can actually be omitted without disrupting the continuity of presentation. This includes the sections marked by asterisks, perhaps, Sections 13.4–13.6 in Chapter 13, and all of Chapter 14. The instructor can also be selective regarding Chapters 11 and 18. As for Chapter 19, it has been included in the book for completeness only.

The book can also be used independently for a one-semester (or even one quarter) course in probability alone. In such a case, one would strive to cover the material in Chapters 1 through 10 with the exclusion, perhaps, of the sections marked by an asterisk. One may also be selective in covering the material in Chapter 9.

In either case, presentation of results involving characteristic functions may be perfunctory only, with emphasis placed on moment-generating functions. One should mention, however, why characteristic functions are introduced in the first place, and therefore what one may be missing by not utilizing this valuable tool.

In closing, it is to be mentioned that this author is fully aware of the fact that the audience for a book of this level has diminished rather than increased since the time of its first edition. He is also cognizant of the trend of having recipes of probability and statistical results parading in textbooks, depriving the reader of the challenge of thinking and reasoning instead delegating the "thinking" to a computer. It is hoped that there is still room for a book of the nature and scope of the one at hand. Indeed, the trend and practices just described should make the availability of a textbook such as this one exceedingly useful if not imperative.

G. G. Roussas
Davis, California
May 1996

Preface to the First Edition

This book is designed for a first-year course in mathematical statistics at the undergraduate level, as well as for first-year graduate students in statistics—or graduate students, in general—with no prior knowledge of statistics. A typical three-semester course in calculus and some familiarity with linear algebra should suffice for the understanding of most of the mathematical aspects of this book. Some advanced calculus—perhaps taken concurrently—would be helpful for the complete appreciation of some fine points.

There are basically two streams of textbooks on mathematical statistics that are currently on the market. One category is the advanced level texts which demonstrate the statistical theories in their full generality and mathematical rigor; for that purpose, they require a high level, mathematical background of the reader (for example, measure theory, real and complex analysis). The other category consists of intermediate level texts, where the concepts are demonstrated in terms of intuitive reasoning, and results are often stated without proofs or with partial proofs that fail to satisfy an inquisitive mind. Thus, readers with a modest background in mathematics and a strong motivation to understand statistical concepts are left somewhere in between. The advanced texts are inaccessible to them, whereas the intermediate texts deliver much less than they hope to learn in a course of mathematical statistics. The present book attempts to bridge the gap between the two categories, so that students without a sophisticated mathematical background can assimilate a fairly broad spectrum of the theorems and results from mathematical statistics. This has been made possible by developing the fundamentals of modern probability theory and the accompanying mathematical ideas at the beginning of this book so as to prepare the reader for an understanding of the material presented in the later chapters.

This book consists of two parts, although it is not formally so divided. Part 1 (Chapters 1–10) deals with probability and distribution theory, whereas Part

2 (Chapters 11–20) is devoted to statistical inference. More precisely, in Part 1 the concepts of a field and σ-field, and also the definition of a random variable as a measurable function, are introduced. This allows us to state and prove fundamental results in their full generality that would otherwise be presented vaguely using statements such as "it may be shown that . . . ," "it can be proved that . . . ," etc. This we consider to be one of the distinctive characteristics of this part. Other important features are as follows: a detailed and systematic discussion of the most useful distributions along with figures and various approximations for several of them; the establishment of several moment and probability inequalities; the systematic employment of characteristic functions—rather than moment generating functions—with all the well-known advantages of the former over the latter; an extensive chapter on limit theorems, including all common modes of convergence and their relationship; a *complete* statement and proof of the Central Limit Theorem (in its classical form); statements of the Laws of Large Numbers and several proofs of the Weak Law of Large Numbers, and further useful limit theorems; and also an extensive chapter on transformations of random variables with numerous illustrative examples discussed in detail.

The second part of the book opens with an extensive chapter on sufficiency. The concept of sufficiency is usually treated only in conjunction with estimation and testing hypotheses problems. In our opinion, this does not do justice to such an important concept as that of sufficiency. Next, the point estimation problem is taken up and is discussed in great detail and as large a generality as is allowed by the level of this book. Special attention is given to estimators derived by the principles of unbiasedness, uniform minimum variance and the maximum likelihood and minimax principles. An abundance of examples is also found in this chapter. The following chapter is devoted to testing hypotheses problems. Here, along with the examples (most of them numerical) and the illustrative figures, the reader finds a discussion of families of probability density functions which have the monotone likelihood ratio property and, in particular, a discussion of exponential families. These latter topics are available only in more advanced texts. Other features are a complete formulation and treatment of the general Linear Hypothesis and the discussion of the Analysis of Variance as an application of it. In many textbooks of about the same level of sophistication as the present book, the above two topics are approached either separately or in the reverse order from the one used here, which is pedagogically unsound, although historically logical. Finally, there are special chapters on sequential procedures, confidence regions—tolerance intervals, the Multivariate Normal distribution, quadratic forms, and nonparametric inference.

A few of the proofs of theorems and some exercises have been drawn from recent publications in journals.

For the convenience of the reader, the book also includes an appendix summarizing all necessary results from vector and matrix algebra.

There are more than 120 examples and applications discussed in detail in the text. Also, there are more than 530 exercises, appearing at the end of the chapters, which are of both theoretical and practical importance.

The careful selection of the material, the inclusion of a large variety of topics, the abundance of examples, and the existence of a host of exercises of both theoretical and applied nature will, we hope, satisfy people of both theoretical and applied inclinations. All the application-oriented reader has to do is to skip some fine points of some of the proofs (or some of the proofs altogether!) when studying the book. On the other hand, the careful handling of these same fine points should offer some satisfaction to the more mathematically inclined readers.

The material of this book has been presented several times to classes of the composition mentioned earlier; that is, classes consisting of relatively mathematically immature, eager, and adventurous sophomores, as well as juniors and seniors, and statistically unsophisticated graduate students. These classes met three hours a week over the academic year, and most of the material was covered in the order in which it is presented with the occasional exception of Chapters 14 and 20, Section 5 of Chapter 5, and Section 3 of Chapter 9. We feel that there is enough material in this book for a three-quarter session if the classes meet three or even four hours a week.

At various stages and times during the organization of this book several students and colleagues helped improve it by their comments. In connection with this, special thanks are due to G. K. Bhattacharyya. His meticulous reading of the manuscripts resulted in many comments and suggestions that helped improve the quality of the text. Also thanks go to B. Lind, K. G. Mehrotra, A. Agresti, and a host of others, too many to be mentioned here. Of course, the responsibility in this book lies with this author alone for all omissions and errors which may still be found.

As the teaching of statistics becomes more widespread and its level of sophistication and mathematical rigor (even among those with limited mathematical training but yet wishing to know "why" and "how") more demanding, we hope that this book will fill a gap and satisfy an existing need.

G. G. R.
Madison, Wisconsin
November 1972

Chapter 1

Basic Concepts of Set Theory

1.1 Some Definitions and Notation

A *set* S is a (well defined) collection of distinct objects which we denote by s. The fact that s is *a member of* S, *an element of* S, or that it *belongs to* S is expressed by writing $s \in S$. The negation of the statement is expressed by writing $s \notin S$. We say that S' *is a subset of* S, or that S' *is contained in* S, and write $S' \subseteq S$, if for every $s \in S'$, we have $s \in S$. S' is said to be a *proper subset of* S, and we write $S' \subset S$, if $S' \subseteq S$ and there exists $s \in S$ such that $s \notin S'$. Sets are denoted by capital letters, while lower case letters are used for elements of sets.

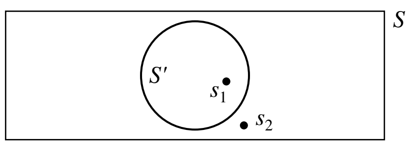

Figure 1.1 $S' \subseteq S$; in fact, $S' \subset S$, since $s_2 \in S$, but $s_2 \notin S'$.

These concepts can be illustrated pictorially by a drawing called a *Venn diagram* (Fig. 1.1). From now on a *basic*, or *universal* set, or *space* (which may be different from situation to situation), to be denoted by S, will be considered and all other sets in question will be subsets of S.

1.1.1 Set Operations

1. The *complement* (with respect to S) of the set A, denoted by A^c, is defined by $A^c = \{s \in S; s \notin A\}$. (See Fig. 1.2.)

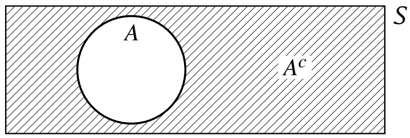

Figure 1.2 A^c is the shaded region.

2. The *union* of the sets A_j, $j = 1, 2, \ldots, n$, to be denoted by

$$A_1 \cup A_2 \cup \cdots \cup A_n \quad \text{or} \quad \bigcup_{j=1}^{n} A_j,$$

is defined by

$$\bigcup_{j=1}^{n} A_j = \left\{s \in S;\ s \in A_j \text{ for } \textit{at least} \text{ one } j = 1,\ 2, \ldots,\ n\right\}.$$

For $n = 2$, this is pictorially illustrated in Fig. 1.3. The definition extends to an infinite number of sets. Thus for denumerably many sets, one has

$$\bigcup_{j=1}^{\infty} A_j = \left\{s \in S;\ s \in A_j \text{ for } \textit{at least one } j = 1,\ 2, \ldots\right\}.$$

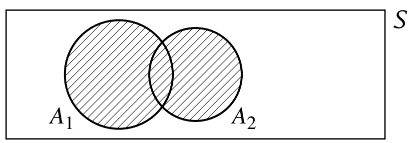

Figure 1.3 $A_1 \cup A_2$ is the shaded region.

3. The *intersection* of the sets A_j, $j = 1, 2, \ldots, n$, to be denoted by

$$A_1 \cap A_2 \cap \cdots \cap A_n \quad \text{or} \quad \bigcap_{j=1}^{n} A_j,$$

is defined by

$$\bigcap_{j=1}^{n} A_j = \left\{s \in S;\ s \in A_j \text{ for } \textit{all } j = 1,\ 2, \ldots,\ n\right\}.$$

For $n = 2$, this is pictorially illustrated in Fig. 1.4. This definition extends to an infinite number of sets. Thus for denumerably many sets, one has

$$\bigcap_{j=1}^{\infty} A_j = \left\{s \in S;\ s \in A_j \text{ for } \textit{all } j = 1,\ 2, \ldots\right\}.$$

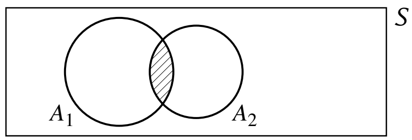

Figure 1.4 $A_1 \cap A_2$ is the shaded region.

4. The *difference* $A_1 - A_2$ is defined by

$$A_1 - A_2 = \{s \in S;\ s \in A_1,\ s \notin A_2\}.$$

Symmetrically,

$$A_2 - A_1 = \{s \in S;\ s \in A_2,\ s \notin A_1\}.$$

Note that $A_1 - A_2 = A_1 \cap A_2^c$, $A_2 - A_1 = A_2 \cap A_1^c$, and that, in general, $A_1 - A_2 \neq A_2 - A_1$. (See Fig. 1.5.)

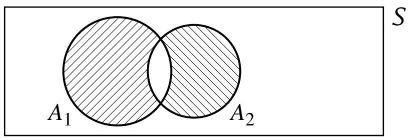

Figure 1.5 $A_1 - A_2$ is ////. $A_2 - A_1$ is \\\\.

5. The *symmetric difference* $A_1 \,\Delta\, A_2$ is defined by

$$A_1 \,\Delta\, A_2 = (A_1 - A_2) \cup (A_2 - A_1).$$

Note that

$$A_1 \,\Delta\, A_2 = (A_1 \cup A_2) - (A_1 \cap A_2).$$

Pictorially, this is shown in Fig. 1.6. It is worthwhile to observe that operations (4) and (5) can be expressed in terms of operations (1), (2), and (3).

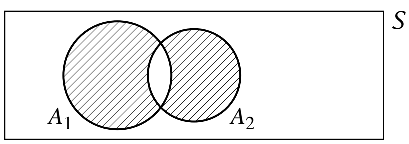

Figure 1.6 $A_1 \,\Delta\, A_2$ is the shaded area.

1.1.2 Further Definitions and Notation

A set which contains no elements is called the *empty set* and is denoted by $\varnothing$. Two sets A_1, A_2 are said to be *disjoint* if $A_1 \cap A_2 = \varnothing$. Two sets A_1, A_2 are said to be *equal*, and we write $A_1 = A_2$, if both $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$. The sets A_j, $j = 1, 2, \ldots$ are said to be *pairwise* or *mutually disjoint* if $A_i \cap A_j = \varnothing$ for all $i \neq j$ (Fig. 1.7). In such a case, it is customary to write

$$A_1 + A_2,\ A_1 + \cdots + A_n = \sum_{j=1}^{n} A_j \quad \text{and} \quad A_1 + A_2 + \cdots = \sum_{j=1}^{\infty} A_j$$

instead of $A_1 \cup A_2$, $\bigcup_{j=1}^{n} A_j$, and $\bigcup_{j=1}^{\infty} A_j$, respectively. We will write $\bigcup_j A_j$, $\sum_j A_j$, $\bigcap_j A_j$, where we do not wish to specify the range of j, which will usually be either the (finite) set $\{1, 2, \ldots, n\}$, or the (infinite) set $\{1, 2, \ldots\}$.

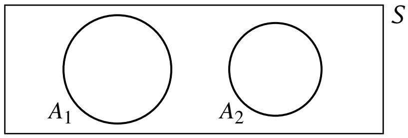

Figure 1.7 A_1 and A_2 are disjoint; that is, $A_1 \cap A_2 = \varnothing$. Also $A_1 \cup A_2 = A_1 + A_2$ for the same reason.

1.1.3 Properties of the Operations on Sets

1. $S^c = \varnothing$, $\varnothing^c = S$, $(A^c)^c = A$.
2. $S \cup A = S$, $\varnothing \cup A = A$, $A \cup A^c = S$, $A \cup A = A$.
3. $S \cap A = A$, $\varnothing \cap A = \varnothing$, $A \cap A^c = \varnothing$, $A \cap A = A$.

The previous statements are all obvious as is the following: $\varnothing \subseteq A$ for every subset A of S. Also

4. $\left.\begin{array}{l} A_1 \cup (A_2 \cup A_3) = (A_1 \cup A_2) \cup A_3 \\ A_1 \cap (A_2 \cap A_3) = (A_1 \cap A_2) \cap A_3 \end{array}\right\}$ (Associative laws)
5. $\left.\begin{array}{l} A_1 \cup A_2 = A_2 \cup A_1 \\ A_1 \cap A_2 = A_2 \cap A_1 \end{array}\right\}$ (Commutative laws)
6. $\left.\begin{array}{l} A \cap (\cup_j A_j) = \cup_j (A \cap A_j) \\ A \cup (\cap_j A_j) = \cap_j (A \cup A_j) \end{array}\right\}$ (Distributive laws)

are easily seen to be true.

The following identity is a useful tool in writing a union of sets as a sum of disjoint sets.

An identity:

$$\bigcup_j A_j = A_1 + A_1^c \cap A_2 + A_1^c \cap A_2^c \cap A_3 + \cdots.$$

There are two more important properties of the operation on sets which relate complementation to union and intersection. They are known as **De Morgan's laws**:

$$\text{i)} \quad \left(\bigcup_j A_j\right)^c = \bigcap_j A_j^c,$$

$$\text{ii)} \quad \left(\bigcap_j A_j\right)^c = \bigcup_j A_j^c.$$

As an example of a set theoretic proof, we prove (i).

PROOF OF (i) We wish to establish

a) $(\bigcup_j A_j)^c \subseteq \bigcap_j A_j^c$ and **b)** $\bigcap_j A_j^c \subseteq (\bigcup_j A_j)^c$.

We will then, by definition, have verified the desired equality of the two sets.

a) Let $s \in (\bigcup_j A_j)^c$. Then $s \notin \bigcup_j A_j$, hence $s \notin A_j$ for any j. Thus $s \in A_j^c$ for every j and therefore $s \in \bigcap_j A_j^c$.

b) Let $s \in \bigcap_j A_j^c$. Then $s \in A_j^c$ for every j and hence $s \notin A_j$ for any j. Then $s \notin \bigcup_j A_j$ and therefore $s \in (\bigcup_j A_j)^c$.

The proof of (ii) is quite similar. ▲

This section is concluded with the following:

DEFINITION 1 The sequence $\{A_n\}$, $n = 1, 2, \ldots$, is said to be a *monotone sequence* of sets if:

i) $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ (that is, A_n is *increasing*, to be denoted by $A_n \uparrow$), or

ii) $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ (that is, A_n is *decreasing*, to be denoted by $A_n \downarrow$).

The *limit* of a monotone sequence is defined as follows:

i) If $A_n \uparrow$, then $\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$, and

ii) If $A_n \downarrow$, then $\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$.

More generally, for any sequence $\{A_n\}$, $n = 1, 2, \ldots$, we define

$$\underline{A} = \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} A_j,$$

and

$$\overline{A} = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j.$$

The sets $\underline{A}$ and $\overline{A}$ are called the *inferior limit* and *superior limit*, respectively, of the sequence $\{A_n\}$. The sequence $\{A_n\}$ has a *limit* if $\underline{A} = \overline{A}$.

Exercises

1.1.1 Let A_j, $j = 1, 2, 3$ be arbitrary subsets of S. Determine whether each of the following statements is correct or incorrect.

i) $(A_1 - A_2) \cup A_2 = A_2$;

ii) $(A_1 \cup A_2) - A_1 = A_2$;

iii) $(A_1 \cap A_2) \cap (A_1 - A_2) = \varnothing$;

iv) $(A_1 \cup A_2) \cap (A_2 \cup A_3) \cap (A_3 \cup A_1) = (A_1 \cap A_2) \cup (A_2 \cap A_3) \cup (A_3 \cap A_1)$.

1.1.2 Let $S = \{(x, y)' \in \mathbb{R}^2; -5 \leq x \leq 5, 0 \leq y \leq 5, x, y = \text{integers}\}$, where prime denotes transpose, and define the subsets $A_j, j = 1, \ldots, 7$ of S as follows:

$$A_1 = \left\{(x, y)' \in S;\ x = y\right\}; \qquad A_2 = \left\{(x, y)' \in S;\ x = -y\right\};$$

$$A_3 = \left\{(x, y)' \in S;\ x^2 = y^2\right\}; \qquad A_4 = \left\{(x, y)' \in S;\ x^2 \leq y^2\right\};$$

$$A_5 = \left\{(x, y)' \in S;\ x^2 + y^2 \leq 4\right\}; \qquad A_6 = \left\{(x, y)' \in S;\ x \leq y^2\right\};$$

$$A_7 = \left\{(x, y)' \in S;\ x^2 \geq y\right\}.$$

List the members of the sets just defined.

1.1.3 Refer to Exercise 1.1.2 and show that:

i) $A_1 \cap \left(\bigcup_{j=2}^{7} A_j\right) = \bigcup_{j=2}^{7} \left(A_1 \cap A_j\right);$

ii) $A_1 \cup \left(\bigcap_{j=2}^{7} A_j\right) = \bigcap_{j=2}^{7} \left(A_1 \cup A_j\right);$

iii) $\left(\bigcup_{j=1}^{7} A_j\right)^c = \bigcap_{j=1}^{7} A_j^c;$

iv) $\left(\bigcap_{j=1}^{7} A_j\right)^c = \bigcup_{j=1}^{7} A_j^c$

by listing the members of each one of the eight sets appearing on either side of each one of the relations (i)–(iv).

1.1.4 Let A, B and C be subsets of S and suppose that $A \subseteq B$ and $B \subseteq C$. Then show that $A \subseteq C$; that is, the subset relationship is transitive. Verify it by taking $A = A_1$, $B = A_3$ and $C = A_4$, where A_1, A_3 and A_4 are defined in Exercise 1.1.2.

1.1.5 Establish the distributive laws stated on page 4.

1.1.6 In terms of the acts A_1, A_2, A_3, and perhaps their complements, express each one of the following acts:

i) $B_i = \{s \in S;\ s$ belongs to exactly i of A_1, A_2, A_3, where $i = 0, 1, 2, 3\}$;

ii) $C = \{s \in S;\ s$ belongs to all of $A_1, A_2, A_3\}$;

iii) $D = \{s \in S;\ s$ belongs to none of $A_1, A_2, A_3\}$;
iv) $E = \{s \in S;\ s$ belongs to at most 2 of $A_1, A_2, A_3\}$;
v) $F = \{s \in S;\ s$ belongs to at least 1 of $A_1, A_2, A_3\}$.

1.1.7 Establish the identity stated on page 4.

1.1.8 Give a detailed proof of the second identity in De Morgan's laws; that is, show that

$$\left(\bigcap_j A_j\right)^c = \bigcup_j A_j^c.$$

1.1.9 Refer to Definition 1 and show that

i) $\underline{A} = \{s \in S;\ s$ belongs to all but finitely many A's$\}$;
ii) $\bar{A} = \{s \in S;\ s$ belongs to infinitely many A's$\}$;
iii) $\underline{A} \subseteq \bar{A}$;
iv) If $\{A_n\}$ is a monotone sequence, then $\underline{A} = \bar{A} = \lim_{n\to\infty} A_n$.

1.1.10 Let $S = \mathbb{R}^2$ and define the subsets $A_n, B_n, n = 1, 2, \ldots$ of S as follows:

$$A_n = \left\{(x, y)' \in \mathbb{R}^2;\ 3 + \frac{1}{n} \le x < 6 - \frac{2}{n},\ 0 \le y \le 2 - \frac{1}{n^2}\right\},$$
$$B_n = \left\{(x, y)' \in \mathbb{R}^2;\ x^2 + y^2 \le \frac{1}{n^3}\right\}.$$

Then show that $A_n \uparrow A$, $B_n \downarrow B$ and identify A and B.

1.1.11 Let $S = \mathbb{R}$ and define the subsets $A_n, B_n, n = 1, 2, \ldots$ of S as follows:

$$A_n = \left\{x \in \mathbb{R};\ -5 + \frac{1}{n} < x < 20 - \frac{1}{n}\right\}, \qquad B_n = \left\{x \in \mathbb{R};\ 0 < x < 7 + \frac{3}{n}\right\}.$$

Then show that $A_n\uparrow$ and $B_n\downarrow$, so that $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$ exist (by Exercise 1.1.9(iv)). Also identify the sets A and B.

1.1.12 Let A and B be subsets of S and for $n = 1, 2, \ldots$, define the sets A_n as follows: $A_{2n-1} = A$, $A_{2n} = B$. Then show that

$$\liminf_{n\to\infty} A_n = A \cap B, \qquad \limsup_{n\to\infty} A_n = A \cup B.$$

1.2* Fields and σ-Fields

In this section, we introduce the concepts of a field and of a σ-field, present a number of examples, and derive some basic results.

DEFINITION 2 A class (set) of subsets of S is said to be a *field*, and is denoted by $\mathcal{F}$, if

($\mathcal{F}$1) $\mathcal{F}$ is a non-empty class.

($\mathcal{F}$2) $A \in \mathcal{F}$ implies that $A^c \in \mathcal{F}$ (that is, $\mathcal{F}$ is closed under complementation).

($\mathcal{F}$3) $A_1, A_2 \in \mathcal{F}$ implies that $A_1 \cup A_2 \in \mathcal{F}$ (that is, $\mathcal{F}$ is closed under pairwise unions).

1.2.1 Consequences of the Definition of a Field

1. $S, \varnothing \in \mathcal{F}$.
2. If $A_j \in \mathcal{F}$, $j = 1, 2, \ldots, n$, then $\bigcup_{j=1}^{n} A_j \in \mathcal{F}$, $\bigcap_{j=1}^{n} A_j \in \mathcal{F}$ for any *finite* n.

(That is, $\mathcal{F}$ is closed under finite unions and intersections. Notice, however, that $A_j \in \mathcal{F}$, $j = 1, 2, \ldots$ need not imply that their union or intersection is in $\mathcal{F}$; for a counterexample, see consequence 2 on page 10.)

PROOF OF (1) AND (2) (1) ($\mathcal{F}$1) implies that there exists $A \in \mathcal{F}$ and ($\mathcal{F}$2) implies that $A^c \in \mathcal{F}$. By ($\mathcal{F}$3), $A \cup A^c = S \in \mathcal{F}$. By ($\mathcal{F}$2), $S^c = \varnothing \in \mathcal{F}$.

(2) The proof will be by induction on n and by one of the De Morgan's laws. By ($\mathcal{F}$3), if $A_1, A_2 \in \mathcal{F}$, then $A_1 \cup A_2 \in \mathcal{F}$; hence the statement for unions is true for $n = 2$. (It is trivially true for $n = 1$.) Now assume the statement for unions is true for $n = k - 1$; that is, if

$$A_1, A_2, \ldots, A_{k-1} \in \mathcal{F}, \quad \text{then} \quad \bigcup_{j=1}^{k-1} A_j \in \mathcal{F}.$$

Consider $A_1, A_2, \ldots, A_k \in \mathcal{F}$. By the associative law for unions of sets,

$$\bigcup_{j=1}^{k} A_j = \left(\bigcup_{j=1}^{k-1} A_j\right) \cup A_k.$$

By the induction hypothesis, $\bigcup_{j=1}^{k-1} A_j \in \mathcal{F}$. Since $A_k \in \mathcal{F}$, ($\mathcal{F}$3) implies that

$$\left(\bigcup_{j=1}^{k-1} A_j\right) \cup A_k = \bigcup_{j=1}^{k} A_j \in \mathcal{F}$$

and by induction, the statement for unions is true for any finite n. By observing that

$$\bigcap_{j=1}^{n} A_j = \left(\bigcup_{j=1}^{n} A_j^c\right)^c,$$

* The reader is reminded that sections marked by an asterisk may be omitted without jeopardizing the understanding of the remaining material.

we see that ($\mathcal{F}2$) and the above statement for unions imply that if $A_1, \ldots, A_n \in \mathcal{F}$, then $\bigcap_{j=1}^{n} A_j \in \mathcal{F}$ for any finite n. ▲

1.2.2 Examples of Fields

1. $C_1 = \{\varnothing, S\}$ is a field (*trivial field*).
2. $C_2 = \{$all subsets of $S\}$ is a field (*discrete field*).
3. $C_3 = \{\varnothing, S, A, A^c\}$, for some $\varnothing \subset A \subset S$, is a field.
4. Let S be infinite (countably so or not) and let C_4 be the class of subsets of S which are finite, or whose complements are finite; that is, $C_4 = \{A \subset S; A$ or A^c is finite$\}$.

As an example, we shall verify that C_4 is a field.

PROOF THAT C_4 IS A FIELD

i) Since $S^c = \varnothing$ is finite, $S \in C_4$, so that C_4 is non-empty.

ii) Suppose that $A \in C_4$. Then A or A^c is finite. If A is finite, then $(A^c)^c = A$ is finite and hence $A^c \in C_4$ also. If A^c is finite, then $A^c \in C_4$.

iii) Suppose that $A_1, A_2 \in C_4$. Then A_1 or A_1^c is finite and A_2 or A_2^c is finite.

a) Suppose that A_1, A_2 are both finite. Then $A_1 \cup A_2$ is finite, so that $A_1 \cup A_2 \in C_4$.

b) Suppose that A_1^c, A_2 are finite. Then $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$ is finite since A_1^c is. Hence $A_1 \cup A_2 \in C_4$.

The other two possibilities follow just as in (b). Hence ($\mathcal{F}1$), ($\mathcal{F}2$), ($\mathcal{F}3$) are satisfied. ▲

We now formulate and prove the following theorems about fields.

THEOREM 1 Let I be any non-empty index set (finite, or countably infinite, or uncountable), and let $\mathcal{F}_j$, $j \in I$ be fields of subsets of S. Define $\mathcal{F}$ by $\mathcal{F} = \bigcap_{j \in I} \mathcal{F}_j = \{A; A \in \mathcal{F}_j$ for all $j \in I\}$. Then $\mathcal{F}$ is a field.

PROOF

i) $S, \varnothing \in \mathcal{F}_j$ for every $j \in I$, so that $S, \varnothing \in \mathcal{F}$ and hence $\mathcal{F}$ is non-empty.

ii) If $A \in \mathcal{F}$, then $A \in \mathcal{F}_j$ for every $j \in I$. Thus $A^c \in \mathcal{F}_j$ for every $j \in I$, so that $A^c \in \mathcal{F}$.

iii) If $A_1, A_2 \in \mathcal{F}$, then $A_1, A_2 \in \mathcal{F}_j$ for every $j \in I$. Then $A_1 \cup A_2 \in \mathcal{F}_j$ for every $j \in I$, and hence $A_1 \cup A_2 \in \mathcal{F}$. ▲

THEOREM 2 Let C be an arbitrary class of subsets of S. Then there is a unique minimal field $\mathcal{F}$ containing C. (We say that $\mathcal{F}$ is *generated* by C and write $\mathcal{F} = \mathcal{F}(C)$.)

PROOF Clearly, C is contained in the discrete field. Next, let $\{\mathcal{F}_j, j \in I\}$ be the class of all fields containing C and define $\mathcal{F}(C)$ by

$$\mathcal{F}(C) = \bigcap_{j \in I} \mathcal{F}_j.$$